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PRINCIPLES OF DIRECT-CURRENT MACHINES

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PRINCIPLES OF DIRECT-CURRENT MACHINES

BY

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FIFTH EDITION

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In Memory of

My Mother

SARAH SUSS LANGSDORF

PREFACE TO THE FIFTH EDITION

The distinctive difference between this edition and the four which preceded it is that it includes a more extensive treatment of the general principles of electricity and magnetism than was possible in the single introductory chapter previously allotted to basic theory. The material presented in the first three chapters of this book is an amplification and extension of the original first chapter, which dealt only with those principles of electric and magnetic circuits, and of electromagnetism, which find immediate application in the theory of direct-current machinery; to this has been added a chapter on electrostatics and the dielectric circuit, and another dealing with the important subject of units and dimensions of electric and magnetic quantities. These five chapters have been planned to meet the requirements of introductory courses of instruction, now given as early as the sophomore year in well-organized electrical engineering curricula. Some of the material that has been included, such as the elementary theory of electric images, is not ordinarily presented to beginners, but there is no part of the subject matter that is beyond the understanding of students who have either completed, or are concurrently pursuing, the usual courses in college physics and calculus. The attempt has been made throughout the book to arrange the material in such a way that an instructor, by judicious omission of parts, may make the remainder fit within the limitations of class schedules without sacrifice of continuity.

The adoption by the International Electrotechnical Commission of the meter-kilogram-second (m.k.s.) system of units made it necessary to emphasize the subject of units in general. There are able writers and teachers who believe that the advantages of the m.k.s. system are so great as to warrant the immediate rejection of any reference to the older c.g.s. system, and who therefore advocate that from now on students be taught exclusively in terms of m.k.s. units. The author has been unable to accept this conclusion, for the reason that for many years

students and young graduates will find it necessary to consult references written in terms of the older systems, and to associate with practicing engineers who may not be familiar with the new system. Accordingly, this book has been designed to present the m.k.s. system in its proper relation to the c.g.s. electrostatic and electromagnetic systems, and to the so-called practical units. In the chapters which deal with the operating characteristics of generators and motors, the formulas have in some cases been kept, as in earlier editions, in terms of centimeters and square centimeters, or inches and square inches, because it is very unlikely that manufacturers will replace these convenient and familiar units by meters and square meters; but the treatment is such that conversion to the new units is readily possible, and some problems have been incorporated to develop facility in doing so.

The development of the m.k.s. system has thrown entirely new light upon the distinction between H (strength of magnetic field) and B (magnetic flux density); and between E (strength of electric field) and D (dielectric flux density). It has likewise forced a more searching analysis of the meaning and dimensions of space permeability, μ_0, and space permittivity, ϵ_0. There has been much error and confusion in books and articles dealing with these relations, not excepting earlier editions of this book. But all of these errors have been eliminated in this edition, as may be seen by referring to the formulas and equations in which μ_0 and ϵ_0 now appear explicitly, whereas formerly they were assigned unit value and thereafter incorrectly ignored.

Special attention is directed to Art. 22 of Chap. V, which includes a very simple and easily understood derivation of the formula for the velocity of propagation of an electromagnetic wave. This formula is essential to a complete understanding of the magnitudes and dimensions of μ_0 and ϵ_0, but beginning students have heretofore been obliged to accept it on faith since the usual derivation employs partial differential equations, a branch of mathematics ordinarily beyond their preparation. The author desires to acknowledge his indebtedness to his friend and colleague, Professor G. E. M. Jauncey, of the Department of Physics of Washington University, for invaluable help in the development of Eq. (56) of Chap. V, which is one of the two equations that lead directly to the desired result.

The chapter on commutation has been entirely rewritten and greatly simplified by discarding the mathematical approach, which has been found to be misleading and undesirable. Some of the remaining chapters have been rearranged and partially rewritten to bring the material up to date and to eliminate ambiguities and minor errors, but the original treatment of generator and motor characteristics has been left substantially unchanged. Entirely new sets of problems have been prepared and have been incorporated at the end of the book.

Sincere thanks are extended to the numerous teachers in this and foreign countries who have from time to time made suggestions for corrections and improvements; and to the officers of manufacturing companies for help in obtaining photographs and drawings.

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April, 1940.

PREFACE TO THE FIRST EDITION

This book has been prepared with the object of placing before junior and senior students of electrical engineering a reasonably complete treatment of the fundamental principles that underlie the design and operation of all types of direct-current machinery. Instead of attempting to touch the "high spots" in the whole field of direct-current engineering, attention has been concentrated upon certain important features that are ordinarily dismissed with little more than passing mention, but which, in the opinion of the author, are vital to a thorough grasp of the subject. For example, the book will be found to contain in Chap. III a full derivation of the rules covering armature windings (following Professor Arnold), in addition to the usual description of typical windings; Chapters VI and VII include a considerable amount of new material concerning the operating characteristics of generators and motors, the treatment being largely graphical and including the use of three-dimensional diagrams for depicting the mutual relationships among all of the variables; and in Chapters VIII and IX there has been developed a much more extensive treatment of the important subject of commutation than has been heretofore easily accessible to students of the type for whom the book is intended. In the selection and arrangement of the material dealing with commutation, care has been exercised to eliminate those minute details and excessive refinements that are more likely to confuse than to clarify.

Although the methods of the calculus have been freely used throughout the book, a conscious effort has been made to give special prominence to the physical concepts of which the equations are merely the short-hand expressions; to this end, the mathematical analysis has been preceded, wherever possible, by a full and copiously illustrated discussion of the physical facts of the problem and their relations to one another. This has been done to counteract the tendency, manifested by many students, to look upon a mathematical solution of a problem as an end complete in itself, apparently without a due realization

that the first essential is a clearly thought out analysis of physical realities. As an example of this procedure, attention is directed to the new material of Article 210 of Chap. XI.[1]

The illustrative problems at the end of each of the first ten chapters have, for the most part, been designed to prevent the practice of feeding figures into one end of a formula and extracting the result (painlessly) from the other end. No attempt has been made to include as complete a set of problems as is desirable in studying the subject, for the reason that each instructor will naturally prepare a set to meet his own needs. Some of the problems at the end of Chapters VI and VII will be found to tax the reasoning powers of the best students, but all of them have been successfully solved in the author's classes. Answers have not been given in the text, but will be supplied upon request to those instructors who ask for them.

It is not to be expected that a new book on direct currents can avoid including much material common to the large number of existing texts on the subject. Such originality as has been brought to bear, aside from that represented by the new matter already referred to, has been exercised in selecting from the vast amount of available material those parts that seem most essential to an orderly presentation of the subject. Numerous well-known texts have been freely drawn upon, with suitable acknowledgment in all essential cases.

That part of Chapter IV which deals with details of the calculation of the magnetization curve and of magnetic leakage, and the part of Chapter VIII in which the formulas for armature inductance are developed, may be omitted without interfering with the continuity of treatment, in case design is taught as a separate course.

In conclusion, the author desires to express his sincere thanks to Professor H. E. Clifford, of Harvard University, who made helpful criticisms and suggestions after reading the original manuscript, and who also assisted in the proof reading; and to the various manufacturers who have kindly contributed illustrations.

ALEXANDER S. LANGSDORF.

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[1] See Article 20, Chap. XI, 4th ed.

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CHAPTER I

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1. Historical.*—The first genuinely scientific study of electricity is recorded in an epoch-making book† published in 1600 by Sir William Gilbert, court physician to Queen Elizabeth. Prior to that time the entire extent of knowledge of this subject was the single fact, probably known in remote antiquity, but first mentioned by Thales of Miletus in 585 B.C., that amber and jet, when rubbed, acquire the property of attracting light materials such as bits of straw and feathers. To Gilbert belongs the credit for showing that this property of amber (Greek *elektron*) is likewise possessed by numerous other substances which he called *electrics*, but which in his listing comprised materials now classed as insulators or nonconductors of electricity. Thus was initiated an era, which lasted for almost 200 years, devoted to the study of static electricity.

Following Gilbert, there was no further experimental advance until Otto von Guericke of Magdeburg constructed the first electrostatic generator, about 1663. Like Gilbert, he had been led to a study of electricity and magnetism in an attempt to account for the forces demanded by the Copernican theory of the solar system; for that reason, his machine consisted of a rotating sphere of sulphur (the spherical shape in imitation of the earth)

* For further details, see E. T. WHITTAKER, "History of the Theories of Aether and Electricity"; J. A. FLEMING, Article on Electricity, *Encyclopaedia Britannica;* L. B. LOEB, "Fundamentals of Electricity and Magnetism."

† "De Magnete, Magneticisque Corporibus, et de Magno Magnete Tellure (On the Magnet, Magnetic Bodies, and the Great Magnet, the Earth)."

which, when rubbed with the dry hand, developed electrical effects of sufficient intensity to permit a wide range of experiments. He found that a feather brought near the electrified sphere became electrified by induction, and he observed the noise and the light produced by an electric spark, as well as the fact that the electrical effects could be transmitted over a 6-ft. length of linen cord.

Largely because of the absence of scientific journals at that time, von Guericke's results remained unknown or were forgotten and were later rediscovered by others. Thus, in 1708, Wall had observed the light and noise of long sparks and likened them to lightning and thunder, and the actual identification of lightning as an electrical discharge was finally established by Franklin's famous kite experiment in 1752. Stephen Gray found (1729) that electrification produced by friction could be transmitted over 50 ft. of moist linen cord provided that the cord was suspended by silk threads, thus rediscovering not only the fact of conduction but, what was even more important, that certain substances are nonconductors or insulators.

In France, Charles François du Fay in 1733 repeated Gray's experiments and greatly extended them. It was he who discovered that vitreous (glassy) substances, electrified by being rubbed with silk, attract resinous substances that have been rubbed with wool, while electrified vitreous bodies repel each other, as do also electrified resinous bodies; but, still more important, he found, contrary to the prior belief that the so-called nonelectrics (that is, conductors) could not be electrified, that all substances, including conductors, can be electrified, the necessary condition in the case of conductors being that they must be suitably insulated. These facts, in conjunction with the possibility of electrification by induction, led Du Fay in 1733 to postulate the *two-fluid theory* of electricity, which assumes that there exist normally in all substances equal quantities of weightless continuous fluids, one called vitreous (or positive), the other resinous (or negative), which annul each other when mixed together. In accordance with this theory, the removal of a portion of one of these fluids leaves the substance charged, owing to the excess of the other; electrification by induction is then accounted for by a separation of the two fluids, which become more or less concentrated at opposite ends of the body.

The two-fluid theory was vigorously opposed by Franklin, who advanced the *one-fluid theory* as a substitute. The one-fluid theory assumes that all substances in the neutral state are permeated by a definite or normal amount of weightless fluid, which, if increased in quantity, gives the equivalent of a positive charge, whereas a decrease produces a negative charge.

The one- and the two-fluid theories remained in controversy for many years, but with the weight of opinion in general inclining toward the acceptance of the two-fluid theory until after Faraday's work had been completed. In any case, the hydraulic terminology employed in both theories has stamped itself indelibly upon all subsequent developments, as is clear from such common expressions as flow of current, electric circuit, charge of a condenser, etc. It was not until after the discovery of X-rays by Roentgen in 1896 that J. J. Thomson was able to prove experimentally that the cathode beam in an X-ray tube consists of minute, swiftly-moving particles, or corpuscles, each carrying a very small negative charge. The name *electron* was given to these particles, and it was found later that, from whatever source they might be derived, all had the same mass and the same charge. The electron seemed to be the essential stuff out of which ordinary matter is built up, and the one-fluid theory of Franklin appeared to be vindicated. However, it has been found that the atomic structure of matter cannot be accounted for in terms of electrons alone. The atom of hydrogen for example, which is the simplest of all the chemical elements, consists of a positively charged nucleus called a *proton* around which rotates a single electron, the charge on the proton being equal and opposite to that of the electron; the electron accounts, however, for only $\frac{1}{1840}$ of the mass of the atom, the proton for all the remainder. Helium, the next element in the periodic table, has a nucleus that was originally conceived of as containing four protons and two electrons, thus yielding a net positive charge of two units, neutralized as to exterior points by two electrons revolving around the nucleus; more recently, however, it has been found that the helium nucleus is in reality composed of two protons and two *neutrons*. The neutron has been found to be a definite entity, having very nearly the same mass as a proton, but without any charge whatever. Quite recently (in 1932) Anderson has demonstrated the existence of

minute positive particles having the same mass as the electron but an equal charge of opposite sign, and the name *positron* has been given to these particles. In effect, therefore, the two-fluid theory has been restored, but with the significant difference that, whereas initially the electric charge was thought of as a homogeneous, weightless, continuous fluid, it has now come to be regarded as made up of discrete, granular particles each having finite but exceedingly minute mass.

For nearly half a century after Du Fay's experiments, work in this field continued to be of the same qualitative kind originated by Gilbert. But, in 1785, Coulomb made a great forward step by introducing quantitative measurements for the first time. Proceeding from earlier studies of the deflection of a compass needle, he was led to the invention of the torsion balance for the measurement of small forces. It is interesting to note that the concept of force as a measurable entity grew out of the formulation of Newton's laws of motion, first published in 1687, but not generally accepted until about 1750. With the aid of the torsion balance, Coulomb was able to determine that electric charges, if on bodies small enough to approximate geometrical points, act upon each other *with a force proportional to their magnitudes and inversely proportional to the square of the distance between them.* Thus stated, the law is known as Coulomb's law, but it is interesting to note that the inverse square relationship had been theoretically deduced as early as 1767 by Joseph Priestley, the discoverer of oxygen.

With the exception of the isolated facts concerning the conduction of electricity that had been discovered by von Guericke and Gray, all the developments from the time of Gilbert to that of Coulomb, and for some years thereafter, dealt with the properties of electricity at rest. Then, in 1799, practically two centuries after Gilbert's book appeared, came the development by Alessandro Volta of the voltaic pile, or galvanic cell, which for the first time made available a source of large quantities of electricity in motion and made studies of the electric current possible. Volta's work was itself an outgrowth of Galvani's observations (1780) on the twitching of the leg muscles of a frog, as induced by touching the nerve with the ends of two dissimilar metals joined at their other ends. The voltaic pile, as finally perfected, consisted of a set of alternate disks of dis-

similar metals separated in pairs by cloths moistened with a conducting salt solution. This was the forerunner of numerous types of primary batteries or cells whose development followed quickly on Volta's discovery, and in the hands of a rapidly increasing number of experimenters served to expand knowledge of the subject at an accelerating rate of increase. Volta himself, however, showed by means of the electroscope that the electricity produced by a galvanic cell is identical with that produced by friction; the differences between what is generally called *static electricity* on the one hand and *dynamic electricity*, or electric current, on the other, may all be attributed to the fact that static electricity consists of small quantities of electricity at rest whereas dynamic electricity consists of large quantities of electricity in motion relative to the observer.

In identifying the polarity of charges derived from batteries, Volta used the electroscope, an instrument originally devised in crude form by Gilbert, but later more highly developed. Quite naturally, therefore, it came to be assumed in terms of the one-fluid theory that the current of supposedly positive fluid flowing from the battery to an external metallic circuit issued from the terminal that showed vitreous, or positive, charge and returned to the battery by way of the negative terminal. This is the convention which is still used everywhere, though it has been known since the discovery of the electron that what actually flows through metallic conductors is a stream of electrons moving, in the circuit external to the battery, from the negative to the positive terminal. The metallic atoms of the conductor from which the electrons are detached are positively charged, but being physically bound they do not move relative to the conductor itself. In liquid conductors, the electrolyte becomes dissociated into positively and negatively charged ions, the former moving to the negative terminal, or cathode, the latter to the positive terminal, or anode. It is clear, therefore, that the long controversy concerning the one- and the two-fluid theories was futile; for there was some truth in both, but neither was wholly correct.

The period from 1799 to 1831 was characterized by the attention paid to the effects of the electric current and by the development of the concepts upon which methods of measuring those effects could be based. Thus, in 1800, Ritter discovered that electrolysis of water yielded oxygen and hydrogen; and, in 1808,

Davy electrolyzed molten alkalies and so discovered sodium and potassium. In 1819, Oersted made the important discovery, first published in July, 1820, that a current flowing in a wire deflects a compass needle in its immediate vicinity, thus disclosing for the first time that there is an intimate relation between electricity and magnetism and that a magnetic field is invariably associated with an electric current; a corollary to the fact that a current exerts a force upon a magnet is that the magnet exerts an equal and opposite force upon the conductor carrying the current, and this mutuality of forces was experimentally verified by Oersted himself.

The first quantitative study of the relation between the force on a magnet and the current producing the force was published jointly by Biot and Savart in October, 1820; they showed by experimental means that, if a current flows in a long straight wire, it exerts a force upon a magnet which varies inversely as the distance of the magnet from the wire; but the first complete analysis of the problem was the work of Ampère, who, in a series of brilliant papers, the first of which appeared in September, 1820, laid the foundations for the entire subject of electrodynamics. It was Ampère who formulated the basic mathematical laws for computing the forces between currents flowing in neighboring circuits, and between currents and magnets, his final results being sufficiently general to include as a special case the experimentally deduced law of Biot and Savart. Ampère discovered that there is attraction between parallel currents flowing in the same direction and repulsion when the currents flow in opposite directions and that a current flowing in a solenoid is the exact equivalent of an appropriately magnetized bar magnet that exactly fits within the solenoid. The latter result led him to suggest that each molecule of a magnetic substance owes its properties to the existence within itself of a small closed circuit in which an electric current is continuously flowing; in terms of present theory, these molecular currents are accounted for by the existence of spinning electrons suitably oriented throughout the mass of a magnetized substance.

In 1827, Ohm published the famous law that bears his name. As now formulated, this law states that the current in a wire is directly proportional to the difference of potential between its ends and inversely proportional to its resistance; but the

terminology actually used by Ohm sounds strange to the modern ear, for the concepts involved in terms like potential difference and resistance were at that time just evolving, and Ohm's ideas were in reality borrowed from the then recently published work of Fourier on the conduction of heat. The equivalent of potential difference as used by Ohm was the temperature difference responsible for flow of heat, and electrical resistance was the analogue of the observed resistance to heat flow in solids.

Then, in 1831, came the epochal discovery by Faraday of the phenomenon known as *electromagnetic induction*, which is the starting point of the electrotechnical development that has continued ever since. In common with other experimenters of that time, Faraday reasoned that if an electric current could produce magnetism, as Oersted had found, it ought to be possible to make a magnetic field produce an electric current. As early as 1821 he had demonstrated that a magnetic field could be made to react upon a current so as to produce rotation of a copper disk in which current had been caused to flow. The converse possibility did not occur to him at that time, however, and other experiments in the intervening years failed for various reasons. In the experiment that finally succeeded, he used an iron ring wound with tw coils A and B, coil A arranged to be connected to a battery and coil B to the two ends of a long copper wire which passed over a compass needle. On making and breaking the contact between the battery and coil A, the needle was deflected in opposite directions. In general, this discovery may be summed up by the statement that, if a current is to be induced in a circuit, the circuit must either be linked with a *changing* magnetic field or must move relatively to an unvarying magnetic field in such a way that the total magnetic field linked with the circuit changes in magnitude from instant to instant. It is this experiment, and others made immediately thereafter, which led to the development of the electric generator and of all the types of electromagnetic machinery with which this book is concerned. For that reason the further discussion of historical development will be continued in the text itself as occasion may require.

2. Unit Quantity of Electricity.—Coulomb's law defining the force between two point charges is expressed by the equation

$$f = \frac{qq'}{\epsilon_0 r^2} \qquad (1)$$

where q and q' are the magnitudes of the two charges, r is the distance between them, f is the force acting along the line connecting the two point charges, and ϵ_a is a proportionality constant. The magnitude and dimensions of ϵ_a depend upon the units in which the charges, their distance apart, and the force between them are severally expressed, as well as upon the medium in which the charges are immersed.

In the *absolute* (*c.g.s.*) *electrostatic* system of units, force is expressed in dynes and distance in centimeters; and in this system it is *arbitrarily assumed* that in free space, or a vacuum, $\epsilon_a = \epsilon_0 = 1$. Consequently, in free space, *so far as numerical magnitudes are concerned*, Eq. (1) may be written

$$f = \frac{qq'}{r^2} \quad \text{dynes} \tag{2}$$

and if q and q' are numerically equal

$$q = \sqrt{fr^2} \tag{3}$$

If the two charges, each represented by q, have a magnitude such that $f = 1$ dyne when $r = 1$ cm., it follows from Eq. (3) that $q = 1$, which leads to the fundamental definition of electrical charge: *Unit electrical charge, or unit quantity, in the absolute electrostatic system is that amount of electricity which, if concentrated at a point* 1 *cm. from an equal point charge, in free space, will experience a force of* 1 *dyne.* The direction of the force is along the line joining the two point charges and must be interpreted as a repulsion if f is positive (when q and q' have like signs) and as an attraction if f is negative (when q and q' have unlike signs).

The unit charge defined in this way is called the *statcoulomb*, the prefix stat indicating that the unit is based upon the effect of stationary charges. It is a wholly arbitrary unit, not in any manner related to the natural unit of charge associated with an electron, as may be seen from the fact that the charge on an electron is found by experiment to be equal to 4.803×10^{-10} statcoulomb.* The statcoulomb is a very small amount of electricity, as may be judged from the following consideration:

* The charge on an electron as originally determined by Millikan is 4.774×10^{-10} e.s.u., but more recent calculations indicate the value 4.803×10^{-10}.

Let two small pith balls a and b (Fig. 1), each having a mass of 10 mg., be suspended in free space on fibers of negligible weight each 10 cm. long, and let a positive charge of 1 statcoulomb be placed on each ball. The system will be in equilibrium when the resultant of the force of repulsion ($qq'/\epsilon_0 r^2 = 1/r^2$ dynes) and the gravitational pull ($mg = 0.010 \times 980 = 9.8$ dynes) acts along the line of the suspending fiber; therefore,

$$\tan\theta = \frac{r/2}{\sqrt{(10)^2 - (r/2)^2}} = \frac{1/r^2}{9.8}$$

whence $r = 1.26$ cm. In other words, the statcoulomb is so small a quantity that the force of repulsion between the two pith balls is small even in comparison with their minute weight.

The preceding calculation* applies only when the charges are in free space, for which $\epsilon_0 = 1$. In air or some other medium the properties of which differ from those of free space, the equation must be written

$$f = \frac{qq'}{\epsilon_a r^2} \tag{4}$$

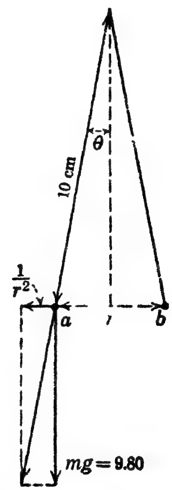


FIG. 1.—Suspended pith balls.

where ϵ_a is the constant for the medium in question; in the case of air it is found from experiment that $\epsilon_a/\epsilon_0 = K = 1.0006$, or so nearly unity that for all practical purposes air may be regarded as indistinguishable from free space.

The statcoulomb is far too small for ordinary use, and so the unit of quantity used in the *practical system of units* is the *coulomb*, which is 3×10^9 times as large as the statcoulomb. The reason for the choice of the factor 3×10^9 will be developed in Chap. V; for the present it will suffice to say that it involves the velocity of light and of electromagnetic radiation in free space. This velocity is usually taken to be 3×10^{10} cm. per sec., though its value as determined by

* The rigorous solution of the problem should include the gravitational force of attraction between the two masses, but this is so small that it has been neglected. The reason for inserting the calculation is that it gives some conception of electrical magnitudes, usually more or less mysterious to beginners, in terms of tangible, physical phenomena.

Michelson is 2.99776×10^{10} with a probable error of 1 part in 22,000.

It is desirable at this point to call attention to the fact that the form of Eq. (2) may become distinctly misleading unless it is remembered that it comes from the *arbitrary assumption* that the *numerical value* of ϵ_0 is taken as unity. It does not necessarily follow that ϵ_0 is itself a mere numeric without dimensions; as a matter of fact, it is seen from Eq. (1) that

$$\epsilon_0 = \frac{qq'}{fr^2}$$

from which it is obvious that the unit in which ϵ_0 is expressible is of the nature

$$\frac{\text{Square of electrical quantity}}{\text{Force} \times \text{square of length}}$$

and this is not necessarily an abstract number like the quotient of two similar quantities. A more extended discussion of this important point is reserved for Chap. V.

3. Electric Field of Force.—If we imagine an electrically charged body isolated in free space, the region surrounding it will have the property of exerting a force upon another electric charge introduced therein. Such a region, in which electrified bodies are acted upon by forces, is called an *electric field*, and it is clear that the intensity of the field at any point will vary in proportion to the force experienced by a test charge placed at that point. *In particular, if the test charge is positive and is* equal to the statcoulomb, the force in dynes with which it is acted upon at any point is defined [in the electrostatic (e.s.) *system] as the intensity of the electric field* at that point; and this definition is not in any way dependent upon the medium in which the force is found to act. If the field intensity at a particular point is E (meaning E dynes per statcoulomb), the force with which the same field would act upon a point charge of q electrostatic units (e.s.u.) at the same point would be Eq dynes.

It is essential to bear in mind that the introduction of a test charge into an electric field may alter the very field strength intended to be measured. Thus, the field may be due to a very small charge distributed, say, on a conducting (metallic) object,

and the introduction of a relatively large unit point charge may so disturb the distribution of the original charge as to make the field intensity at the test charge materially different from what it was before the introduction of the unit charge. For this reason it is best to think of the test procedure as carried out by means of an infinitesimal point charge which will not alter the initial field distribution; the force acting on the minute test charge Δq will be Δf, and the limit of the ratio $\Delta f/\Delta q$ as Δq approaches zero will be E, the force per unit charge.

In general, the intensity E will vary both in magnitude and direction from point to point in an electric field, though it is possible by suitably distributing the inducing charge to make the field it produces constant in magnitude and uniform in direction, at least within a limited region. Such a field is said to be a *uniform* field. Field intensity, being defined in general as force per unit charge, is a *vector* quantity in the same way that forces in general are vector quantities.

If there are several charges in a given region of free space, the field intensity at any point may be computed by the vectorial combination of the separate field intensities due to each charge considered as acting alone. This procedure is an example of the *principle of superposition*.

4. Lines and Tubes of Electric Force.—At each point in an electric field a positive test charge will be acted upon by one, and only one, force having definite magnitude and direction. This is true even though the field is originally produced by a number of separate charged objects or points arranged at random. Each of these charges, considered as acting alone, will act upon the test charge with a definite force acting in a definite direction, but the resultant of all of them will be unique as to magnitude and direction; however, though the field intensity is itself definite *and single-valued as to magnitude and direction at any point, it* may be resolved at will into as many components as may be desired for purposes of calculation.

It follows from the preceding discussion that the electric field intensity at every point can be represented by a straight line or vector of which the length is proportional to the magnitude of the force and the direction coincides with that of the force. If curves are now drawn in such a manner that their tangents are at every point in the direction of the intensity vector at that

point, the curves will be *lines of electric force.* It is obviously possible to draw an infinite number of such lines in an electric field; for there exists an infinite number of points that do not lie upon one and the same line of force, and through each of these points a line of force may be drawn.

It is a common mistake to think of a line of force as the curve that would be described by a positive point charge free to move in the electric field. This notion would be correct only if the point charge were entirely devoid of inertia; but inasmuch as the smallest charge known, namely, that of the positron or electron, possesses mass and therefore inertia, the idea must be dismissed as inaccurate and misleading.

It is clear that lines of force cannot intersect; for if they did, each of the intersecting lines would have a different tangent at the point of intersection, which would imply that a test charge placed at that point would simultaneously experience *resultant* forces in more than one direction—a condition that is clearly impossible.

It is of course impossible to draw the infinite number of lines of force that exist in any field. But the field can be mapped by drawing a selected number of them, just as in a geographical map we are accustomed to draw only a few of the infinity of possible meridians and parallels. Accordingly, it is conventional practice to draw the lines of force in such manner that at any point where the field intensity has the magnitude E the number of lines of force per unit area, taken at right angles to the direction of E, is numerically equal to E. A bundle of lines of force will therefore converge as the field intensity increases and diverge as it decreases.

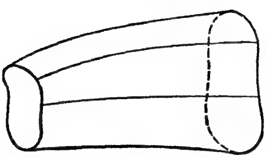


FIG. 2.—Tube of force

The lateral walls of a bundle of lines of force constitute a tubular surface the elements of which are themselves lines of force, as indicated in Fig. 2. It follows, therefore, since lines of force cannot intersect, that the same total number of lines crosses all sections of a tube.

5. Potential. Difference of Potential.—In Fig. 3, let q and q' represent positive point charges in air or in vacuum, separated by a distance x cm., the magnitudes of the charges being expressed in statcoulombs. Each charge will repel the other with a force

$$f = \frac{qq'}{\epsilon_0 r^2} \quad \text{dynes}$$

and if, under the influence of this force, charge q' moves to the right through a distance dx, work will be done upon it to the extent of

$$dW = \frac{qq'}{\epsilon_0 r^2} \cdot dx \quad \text{ergs*}$$

The entire amount of work done when the two charges separate to an infinite distance, starting from an initial separation of r cm., is

$$W = \int_r^\infty \frac{qq'}{\epsilon_0 x^2} dx = \frac{qq'}{\epsilon_0 r} \quad \text{ergs} \tag{5}$$

Since no work has been done upon the system by any outside

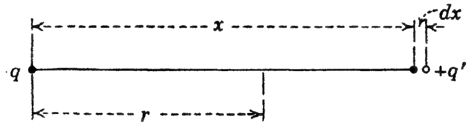


FIG. 3.—Potential energy of two charges.

agency during this process, the movement of q' having been the result of the mutual repulsion between itself and q, the work represented by the expression $qq'/\epsilon_0 r$ must have come from the system itself and therefore represents the *stored* or *potential* energy of the two charges in each other's presence.

Suppose now that q and q' are originally at an infinite distance apart, so that their mutual repulsion is zero. Let q' be carried nearer and nearer to q along the straight line joining them, against the ever-increasing force of repulsion that q' encounters, this force acting to the right in Fig. 3 whereas the motion is toward the left. The work that must be done by some *external agency* to carry q' from an infinite distance to a point distant r cm. from q is

$$= \int_\infty^r \frac{qq'}{\epsilon_0 x^2}(-dx) = \frac{qq}{\epsilon_0 r}$$

* The erg is the c.g.s. unit of work and is equivalent to the work done when a force of one dyne acts through a distance of one centimeter measured in the direction of the force; otherwise expressed, an erg is equivalent to a centimeter-dyne (comparable to the foot-pound unit in British units).

this expression for the stored energy being the same in magnitude as in the first case [Eq. (5)].

If now $q' = 1$ in these expressions, or the moving charge is 1 statcoulomb, the expression $W = q/\epsilon_0 r$ is the work required to bring a unit positive charge from an infinite distance to a point distant r cm. from q (in vacuum or air) and is called the *absolute potential* of the point distant r cm. from charge q.

The work required to move a unit positive charge (or any multiple thereof) from one point to another in an electric field is independent of the path followed, as may be shown with the aid of Fig. 4. Let P_1 be the initial, and P_2 the final, position of the unit charge, and let the path between them be any curve whatsoever. At any point on the curve, distant r cm. from q, the force

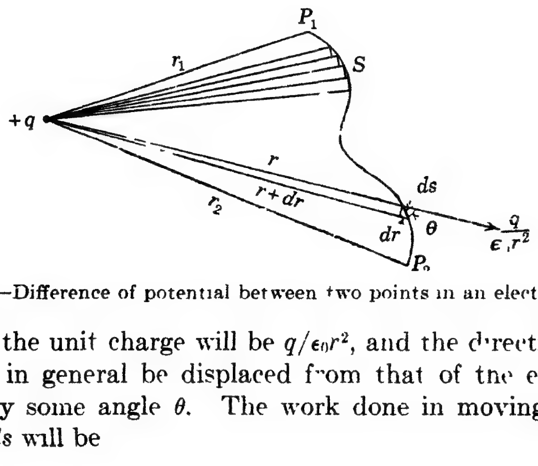


FIG. 4.—Difference of potential between two points in an electric field

acting on the unit charge will be $q/\epsilon_0 r^2$, and the direction of this force will in general be displaced from that of the elementary path ds by some angle θ. The work done in moving over the distance ds will be

$$dW = \frac{q}{\epsilon_0 r^2} ds \cos \theta = \frac{q}{\epsilon_0 r^2} dr$$

and the total work in moving from P_1 to P_2 will be

$$W_{1-2} = \int_{r_1}^{r_2} \frac{q}{\epsilon_0 r^2} dr = \frac{q}{\epsilon_0 r_1} - \frac{q}{\epsilon_0 r_2} \qquad (6)$$

But $q/\epsilon_0 r_1$ and $q/\epsilon_0 r_2$ are what have been called the absolute potentials at P_1 and at P_2. Hence, the work done in the movement of the unit positive charge from the one point to the other is simply the *difference of electrical potential* between the two points and is independent of the path followed by the unit charge in the travel from the one point to the other.

The fact that the shape of the path has nothing to do with the result may be seen in another very simple way. For example, in Fig. 4, the actual path may be resolved into an infinite number of infinitesimal components, as shown at S, one set of these components consisting of straight lines radiating from charge q, the other set consisting of arcs of circles having q as the center. Since the direction of the force due to q is always perpendicular to these circular arcs, no work is required to move the unit charge along them, so that the entire amount of work is accounted for by the radial motion, exactly as in Fig. 3.

It follows from the analysis leading to Eq. (6) that, if the potential of the terminal point of the travel is higher (that is, numerically greater) than that of the starting point, work must be done *by an outside agency* to produce the motion of the unit charge and the work so performed reappears as increased potential energy of the system composed of the two charges. If, on the other hand, the potential of the terminal point of the travel is less than that of the starting point, work must be done by the system itself at the expense of its stored energy. Thus, if the system is left unrestrained, the stored energy will be dissipated by the separation of the charges under the influence of their mutual repulsion. The close analogy to the case of a weight that has been lifted from a lower to a higher level is obvious; another mechanical analogy relates the electrical case to the compression and release of a coiled spring.

The absolute electrostatic potential at a point in an electric field is therefore measured by the *work* (in ergs) required to carry a *unit charge* (the statcoulomb) from an infinite distance to the point in question. The name given to unit potential in this system of units is the *statvolt*, and this is clearly of the nature ergs per statcoulomb, or, in general, *work per unit charge*. Since *the potential at a point in a field is in no way dependent upon* direction, it is a *scalar* quantity, as distinguished from field intensity, which has been shown to be a vector quantity.

If in the case of a given electric field the test charge, instead of being a statcoulomb, is a coulomb, the work required to move it will be 3×10^9 times larger.* But in the system of

* The use of a charge as large as the coulomb would so distort the original field as to change the conditions to be measured. The actual test charge must be very minute, and the effect of a larger charge computed by proportion.

units to which the coulomb belongs the unit of work is the *joule*, which is 10^7 ergs. By analogy with the definition of the statvolt, *the potential at a point, measured in practical units, is the work in joules required to carry 1 coulomb from infinity to the point in question*, and the unit potential in the practical system is called the *volt*.

If, for example, the absolute potential at a given point is 1 statvolt, the work required to carry a coulomb from infinity to that point would be 3×10^9 ergs, or 300 joules; consequently, by definition, the potential at the point is 300 volts, from which it follows that

$$1 \text{ statvolt} \equiv 300 \text{ volts}$$

From these considerations it is seen that the difference of potential between two points in an electrical circuit, measured in

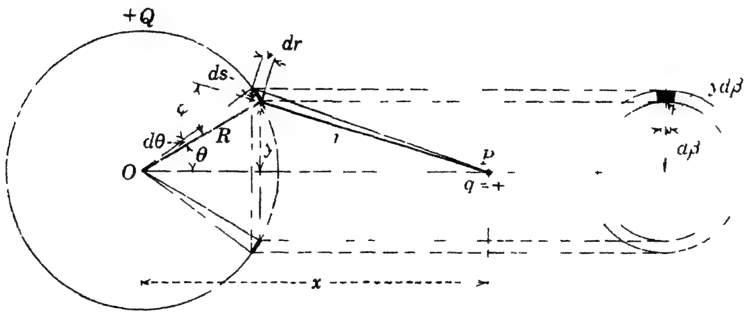


FIG. 5 Potential of a charged sphere

volts, means the work in joules required to carry a unit charge, the coulomb, from the one point to the other. This statement is merely a matter of definition that follows from the basic definition of unit electrical quantity or charge, and from the experimentally derived laws concerning the forces between charges. It is customary, in discussing the flow of current in an electric circuit, to draw an analogy between the voltage of the circuit and the pressure that exists when fluid flows in a hydraulic circuit. The analogy is particularly satisfactory if the hydraulic pressure, or pressure head, is expressed in feet or inches, for thus the idea of lifting a unit mass of fluid through a specified distance is conveyed.

For example, consider a charged metallic sphere (Fig. 5) isolated in free space, having a radius R cm. and an initial charge

of $+Q$ units that, because of symmetry, is uniformly distributed over the surface; the charge per unit area of the sphere is then $\sigma = Q/4\pi R^2$. At point P, distant x cm. from the center of the sphere, there is a positive unit charge, and it will be assumed that the entire system is immersed in a vacuum in which $\epsilon_0 = 1$. It is required to find the amount of work necessary to move the unit charge from an infinite distance to the surface of the sphere.

Consider an annular ring on the surface of the sphere intercepted between two coaxial right cones of which the vertices are at the center of the sphere and the semiangles are θ and $\theta + d\theta$. The length of arc of the annulus is $ds = R\,d\theta$, the radius of the ring is $y = R \sin\theta$, and the area of the ring is $2\pi y\,ds$. The side view of the annulus is indicated at the right-hand side of the diagram, and an elementary sector of the ring, having an area $y d\beta\,ds$, will carry a charge $\sigma y d\beta\,ds$. The potential at point P due to this elementary charge will be

$$d^2W = \frac{\sigma y d\beta\,ds}{\epsilon_0 r}$$

and the potential due to the entire annulus is

$$dW = \frac{\sigma y\,ds}{\epsilon_0 r}\int_0^{2\pi} d\beta = \frac{2\pi\sigma y\,ds}{\epsilon_0 r}$$

the simple summation being possible for the reason that potential is a scalar quantity, independent of directional effects.

Inspection of the diagram shows that $ds = dr \sin\varphi$, $y = R \sin\theta$, and $\sin\theta/\sin\varphi = r/x$, whence

$$dW = \frac{2\pi\sigma}{\epsilon_0 r} R\,dr \frac{\sin\theta}{\sin\varphi} = \frac{2\pi\sigma}{\epsilon_0 r} R\,dr \frac{r}{x} = \frac{2\pi\sigma R}{\epsilon_0 x} dr$$

and

$$W = \frac{2\pi\sigma R}{\epsilon_0 x}\int_{r=x-R}^{r=x+R} dr = \frac{4\pi\sigma R^2}{\epsilon_0 x} = \frac{Q}{\epsilon_0 x} \qquad (7)$$

Equation (7) states that so far as concerns points external to the sphere the potential is the same as if the charge on the sphere were concentrated at its center. Accordingly, if x is made equal to R, the potential at the surface of the sphere becomes $Q/\epsilon_0 R$. The physical meaning of this result is that $Q/\epsilon_0 R$ is the amount of work required to put the charge Q on the sphere, starting from an originally uncharged condition. It is convenient

to think of the process as having been effected by adding successive increments of charge to the sphere, each increment being brought from an infinite distance; as each new element of charge is made to approach the sphere, and after it has been placed thereon, it experiences a force of repulsion due to all the charges already there. Thus the surface of the sphere is under the influence of forces that act exactly like an internal pressure tending to enlarge it; in fact, if the sphere is a soap bubble, it will expand on being charged.

The concept of potential and of difference of potential in the form here presented was developed by C. F. Gauss in 1839, some years after the original formulation of Ohm's law which, as it is now stated, is given in terms of Gaussian potential. Further consideration of some important properties of the potential function $q/\epsilon_0 r$ will be presented in a later chapter. The subject has been introduced and treated at some length at this stage because a thorough grasp of the meaning of the terms potential and potential difference is of vital importance to a correct understanding of the flow of current in ordinary circuits.

Of the two terms, potential and difference of potential, the latter is much the more important. Potential (or absolute potential) involves the notion of bringing a test charge from an infinite distance or from a region where there are no electric forces; since infinity is inaccessible and we have no means of knowing the charge or potential of the earth, it is necessary to be content with the arbitrary assignment of zero potential to the earth and to reckon all other potentials with that as a datum. Consequently, when the potential of a charged body is expressed in volts without other qualification, it is to be understood that the difference of potential between the body and the earth is equal to the given amount. In the same way, an object that is conductively connected to the earth is said to have zero potential by reason of being grounded.*

* It does not necessarily follow that all parts of the earth's surface are always at the same potential. Some parts may be so constituted, or may be so dry, as to be fairly good insulators, capable of being charged to a higher (or lower) potential than neighboring parts that are permanently moist and conducting. For this reason, ground connections intended to hold potentials to the arbitrary zero value must be carefully made in permanently moist soil.

6. Electromotive Force.—Consider two insulated metallic objects, one carrying a charge $+Q_1$, the other a charge $-Q_2$. There is a difference of potential between them measured by the amount of work required to carry a unit positive charge from $-Q_2$ to $+Q_1$, the work in such a case being due to overcoming simultaneously the repulsion of Q_1 and the attraction of Q_2. In terms of the electron theory, there is a deficit of electrons on the positively charged body, and a surplus on the negatively charged body, with a resultant tendency for the surplus electrons in Q_2 to be repelled toward Q_1. If the two bodies are connected by a partially conducting fiber, say a slightly moist string, there will take place a transfer of electrons from Q_2 to Q_1, and in the interval during which the transfer is being effected there is a current of electricity in the connecting fiber. In accordance with the commonly used convention, current is said to flow in the direction from $+Q_1$ to $-Q_2$. The current will continue until there is no longer any difference of potential between the two bodies, and the final charge, distributed over the entire system, will be $Q_1 - Q_2$, which will be zero if the two charges are originally numerically equal. The hydraulic analogy to which this case may be likened is that of two reservoirs, having different surface elevations, connected by a pipe; in the latter case water will be transferred from one reservoir to the other until both are at the same level. The speed with which the final equalization occurs will depend upon the conducting qualities of the connecting wire (or pipe)- or, inversely, upon its resistance to the flow.

The maintenance of a steady flow of current between the two charged bodies requires the presence of some agency that will maintain the original difference of potential between them, notwithstanding the tendency of the current to equalize that difference of potential. In other words, if the difference of potential is to be maintained, there must exist some form of electron pump, just as in the case of the two reservoirs some form of hydraulic pump is required to maintain the difference of levels which produces the flow of water. Numerous forms of electron pumps exist, all requiring the expenditure of energy for their operation; they may be classified as follows:

1. *Electrostatic generators*, depending upon frictional or induction effects, all stemming from the original sulphur-ball machine of von Guericke.

2. *Electrochemical cells* or batteries, all traceable to Volta's pile.

3. *Electromagnetic generators*, which are developments of Faraday's discovery.

4. *Thermocouples*, consisting of a pair of dissimilar metals which when heated at their junction develop a difference of potential between their other ends. This effect was discovered by Seebeck in 1826.

5. *Thermionic tubes*, which date from the observations of Elster and Geitel in 1880 and which were independently noted by Edison at about the same time.

6. The *photoelectric effect*, in accordance with which light falling upon a metal may under certain conditions cause the metal to emit electrons; this effect was first observed by Hertz, in 1887, while making the original experiments that led to the development of the radio.

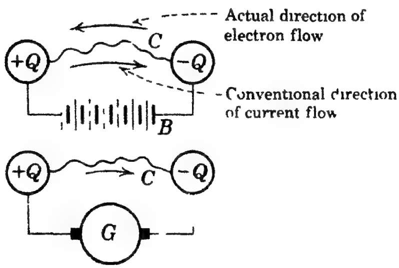


FIG. 6.—E.m.f. of battery or generator

Of these six classifications, the two which are particularly useful in "heavy" electrical engineering are those listed as (2) and (3).

To return to the case of the two charged bodies, let them be the terminals of a battery B or a generator G, as represented in Fig. 6. Before they are connected by the conductor C, the electrochemical action of the battery or the electromagnetic effect of the generator produces a transfer of electrons in such manner that one terminal acquires a surplus of them, hence is negatively charged, whereas the other has electrons extracted from it and becomes positively charged. Thus a difference of potential is established between the two terminals that tends to drive the surplus electrons on the negative terminal back to the positive terminal through the battery (or generator); but this effect cannot occur so long as the chemical action in the battery or the electromagnetic effect in the generator is maintained, and so a condition of equilibrium is attained such that the difference of potential between the terminals is exactly balanced by the *electromotive force* of the battery (or generator). Electromotive

force (hereafter abbreviated as *e.m.f.*) is therefore of the same nature as potential difference (usually abbreviated as *p.d.*), both being measured in volts in the practical system of units and both being of the nature *work per unit charge*. It is for this reason that the name electromotive force is unfortunate, because e.m.f. is decidedly *not a force*, for force is measured in quite different units. Underlying the term there is, in the case of the battery, the entire complex of phenomena that results in the liberation of electrons from the electrodes and, in the case of the generator, the electromagnetic induction that liberates electrons from the atoms of the conductor moving in a magnetic field; in both cases there are forces of repulsion between the electrons thus liberated, but these forces are not the e.m.f. itself.

When the terminals are connected by the conductor C, Fig. 6, the excess electrons on the negative terminal tend to equalize the deficit on the positive terminal by a flow through C from the negative to the positive terminal, and the e.m.f. of the battery (or generator) then tends to maintain the original conditions by driving a fresh supply of electrons into the circulation. In general, however, the potential difference between the terminals is smaller when current flows than when the circuit is open, for the reason that part of the e.m.f. of the battery is consumed in driving the current through the internal resistance of the battery itself, there being left only the remainder to appear as potential difference between the terminals.

In the case of the battery, the e.m.f. is equal to the potential difference between the terminals when the battery is on open circuit, provided that the battery is not polarized. The same equality will exist in the case of the generator only if it is one of the particular types to be described later.

7. Unit Current. The concept of current in an electrical circuit is derived directly from the familiar one of the flow of water in streams or in pipes. It is simple to visualize hydraulic flow in terms of a particular number of gallons or cubic feet of the fluid material passing a given cross-section of the stream in a unit of time; in other words, to think of it as *quantity per second*, which may also be expressed as the time rate of change of quantity.

Thus, if a *steady* current of electricity is flowing in a conductor so that Q coulombs pass a given cross-section in t seconds, the current is said to be equal to

$$I = \frac{Q}{t} \text{ amp.} \tag{8}$$

If, in particular, $Q = 1$ and $t = 1$, I will be equal to unity, from which it follows that unit current, the *ampere*, may be defined as *the unvarying current equivalent to the flow of* 1 *coulomb per sec.* across each section of the conductor taken at right angles to the direction of the flow.

In general, current flow is not steady; the magnitude of the current may change from instant to instant. In such a case it is possible to define the instantaneous current by the equation

$$i = \frac{dq}{dt} \tag{9}$$

where dq is the infinitesimal quantity (in coulombs) that passes a given section of the conductor in the infinitesimal time dt sec.

This definition of unit current (the ampere) thus depends upon that of the coulomb, and the latter in turn depends upon that of the statcoulomb which is fixed in terms of the force between static charges. In other words, the ampere, which measures a moving charge, can be tied to the static effects of charges at rest. It will appear later that the ampere can also be defined in terms of the magnetic effect of electricity in motion (the electrostatic and the electromagnetic effects and the measurements of these effects) being thus linked together.

Measurements of current based upon these fundamental relations present such great experimental difficulties that the actual determination of unit current is made to depend upon the electrolytic effect of current flow in solutions. It has already been mentioned that as early as 1800 Ritter had observed that oxygen and hydrogen are evolved from a salt solution; he also observed the deposition of copper upon one of the electrodes immersed in a solution of a copper salt. A quantitative study of these phenomena, begun by Faraday in 1834, resulted in the formulation of the fundamental law that 96,500 coulombs will liberate 1 gram-equivalent of any substance from solution. In the case of silver, for example, which has an atomic weight of 108, 1 amp. (1 coulomb per sec.) will deposit from solution 108/96,500 = 0.001118 g. of silver per second. The legal definition of the ampere, as fixed in 1894 by an act of Congress, is:

The unit of current is the practical equivalent of the unvarying current which, when passed through a solution of nitrate of silver in water in accordance with standard specifications, deposits silver at the rate of 0.001118 *g. per sec.*

8. Ohm's Law.—The experimental work performed by Ohm showed that the potential difference between the terminals of a conductor is directly proportional to the current flowing through it. That is, if the potential difference is V volts and the current is I amp., the ratio

$$\frac{V}{I} = \text{a constant}$$

provided that the conductor is metallic and is maintained at a constant temperature. The constant in this relation is found to be directly proportional to the length of the conductor and inversely proportional to its cross-section; it is called the *resistance* and is measured in terms of a unit called the *ohm*. That is,

$$\frac{V\text{ (in volts)}}{I\text{ (in amperes)}} = R\text{ (in ohms)} \tag{10}$$

This equation and its equivalents $V = IR$ and $I = V/R$ are the usual mathematical statements of Ohm's law.

Inasmuch as resistance is the ratio of potential difference to current, the ohm is a unit the dimensions of which are fixed in terms of those of the volt and the ampere, and it is therefore a derived unit if the volt and the ampere are regarded as basic. It is seen from Eq. (10) that, if V and I are both unity, $R = 1$, from which the definition of the ohm may be formulated as *the resistance that will develop a potential difference of* 1 *volt between its terminals when it is made to carry a current of* 1 *amp.*

Experiment has shown that a column of mercury 106.3 cm. long, having constant cross-sectional area, weighing 14.4521 g., and held at the temperature of melting ice (0°C.), has a resistance of 1 (international) ohm.* It is therefore comparatively easy to

* As a result of careful checking of the international electrical units under the direction of the International Conference of Weights and Measures, it is expected that there will be a slight change in the ohm and that the length of the mercury column will be fixed as approximately 106.25 cm., a reduction of about 0.05 per cent from the figure given above. (See A. E. KENNELLY, Recent Developments in Electrical Units, *Elec. Eng.*, February, 1939, p. 78.)

reproduce the standard ohm; and for this reason, though the ohm is subsidiary to the fundamental definitions of potential difference and current, it is commonly employed as a primary standard in making precise measurements.

Since the volt, the ampere, and the ohm are definitely related in accordance with Ohm's law, the definition of any two of them inexorably fixes the third. Although the ampere is fixed by law in terms of the electrochemical equivalent of silver, measurements of current by means of the silver voltameter are difficult. But by means of such measurements it has been found that the normal Weston cell has an e.m.f. of 1.01830 volts; and since this device is readily reproducible, the volt can be taken as 1/1.01830 of the e.m.f. of this cell. In actual practice, therefore, the combination of a Weston cell and a standard resistor provides the simplest and most convenient means for determining current strength with precision.

In some cases that arise in the measurement of resistance, the ohm is either inconveniently small or large. Thus, in the case of the insulation of electrical machines and instruments, the resistance is of the order of millions of ohms; in the case of short lengths of metal having large cross-section the resistance may be only a few millionths of an ohm. In the former case the unit employed is the *megohm* (10^6 ohms); in the latter case the unit is the *microhm* (10^{-6} ohm).

It is not uncommon to find that students fall into the error of defining the three units, the volt, the ampere, and the ohm, each in terms of the others. Thus, though it is correct to define the ohm as that resistance which requires a difference of potential of 1 volt between its terminals in order that a current of 1 amp. may flow through it, it is not then legitimate to follow with the inverted definition that the volt is the potential difference which will maintain 1 amp. through a resistance of 1 ohm; for the two statements taken together constitute a "vicious circle."

Simple as Ohm's law really is, experience shows that beginners commonly misuse it to a greater extent than any other law. The law does *not* say, for example, that a potential difference of V volts applied to the terminals of *any* circuit of resistance R will produce a current $I = V/R$ amp. In order that this relation may be true, two conditions must be fulfilled, namely: (1) The potential difference must be maintained, *after* the current has

been established, at the same value it had before current is caused to flow. (2) The circuit of which the resistance is R ohms must be *passive* in the sense that it does not include any active e.m.f. due to any cause whatsoever.

Consider, for example, the simple circuit indicated in Fig. 7, which represents a storage-battery cell having an e.m.f. E = 2.2 volts which is about to be connected to a passive resistor of which the resistance is 0.032 ohm. After closing the switch, the current would be 2.2/0.032 = 68.75 amp. *only* if the e.m.f. remains constant, *and* if the cell has no internal resistance. Actually, such a cell will maintain constant e.m.f. until it approaches exhaustion, but it has an internal resistance of, say, 0.008 ohm. The current in the circuit is therefore

$$I = \frac{2.2}{0.032 + 0.008} = 55 \text{ amp.}$$

and by Ohm's law the potential difference at the terminals of the resistor (which are also the terminals of the cell) becomes

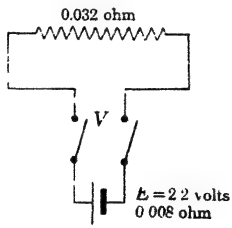


FIG. 7.—Simple circuit illustrating Ohm's law.

$$V = IR = 55 \times 0.032 = 1.76 \text{ volts}$$

The difference between the cell e.m.f. and its terminal voltage, or 2.2 − 1.76 = 0.44 volt, is accounted for as the potential difference required to maintain the current through the resistance of the cell (55 amp. × 0.008 ohm = 0.44 volt); it is this amount which is commonly described as the *drop* or *fall of potential* through the battery resistance.

This example serves to show that Ohm's law applies to the whole of such a simple series circuit as is represented in Fig. 8 and that in such cases the current may be computed by dividing the total e.m.f. by the total resistance; that is, in a circuit in which several cells (or other sources of e.m.f.) in series are connected to a number of resistors, also in series, the current is

$$I = \frac{\Sigma E}{\Sigma R} = \frac{E_1 + E_2 + E_3 + \cdots}{R_1 + R_2 + R_3 + \cdots} \qquad (11)$$

where ΣE represents the algebraic sum (that is, taken with due regard to sign) of all the separate e.m.fs. in series and ΣR is the sum of all the resistances in series.

The last equation may be changed to the form

$$\Sigma E = IR_1 + IR_2 + IR_3 + \cdots \tag{12}$$

which states that the *net e.m.f.* in the circuit as a whole is the sum of the potential drops (IR_1, IR_2, etc.) in each part of the

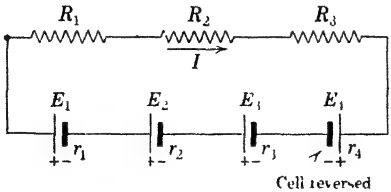


FIG. 8.—E.m.f.s in a series circuit.

circuit. The phrase "net e.m.f." has been used, rather than total e.m.f., for it is possible that one or more cells of a battery may be connected so that the polarity is reversed with respect to the others (this condition is indicated in Fig. 8, where E_4 is in opposition to the other cells).

Each of the terms (such as IR_1) on the right-hand side of Eq. (12) represents the difference of potential between the terminals of an individual resistor. The potential is higher at one terminal than at the other, and the *fall* or *drop of potential* takes place in the direction of the flow of current. The successive drops in the several resistors are indicated in Fig. 9, beginning with R_1 in Fig. 8 and proceeding in cyclical order around the circuit in the direction of the current flow. The *potential diagram* thus formed in Fig. 9 must be closed by a line representing $\Sigma E = E_1 + E_2 + E_3 - E_4$ in accordance with Eq. (12). Otherwise stated, if the potential differences IR_1, IR_2, etc., are considered as *drops*, or *falls*, of potential, the e.m.f.s. acting in the direction of the current flow must each be regarded as a *rise* of potential.

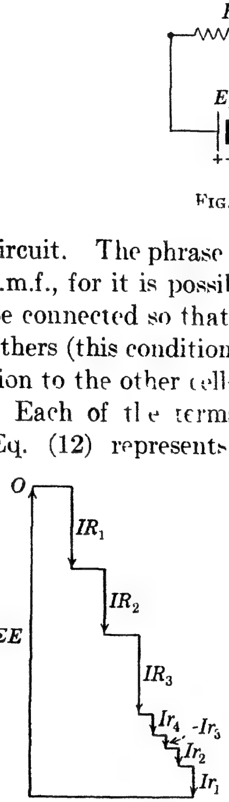


FIG. 9.—Potential diagram corresponding to Fig. 8.

9. Generalized Ohm's Law.—Let Fig. 10*a* represent a simple circuit consisting of a battery (or other source of e.m.f.) of which the e.m.f. is E volts and the internal resistance is r ohms; and let the external circuit be a conductor of resistance R ohms. The

total resistance is $(r + R)$ ohms, and by Ohm's law the current through all parts of the circuit will be

$$I = \frac{E}{r + R} \quad \text{amp.}$$

This equation may be written

$$E = Ir + IR$$

and in this form asserts that the entire e.m.f. of the source is used in supplying (1) the drop of potential in the source, represented by Ir, which is analogous to the drop of pressure that occurs in a pump when it is causing a circulation of water; and (2) the drop of potential in the external circuit, represented by the term IR. The term IR is obviously equal to the difference of potential V that exists between the terminals of the source,

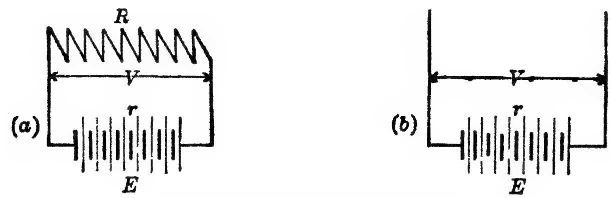


FIG 10—Simple series circuit.

and V can be measured by a voltmeter. The expression can then be written

$$V = IR = E - Ir \tag{13}$$

which states that the difference of potential between the terminals is equal to the e.m.f. of the battery less the drop of potential in the battery; in the particular case where the external circuit is open (R equal to infinity Fig. 10b), the current will be zero, whence $V = E$. Equation (12) can also be written

$$E - Ir - IR = 0 \tag{14}$$

which may be interpreted, in accordance with Art. 8, to mean that there is a rise of potential E (taken as positive) through the source, caused by the active e.m.f., and a drop, or fall, of potential Ir and IR (taken as negative) through the passive resistances r and R.

The preceding discussion refers to the conditions in a circuit containing an active source of e.m.f. and a passive receiver circuit. Suppose, however, that we have a storage battery (Fig. 10*b*) whose internal e.m.f. is E volts that is being charged from some other source through a circuit between the terminals of which there is maintained a constant difference of potential of V volts. As the active e.m.f. of the battery, E, opposes V, V must be great enough to overcome E and at the same time supply the drop of potential through the battery resistance r; that is,

$$V = E + Ir \tag{15}$$

Equations (13), (14), (15) are more general forms of Ohm's

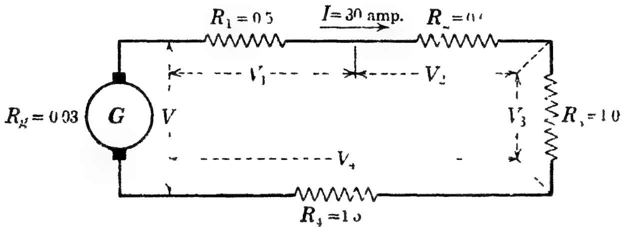


FIG. 11.—Resistors in series.

law than Eq. (10) and are therefore occasionally referred to as the generalized Ohm's law.

Though simple series circuits of the type discussed above are often encountered in engineering practice, most circuits are somewhat more complex and consist of series circuits, parallel circuits, and combination or series-parallel circuits.

10. Series Circuits.—When a number of resistors are connected in series, in the manner shown in Fig. 8, the total resistance of the entire circuit is the sum of the individual resistances, or

$$R = R_1 + R_2 + R_3 + \cdots \tag{16}$$

If a steady current of I amp. flows in such a circuit owing to a steady potential difference of V volts maintained across its terminals,

$$\begin{aligned} V = IR &= IR_1 + IR_2 + IR_3 + \cdots \\ &= V_1 + V_2 + V_3 + \cdots \end{aligned} \tag{17}$$

where V_1, V_2, ... are the drops of potential across the individual resistors.

For example, in the series circuit of Fig. 11, the total resistance of the circuit external to the generator G is $R = R_1 + R_2 + R_3 + R_4 = 3.6$ ohms, and the potential difference between the main terminals is $V = IR = 108$ volts. The drops of potential across the individual resistors R_1, R_2, R_3, R_4, are $V_1 = 15$, $V_2 = 18$, $V_3 = 30$, $V_4 = 45$ volts. The drop of potential in the generator itself is $IR_g = 30 \times 0.03 = 0.9$ volt, so that the generated e.m.f. $E = V + IR_g = 108.9$ volts. This expression is equivalent to

$$E = I(R_g + R_1 + R_2 + R_3 + R_4) = 30 \times 3.63 = 108.9$$

which agrees with the original statement of Ohm's law.

11. Parallel Circuits.—When a number of passive resistors (in the sense that they do not contain any sources of active e.m.f.) are connected in *parallel*, as in Fig. 12, with a difference of potential of V volts maintained across their common terminals a, b, the individual currents are

$$I_1 = \frac{V}{R_1}$$
$$I_2 = \frac{V}{R_2}$$
$$I_3 = \frac{V}{R_3}$$

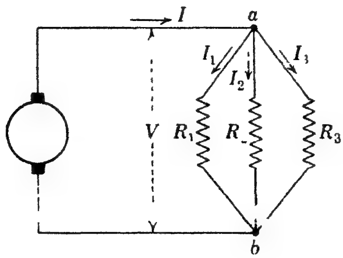


FIG. 12 Resistors in parallel.

and the total current supplied to the group is

$$I = I_1 + I_2 + I_3 = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) \tag{18}$$

If the separate resistors R_1, R_2, R_3 were concealed from view so that their actual arrangement was unknown, the fact that the potential difference V volts produces a line current I amp. would at once lead to the conclusion that the circuit, however constituted, has an *equivalent resistance* R, such that

$$I = \frac{V}{R} = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

whence

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \tag{19}$$

The reciprocal of resistance, such as $1/R$, is called *conductance;* and the unit of conductance is called the *mho* (ohm spelled backward) or the *siemens.** Conductance is represented by the letter G (or g); hence, the conductance of a group of conductors in parallel is

$$G = G_1 + G_2 + G_3 + \cdots \tag{20}$$

where G_1, G_2, . . . , are the individual conductances. Consequently, from Eq. (18),

$$I = I_1 + I_2 + I_3 + \cdots = V(G_1 + G_2 + G_3 + \cdots) \tag{21}$$

If two resistors having resistances R_1 and R_2 are in parallel,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

or their equivalent resistance is

$$R = \frac{R_1R_2}{R_1 + R_2} \tag{22}$$

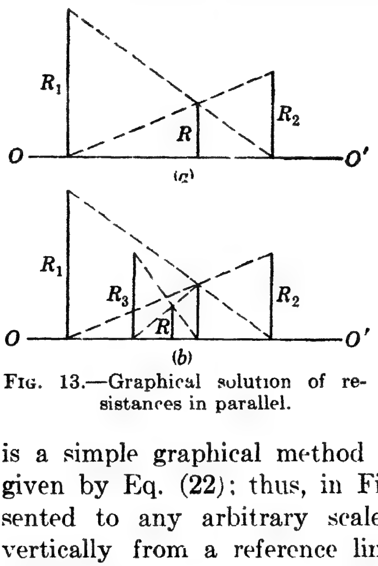


FIG. 13.—Graphical solution of resistances in parallel.

That is, the single resistance that is equivalent to two passive resistances in parallel is *equal to their product divided by their sum*, which is always less than either of the two resistances taken separately. There is a simple graphical method for the solution of the relation given by Eq. (22); thus, in Fig. 13*a*, let R_1 and R_2 be represented to any arbitrary scale by two parallel lines drawn vertically from a reference line OO', and at any convenient distance apart. The equivalent resistance R is then given† to scale by the length measured vertically from OO' to the intersection of the diagonals (drawn as dashed lines). Similarly, if there is a third resistor R_3 in parallel with the other two, the new equivalent resistance can be found in the manner indicated in Fig. 13*b*.

* The name siemens (in honor of Ernst Werner von Siemens) has recently been adopted as the unit of conductance by the International Electrotechnical Commission (I.E.C.).

† The proof is left to the student.

When two passive resistors R_1 and R_2 are in parallel and form part of a circuit in which a current is flowing, the main current will divide between them, a part I_1 flowing through R_1, the remainder, or I_2, through R_2. Since both resistors have their terminals in common, the drop of potential is the same across each of them or

$$I_1R_1 = I_2R_2$$

and

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}$$

from which it may be concluded that the currents through the two resistors are inversely as their resistances. This relation holds true only if the two resistors do not include any active e.m.fs., such as would be due to a battery, thermoelectric effect, or other source.

12. Series-parallel Circuits.—A series-parallel circuit is one in which a group of parallel-connected resistors is in series with other resistors some of which may themselves be connected in parallel groups, as indicated in Fig. 14. Each parallel-connected group, provided that it includes no source of active e.m.f., may be replaced by a single equivalent resistance, with the result that the entire circuit becomes equivalent to a simple series circuit. Thus, in Fig. 14, the left-hand pair of resistors is equivalent to a single resistance $(3 \times 6)/(3 + 6) = 2$ ohms; the right-hand group has an equivalent resistance R such that

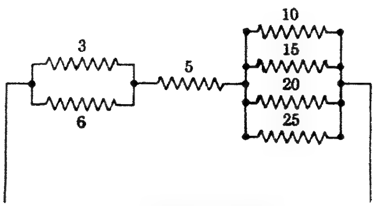


FIG. 14.—Series-parallel circuit.

$$\frac{1}{R} = \frac{1}{10} + \frac{1}{15} + \frac{1}{20} + \frac{1}{25} = 0.2567$$
$$R = 3.896$$

and the resistance of the entire circuit is $2 + 5 + 3.896 = 10.896$ ohms.

13. Kirchhoff's Laws.—In the discussions in the preceding articles, two principles are used without any attempt to state them explicitly. Thus, in Art. 11, Eq. (18) makes use of the fact that the total current supplied to the group of three parallel-

connected resistors in Fig. 12 is the sum of the individual currents. This fact is an illustration of the first of two laws formulated by Kirchhoff about 1842.

1. *The algebraic sum of the currents at any junction in a circuit is zero.*

Otherwise expressed, the sum of all currents *entering* a junction point must be equal to the sum of all currents *leaving* the same point; for otherwise the electrical charge at the junction would either continuously increase or continuously decrease, and the potential of the point would change correspondingly. No effect of this kind is observed in actual circuits under steady conditions.

2. *The algebraic sum of all the e.m.fs. and potential drops in a closed circuit is zero.*

The reasons underlying this law are implicit in that part of the discussion of Art. 8 which relates to Fig. 9; for, in tracing through a closed circuit like that in Fig. 8, the total fall of potential due to the flow of current through the resistance must in accordance with Ohm's law be numerically equal to the total rise of potential due to the active e.m.fs. present in the closed circuit. Kirchhoff's second law is in fact a more general law than that of Ohm and includes Ohm's law as a special case. For example, Ohm's law, considered by itself, is insufficient to determine the current flow in circuits like those of Figs. 15 and 18, which are typical of even more elaborate networks that occur in practice; but the solution of all such networks is readily possible by the aid of Kirchhoff's laws.

Before taking up specific examples, it is important to have clearly in mind certain procedures the observance of which not only facilitates the actual work but eliminates the possibility of serious error. Thus, in solving network problems, it is absolutely essential to make a clear diagram of the circuit, to assign to the current in each link an appropriate symbol, and to indicate with an arrow the direction in which such current is *assumed* to flow. If sources of active e.m.f., such as batteries, generators, or thermocouples, are present, they must be shown with plus and minus signs to indicate their polarities. Then, to write the equations based upon the second law, trace through any one of the closed loops in either of the two possible directions, and assign to each active e.m.f. a positive sign if the source is traversed from its negative to its positive terminal (a rise of potential

being thus indicated) and a negative sign if it is traversed from its positive to its negative terminal; each fall of potential due to a current assumed to flow in the same direction as the traverse must be given a negative sign, and each rise of potential due to a current assumed to flow in the direction opposite to the traverse must be given a positive sign. If, after completing the solution of the equations based upon these rules, one or more of the unknown currents come out with negative signs, the meaning is that these currents actually flow in a direction opposite to the assumed direction.

Example 1.—Figure 15 represents a storage battery connected to a *three-wire system*, the "neutral" wire c being connected to the middle point of the battery. Each half of the battery has an e.m.f. of 115 volts, directed as shown. A and B are the loads, which are supposed to consist of lamps,

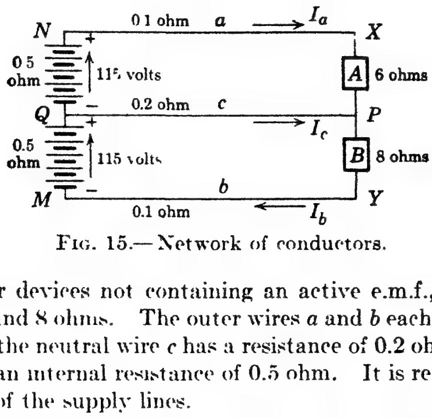


FIG. 15.—Network of conductors.

heaters, or other devices not containing an active e.m.f., and which have resistances of 6 and 8 ohms. The outer wires a and b each have a resistance of 0.1 ohm, and the neutral wire c has a resistance of 0.2 ohm. Each half of the battery has an internal resistance of 0.5 ohm. It is required to find the current in each of the supply lines.

Since there are three unknown quantities, three independent equations are necessary to obtain the solution. One of these equations may be obtained by applying the first law to the junction point P, and the two others by applying the second law, first to the upper and then to the lower loop of the circuit diagram. The three equations thus obtained are

$$I_a + I_c = I_b$$
$$+115 - I_a(0.5 + 0.1 + 6) + 0.2I_c = 0$$
$$+115 - 0.2I_c - I_b(8 + 0.1 + 0.5) = 0$$

From these three independent equations, it is easily found that $I_a = 17.31$, $I_b = 13.47$, and $I_c = -3.84$. The negative sign of I_c merely means that I_c actually flows in a direction opposite to the assumed direction.

In writing the equations based upon the second law, confusion may result unless care is exercised in selecting the closed circuits to which this law applies. For example, in Fig. 15, one might use the circuit $MQNXPYM$.

The equation based upon the second law is

$$+115 - I_b(8 + 0.1 + 0.5) + 115 - I_a(0.5 + 0.1 + 6) = 0$$

This equation is not independent of the others, for it is obtainable by adding the last two of the original group of three equations. Cases may arise where the lack of independence of one or more of the equations is not so readily apparent as in this instance; and if the equations are not truly independent, the result of a long process of reduction may be a mere identity. A straightforward solution of the most complicated networks is always possible if the following considerations are observed:

1. The number of independent equations obtained by applying the first and second laws must be equal to the number of unknown quantities in the circuit.

2. The application of the first law to all junction points in the network, any points that yield duplicate equations being omitted, will give relations that are independent of one another.

3. The necessary remaining equations must be set up by applying the second law to closed loops, each of which comprises some part of the network that is not included in the other loops selected in making up the equations.

It is useful to observe that the number of unknown quantities, and therefore the number of equations to be written, may be reduced by a judicious use of the first law. Thus, in Fig. 15 it is possible to replace I by its equivalent $I_a - I_b$, in which case the second law, applied to the upper and lower loops, will yield two equations involving two unknowns I_a and I_b. The saving is insignificant in this particular instance but may be considerable in more complicated circuits.

The construction of a potential diagram of the kind already indicated in Fig. 9 is very helpful in writing down and checking the equations involved in network problems. Thus, if in Fig. 15 it is arbitrarily assumed that the bottom terminal M of the lower battery is at zero potential (as though this point were grounded), the upper terminal Q of this battery will have a potential $+115$ if no current is flowing and something less than this when there is a current, the difference representing the drop of potential in the battery, in this case amounting to $0.5I_b$ volts. In Fig. 16, let the vertical line OV represent a scale of potential (volts), O being the origin and the positive direction being upward; the potential of point Q will be as shown. Between Q and P there must be a fall of potential in accordance with the assumed direction of current flow in conductor c, and the potential of point P must be lower than that of Q by $0.2I_c$ volts; similarly, there must be a fall of potential of $8I_b$ volts from P to Y, and a further fall of $0.1I_b$ volts from Y to O, to conform to the assumed direction of current I_b; at the point O, at which the construction was started, there must be a final closure in agreement with Kirchhoff's second law. In the same way, there is a rise of potential from Q to N, which will be equal to 115 volts less the drop $0.5I_a$ in battery QN; there is a drop of $0.1I_a$ volts from N to X, and of $6I_a$ volts from X to P, at which point the upper polygon must close, for the potential

of point P has already been fixed. The diagram of Fig. 16 is helpful, not only in setting up the equations necessary to a solution of the problem, but also in visualizing their physical meaning; and it shows in a convincing manner how changes in the circuit constants affect the relative values of the potential differences PX and PY and how under conditions that may arise in practice one or the other of these potential differences may exceed the e.m.f. of the battery, or generator, on its side of the circuit.

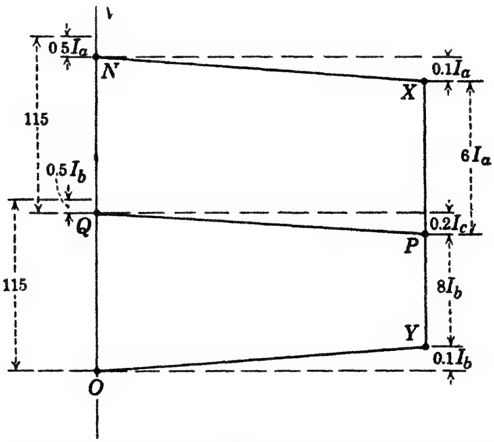


FIG. 16.—Potential diagram of circuit of Fig. 15.

Example 2.—To illustrate the effect of active e.m.fs. in one or more of the branches of a divided circuit, consider the circuits represented in parts (a), (b), and (c) of Fig. 17.

a. The total resistance of the circuit of Fig. 17a is

$$R_t = R + \frac{R_1 R_2}{R_1 + R_2}$$

and the total current is

$$I = \frac{V}{R_t} = \frac{V}{R + \dfrac{R_1 R_2}{R_1 + R_2}}$$

Further,

$$I_1 = \frac{V - IR}{R_1} = \frac{V}{R_t} \frac{R_2}{R_1 + R_2}$$

$$I_2 = \frac{V - IR}{R_2} = \frac{V}{R_t} \frac{R_1}{R_1 + R_2}$$

$$\frac{I_1}{I_2} = \frac{R_2}{R_1}$$

b. Applying Kirchhoff's laws to Fig. 17*b*,

$$
\begin{aligned}
I &= I_1 + I_2 \\
V &= IR + I_1R_1 + E_1 \\
V &= IR + I_2R_2
\end{aligned}
$$

whence

$$
I_1 = \frac{VR_2 - E_1(R + R_2)}{R(R_1 + R_2) + R_1R_2}
$$

$$
I_2 = \frac{VR_1 + E_1R}{R(R_1 + R_2) + R_1R_2}
$$

$$
I = \frac{V - E_1\dfrac{R_2}{R_1 + R_2}}{R + \dfrac{R_1R_2}{R_1 + R_2}}
$$

From these equations it is clear that the branch currents I_1 and I_2 are no longer inversely proportional to the resistances of the branch paths and also

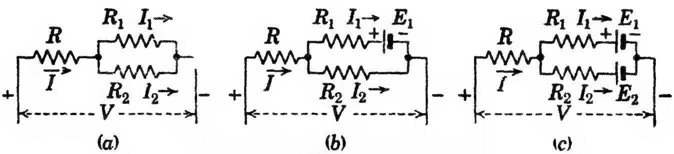


FIG. 17.—Series-parallel circuit.

that the total current is not equal to the impressed voltage V divided by the resistance as computed in (*a*).

c. Applying Kirchhoff's laws to Fig. 17c,

$$
\begin{aligned}
I &= I_1 + I_2 \\
V &= IR + I_1R_1 + E_1 \\
V &= IR + I_2R_2 + E_2
\end{aligned}
$$

whence

$$
I_1 = \frac{VR_2 - E_1(R + R_2) + E_2R}{R(R_1 + R_2) + R_1R_2}
$$

$$
I_2 = \frac{VR_1 - E_2(R + R_1) + E_1R}{R(R_1 + R_2) + R_1R_2}
$$

$$
I = \frac{V(R_1 + R_2) - E_1R_2 - E_2R_1}{R(R_1 + R_2) + R_1R_2}
$$

Here also it is plain that the division of current between the parallel branches does not follow the inverse resistance law (namely, $I_1/I_2 = R_2/R_1$) when E_1 and E_2 have different values. But if E_1 and E_2 are *numerically equal* and are of the *same sign*, that is, if

$$
E_1 = E_2 = E
$$

it follows that

$$\frac{I_1}{I_2} = \frac{VR_2 - E(R + R_2) + ER}{VR_1 - E(R + R_1) + ER} = \frac{R_2}{R_1}$$

and

$$I = \frac{V - E}{R + \dfrac{R_1R_2}{R_1 + R_2}}$$

in which case it is seen that it becomes permissible to combine the resistances in accordance with equations like (15) and (18).

Example 3.—Consider the somewhat more complicated circuit of Fig. 18, and let it be required to find the current in each branch. As there are 11 unknowns, 11 simultaneous equations would have to be solved if the methods previously described were followed. It is possible to reduce the labor to an appreciable extent by observing that the eight resistors at

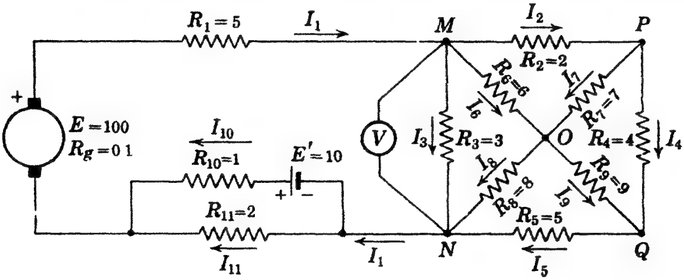


FIG. 18.—Network—series-parallel connections.

the right-hand side of the diagram are equivalent to a single resistor R, the resistance of which must satisfy the relation

$$R = \frac{V}{I_1}$$

Considering for the moment that V and I_1 are known, the currents in the eight branches of the cross can be found by solving only eight equations, as follows:

On applying the first law to the junctions M, N, P, Q, there result

$$\begin{aligned}
(M)\ I_1 &= I_2 + I_6 + I_3\\
(N)\ I_1 &= I_3 + I_8 + I_5\\
(P)\ I_2 &= I_4 + I_7\\
(Q)\ I_5 &= I_4 + I_9
\end{aligned}$$

and on applying the second law to the loops indicated there result

$$\begin{aligned}
(MON)\ \ -6I_6 - 8I_8 + 3I_3 &= 0\\
(MPO)\ \ -2I_2 - 7I_7 + 6I_6 &= 0\\
(PQO)\ \ -4I_4 + 9I_9 + 7I_7 &= 0\\
(QNO)\ \ -5I_5 + 8I_8 - 9I_9 &= 0
\end{aligned}$$

(Note: Applying the first law to the junction O does not yield an independent equation.)

To solve these equations, it is to be noted that the drop of potential from M to N, or $I_3R_3 = 3I_3$, must be equal to V; that is,

$$I_3 = \frac{V}{3}$$

Upon substituting this value of I_3, the remaining values of current may be found in terms of V by successive elimination; in this manner it is found that

$$\frac{V}{I_1} = R = 1.985 \quad \text{ohms}$$

and this is the *equivalent resistance* of the network at the right-hand side of the diagram, measured between the points M and N.

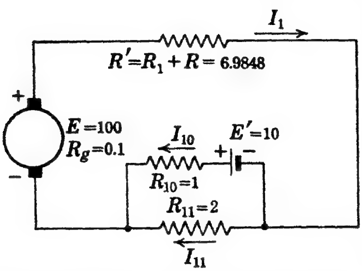


Fig. 19.—Circuit equivalent to that of Fig. 18.

The entire circuit of Fig. 18 may then be replaced by the equivalent circuit of Fig. 19, and, on applying Kirchhoff's laws to this equivalent circuit,

$$\begin{aligned} I_1 &= I_{10} + I_{11} \\ E + E' - I_1(R_g + R') - I_{10}R_{10} &= 0 \\ E - I_1(R_g + R') - I_{11}R_{11} &= 0 \end{aligned}$$

or

$$\begin{aligned} 110 - 7.0848I_1 - I_{10} &= 0 \\ 100 - 7.0848I_1 - 2I_{11} &= 0 \end{aligned}$$

whence $I_1 = 13.76$, $I_{10} = 12.51$, $I_{11} = 1.25$. The remaining values are left for the student to determine.

The preceding solution may be still further simplified by means of the procedure described in the next article.

14. Delta and Star Connections.—In the solution of the circuits sketched in Fig. 18, the determination of the single resistance R, equivalent to the eight interconnected resistors at the right of the diagram, is quite tedious when the calculations are based solely on the use of Kirchhoff's laws. In such cases, provided that passive resistors only are involved (that is, no active e.m.fs. in any of the branches), the calculations can be materially simplified by applying the following facts:

1. A Δ (*delta*, or *mesh*) arrangement of three resistors (Fig. 20) can always be replaced by an equivalent set of three resistors connected in Y or *star* (Fig. 21).

2. Conversely, a Y-connected set of three resistors can be replaced by an equivalent Δ.

Thus, in Fig. 20, the resistance measured between terminals a and b is due to R_{ab} in parallel with $(R_{bc} + R_{ca})$ and is therefore equal to $\frac{R_{ab}(R_{bc} + R_{ca})}{R_{ab} + R_{bc} + R_{ca}}$. In Fig. 21, the resistance between

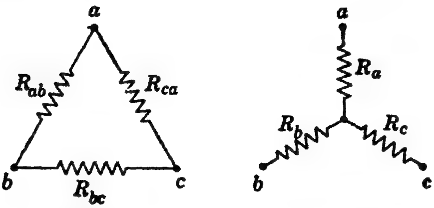


FIG. 20. FIG. 21.
FIGS. 20 and 21.—Delta- and star-connections.

terminals a and b is simply $R_a + R_b$; and if the two circuits are to be equivalent, it is necessary merely to make

$$R_a + R_b = \frac{R_{ab}(R_{bc} + R_{ca})}{R_{ab} + R_{bc} + R_{ca}} = \frac{R_{ab}(R_{bc} + R_{ca})}{R} \tag{23}$$

where

$$R = R_{ab} + R_{bc} + R_{ca} \tag{24}$$

Upon proceeding in the same way with terminals b-c and then with c-a, two additional equations result,

$$R_b + R_c = \frac{R_{bc}(R_{ca} + R_{ab})}{R} \tag{25}$$

$$R_c + R_a = \frac{R_{ca}(R_{ab} + R_{bc})}{R} \tag{26}$$

and on solving Eqs. (23), (25), (26) for R_a, R_b, R_c, the results are

$$\left.\begin{aligned} R_a &= \frac{R_{ab}R_{ca}}{R} \\ R_b &= \frac{R_{bc}R_{ab}}{R} \\ R_c &= \frac{R_{ca}R_{bc}}{R} \end{aligned}\right\} \tag{27}$$

which give the resistances of a set of Y-connected resistors equivalent to a given set of Δ-connected resistors. (Note the

simple cyclical arrangement of the subscripts, which proceed in the order $a, b, c, a, \ldots$, in pairs, it being thus possible to write the results without conscious effort of memory. The resistance of any one of the star branches is the product of the two Δ-branch resistances meeting at the corresponding terminal, divided by the sum of all three Δ resistances.)

Conversely, if the three branches of a given Y-connected set have resistances R_a, R_b, R_c, the equivalent set of Δ-connected

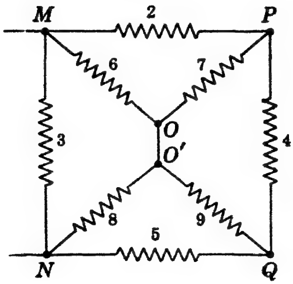


FIG. 22.

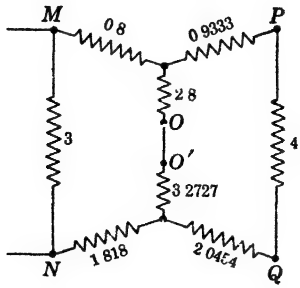


FIG. 23.

FIGS. 22 and 23.—Simplification of network.

resistances may be found by solving the three equations (27) for R_{ab}, R_{bc}, R_{ca}; the results are

$$\left.\begin{aligned} R_{ab} &= \frac{R_p^2}{R_c} \\ R_{bc} &= \frac{R_p^2}{R_a} \\ R_{ca} &= \frac{R_p^2}{R_b} \end{aligned}\right\} \tag{28}$$

where

$$R_p^2 = R_aR_b + R_cR_a + R_bR_c \tag{29}$$

In each of the equations (28) the numerator is the sum of the products, in pairs, formed from the three Y-connected resistors; and the denominator is in each case the resistance of the Y branch opposite to the side of the delta whose resistance is to be found.

Example.—The square array at the right-hand side of Fig. 18, when redrawn in the manner shown in Fig. 22 (the connecting link marked OO' having zero resistance), is seen to include the two deltas OMP and $O'NQ$, each of which can be replaced by the equivalent stars shown in Fig. 23; the

resultant diagram (Fig. 23) is a simple series-parallel circuit of which the equivalent resistance is readily found to be 1.985 ohms, thus checking the result previously obtained by a much longer procedure.

15. Power and Energy in Continuous-current Circuits.—When a steady current of I amp. flows through a conductor having resistance R ohms, the difference of potential between the terminals of the conductor is $V = IR$ volts, in accordance with Ohm's law. In an interval of time t sec. there is transferred from end to end of the conductor a quantity of electricity $Q = It$ coulombs, and in accordance with the definition of potential difference the work done in the time t is

$$W = VQ \quad \text{joules} \tag{30}$$

The *rate of doing work*, which by definition is the *power*, is

$$P = \frac{W}{t} = V\frac{Q}{t} = VI \quad \text{joules per sec.} \tag{31}$$

since Q/t, the number of coulombs per second, is by definition the current in amperes.

In the practical system of units, the joule per second is equivalent to the *watt;* hence,

$$P = VI \quad \text{watts} \tag{32}$$

or, in words,

Power (in watts) = *potential difference* (in volts) × *current* (in amperes)

In systems supplying large amounts of power, the watt is inconveniently small, and the *kilowatt*, which is equivalent to 1000 watts, is used; that is,

$$\text{Power (in kilowatts)} = \frac{\text{volts} \times \text{amperes}}{1000}$$

If the voltage is high, the last expression may be written

$$\text{Power (in kilowatts)} = \frac{\text{volts}}{1000} \times \text{amperes} = \text{kilovolts} \times \text{amperes}$$

the *kilovolt* being equivalent to 1000 volts. Usually, however, in continuous-current systems of the type chiefly considered in this

book, the voltage is not sufficiently high to warrant the use of the kilovolt unit.

In deriving the relations expressed by Eqs. (30), (31), and (32), it is assumed that the current flowing in the circuit has the steady value I amp., so that the quantity of electricity transferred per second is $Q = I$ coulombs. However, the current in a circuit is not necessarily steady but may vary in magnitude from instant to instant; during an infinitesimal time dt, the quantity of electricity passing a given cross-section of the conductor will be dq coulombs, the corresponding instantaneous value of current will be $i = dq/dt$ amp., and the work done in time dt will be

$$dW = v\,dq = vi\,dt = p\,dt \tag{33}$$

where p is the instantaneous value of power; in the time interval from $t = t_1$ to $t = t_2$ the total work is

$$W = \int_{t_1}^{t_2} p\,dt = \int_{t_1}^{t_2} vi\,dt \tag{34}$$

In case v and i are both unvarying in magnitude and are equal to V and I, Eq. (34) becomes

$$W = VI(t_2 - t_1) = VIt = Pt \tag{35}$$

where $t = t_2 - t_1$. But even though both v and i vary from instant to instant, it follows from Eq. (33) that $dW/dt = p = vi$, the instantaneous rate of doing work being thus equal to the corresponding value of instantaneous power. In general, power is merely the *time rate* at which work is done. When it is stated, for instance, that a certain generator has a power rating of 500 kw., the meaning is that when the generator is operating at its normal load it is delivering 500,000 joules of energy during each second.

Since $W = Pt$, which states that energy in joules is equal to the product of watts by seconds, it follows that

$$1\ \textit{joule} = 1\ \textit{watt-second}$$

the expression "watt-second" meaning that power at the rate of x watts is continued during $1/x$ sec. The watt-second is too small for convenience in practical use, and so it is usually replaced by the *kilowatt-hour* (kw.-hr.), and this is the unit commonly used in determining charges for electricity supply. It is impor-

tant to realize that the charge is made for *energy* supplied during a definite time interval and that the time rate at which energy is used at any instant is immaterial except as it affects the total amount of energy consumed; in practice, the maximum rate at which energy may be absorbed is limited in order to avoid overloading the supply system.

16. Joule's Law.—It has been shown in the preceding article that when a steady current of I amp. flows in a conductor having a resistance of R ohms, the power consumed is, by Eq. (31),

$$P = VI \quad \text{watts}$$

where

$$V = IR \quad \text{volts}$$

in accordance with Ohm's law. Upon substituting this value of V in Eq. (31), it follows that

$$P = I^2R \quad \text{watts} \tag{36}$$

and the energy supplied in a time t sec., provided that I remains constant, is

$$W = Pt = I^2Rt \quad \text{joules} \tag{37}$$

The power and energy relations embodied in Eqs. (36) and (37) are the mathematical expressions of *Joule's law.* It was Joule who in 1843 discovered that the heating effect of a current in a wire (an effect previously noted by others) is proportional to the square of the current, to the resistance, and to the time, and he used these facts to determine the mechanical equivalent of heat. It has been found by experiment that the amount of heat required to raise the temperature of 1 g. of water through 1°C., which is by definition 1 cal., is equivalent to 4.19 joules (or 4.19×10^7 ergs). The heating effect of a current is therefore given by

$$H = \frac{1}{4.19}I^2Rt = 0.24I^2Rt \quad \text{cal.} \tag{38}$$

With reference to the battery circuit shown in Fig. 10 and discussed in Art. 9, the e.m.f. of the battery is given by

$$E = Ir + IR$$

when it is discharging through the external resistance R; multiplying this equation by I,

$$P = EI = I^2r + I^2R$$

which states that, of the total power $P = EI$ developed by the battery, a part I^2r is lost in the internal resistance of the battery itself, and the remainder I^2R is the power developed in the external circuit R. The power in the external circuit is clearly equal to zero when $R = 0$, that is, when the battery is completely *short-circuited*; and it is also equal to zero when $R = \infty$, for this condition means that the circuit is open and that $I = 0$. Between these two values of the external resistance there must be some particular value for which the power supplied to the external circuit (P_R) is a maximum. Since

$$P_R = I^2R = EI - I^2r$$

the condition for maximum value of P_R is that

$$\frac{dP_R}{dI} = E - 2Ir = 0$$

or

$$I = \frac{E}{2r}$$

But since

$$I = \frac{E}{r + R}$$

it follows that

$$2r = r + R$$

or

$$R = r$$

is the condition for maximum output; in other words, when the external resistance is equal to the internal resistance, the power supplied to the external circuit has maximum value. Under this condition the power lost in the battery itself is equal to the power delivered to the external circuit, and so the efficiency is only 50 per cent; moreover, the terminal voltage, which is equal to IR, is only half the e.m.f. E of the battery, the other half being consumed in overcoming the internal resistance of the battery. For these reasons the condition for maximum output in such a circuit has no practical importance.

In general, if a battery (or other source) having an e.m.f. E volts and an internal resistance r ohms delivers a current of I amp. to a receiver circuit such as a motor, the circuit *as a whole* must

have, in accordance with Ohm's law, an *equivalent resistance* given by

$$\frac{E}{I} = R + r$$

where R is a *fictitious resistance* equivalent in its effect to the entire load represented by the motor; thus, since

$$\text{Total power} = EI = I^2R + I^2r$$

the part represented by I^2R is the power taken by the motor; a part of this power will be consumed in overcoming the frictional and other losses inherent in the motor, but the balance will be converted into useful mechanical power at the motor shaft. The concept that electrical resistance may be considered as a substitute for mechanical load has important applications in the analysis of motor performance. An extension of this idea occurs in case the value of R computed from the relation $E/I = R + r$ is found to have a *negative* value. The power I^2R will then also be negative; and since $+I^2R$ must be interpreted to mean power *absorbed*, $-I^2R$ must mean power *developed*. In other words, a source of power may be considered to possess negative equivalent resistance.

It is both interesting and important that when a given current I divides between two parallel-connected resistors so that the inverse resistance law $I_1/I_2 = R_2/R_1$ is satisfied, the total loss of power is a minimum. The loss of power in any case is

$$P = I_1^2R_1 + I_2^2R_2$$

and if I_2 is replaced by $I - I_1$, the result is

$$P = I_1^2R_1 + (I - I_1)^2R_2$$

Upon differentiating P with respect to I_1 and equating the result to zero, there is obtained as the condition for minimum value of P

$$I_1R_1 = (I - I_1)R_2 = I_2R_2$$

which expresses the inverse resistance law. The fact that the current automatically divides in such a way that the power loss is a minimum is a manifestation of a general law known as the *law of least action*. Among other examples of this law is the fact that a ray of light passing from a point in one medium to

another point in a different medium follows a path which requires the least time; another example, referred to in later chapters, is that a magnetic field produced by a given excitation has the largest possible value.

17. Nature of Resistance.—The fact that electricity was originally considered to be a continuous (though imponderable) fluid substance, combined with the experimental observation that a current produces heat in the conductor in which it flows, gave rise to the notion of electrical resistance as a property quite analogous to the frictional resistance encountered by a material fluid flowing in a pipe. However, the analogy is not complete. For though in the case of water flowing at moderate velocity the retarding effect is largely due to the skin friction of the layers adjacent to the wall of the pipe, there is no such effect in the case of *steady* flow of electricity in a wire; instead, the frictional effect is uniform all over the cross-section of the wire right up to its boundary surface. Moreover, in terms of the modern electron theory, the current flow is *not* to be thought of as a continuous stream of electrons that enter the wire at one end and leave the wire at the other end, meanwhile bumping their way through the atomic interstices of the conducting material.

In accordance with currently accepted views, the atom is regarded as a nucleus composed of protons and neutrons surrounded by one or more rings or shells of electrons which are more or less tightly bound to the nucleus. Materials in which some of the outer electrons are readily detachable by moderate electric forces are then classifiable as *conductors*, and those in which the electrons are so firmly held that they can be detached only by very intense electric fields are called *insulators*. Accordingly, if a conductor forms part of a closed circuit that is acted upon by an e.m.f., the electrons that have been detached from one atom move to an adjacent atom which has in its turn lost electrons to another.

It is therefore apparent that there is no sharp line of demarcation between conductors and insulators, the distinction between them residing in the nature of the bond that holds the electrons to the atom and upon the intensity of the electric field required to effect a separation of the electrons. In general, this bond is weakest in the metals, which are accordingly the best conductors,

silver ranking first, with copper next in order. Vitreous materials such as glass and quartz, and resinous and organic substances like amber, bakelite, rubber, oil, and silk have good insulating qualities because of the firmness with which the electrons are bound to the atom; and the same is true of dry air at ordinary pressure and temperature. It is obvious that when an insulated conductor, such as a bare copper wire surrounded by air, is subjected to a difference of potential between its terminals the surrounding insulating medium is subjected to electrical forces in the same way as are the atoms of the conductor itself; but whereas the loosely held electrons of the metal are free to move under the influence of these forces, the atoms of the insulator can only undergo an elastic stretch or deformation until at a limiting potential difference there is an actual disruption. For these reasons an insulated conductor cannot be regarded as a rigid pipe carrying an incompressible electric fluid, as is customary in thinking of electric current in terms of the hydraulic analogy; rather must it be regarded as a channel bored in a surrounding elastic medium. In other words, the analogy according to which an electric current is regarded as similar to water flowing in a steel pipe must be replaced by one in which it is regarded as similar to water flowing in a thick rubber tube.

18. Calculation of Resistance.—Ohm's original researches indicated that the resistance of a homogeneous conductor of uniform cross-section is directly proportional to its length and inversely proportional to its cross-section, or

$$R = \rho\frac{l}{a} \quad \text{ohms} \tag{39}$$

where ρ is a proportionality constant the value of which, for a given material, is dependent upon the units employed in measuring the length l and the cross-section a. This constant ρ is called the *specific resistance* or, more concisely, the *resistivity.*

The fundamental resistivity used in the computation of the resistance of copper conductors is the International Annealed Copper Standard,* which assigns a resistance, at 20°C., of

* Adopted by the I.E.C. in 1913. See also *A.I.E.E. Standards*, No. 30, April, 1937, in which the resistance of annealed copper at 20°C. is given as $\frac{1}{58}$ ohm = 0.017241 . . . ohm for a length of 1 m. and a uniform cross-section of 1 sq. mm.

0.15328 ohm to an annealed copper conductor of uniform cross-section, 1 m. long and weighing 1 g., the copper having a density of 8.89. *If the cross-section of this wire is designated as a* sq. cm., its volume is $100a$ cu. cm., and its weight is $889a = 1$ g., whence $a = 1/889$ sq. cm. If, therefore, the length in Eq. (39) is expressed in centimeters and the area in square centimeters,

$$R = \rho \frac{100}{1/889} = 0.15328$$

Hence, $\rho = 1.7241 \times 10^{-6}$, which is the resistance at 20°C. of a conductor 1 cm. long and 1 sq. cm. in cross-section, generally referred to as a *centimeter-cube* (though the cubic form is not essential since the cross-section may have any shape, provided that its area is 1 sq. cm.).

Conductance (in mhos, or siemens) having been defined as the reciprocal of resistance (in ohms), Eq. (39) is equivalent to

$$G = \frac{1}{R} = \frac{1}{\rho}\frac{a}{l} = \gamma\frac{a}{l} \tag{40}$$

where $\gamma = 1/\rho$ is called the *specific conductance* or the *conductivity.* The conductivity of standard annealed copper at 20°C. is therefore $10^6/1.7241 = 580{,}000$ mhos (or siemens) per centimeter-cube.

In American practice it is customary to express lengths of conductors in feet (occasionally in inches or in miles) and cross-sections in *circular mils* (and occasionally in *square mils*). The *mil* is defined as the one-thousandth part of an inch (0.001 in.), the circular mil (cir. mil) as the area of a circle having a diameter of 1 mil, and the square mil as the area of a square whose sides are 1 mil in length. Hence,

$$1 \text{ cir. mil} \equiv \frac{\pi}{4} \times 10^{-6} \text{ sq. in.}$$

$$1 \text{ sq. mil} \equiv 10^{-6} \text{ sq. in.}$$

By analogy with the definition of the circular mil, a *circular inch* may be defined as the area of a circle having a diameter of 1 in. The area of a circular inch being $\pi/4$ sq. in., it follows that a circular inch contains exactly 10^6 circular mils. In general, if a wire has a diameter of d mils (or $d \times 10^{-3}$ in.), its area is $(\pi/4)d^2 \times 10^{-6}$ sq. in., which is equivalent to d^2 cir. mils; in other words, *the area of a circular wire, expressed in circular mils, is simply the square of its diameter expressed in mils.*

If the length of a conductor is expressed in feet and its section in circular mils, its resistance can be computed by Eq. (39) provided that the resistivity ρ is expressed in ohms per circular-mil-foot, a term that signifies a length of 1 ft. and a cross-section of 1 cir. mil. If the dimensions in terms of these units are l', a', and ρ', the resistance is $\rho' l'/a'$ ohms, which is necessarily the same as the resistance of the same wire characterized by the dimensions l cm. and a sq. cm. and resistivity ρ ohms per cm.-cube; consequently,

$$R = \rho'\frac{l'}{a'} = \rho\frac{l}{a}$$

or

$$\rho' = \rho\left(\frac{l}{l'}\right)\left(\frac{a'}{a}\right) = 1.7241 \times 10^{-6} \times (12 \times 2.54) \times \frac{(4/\pi) \times 10^6}{(2.54)^2}$$
$$= 10.37 \quad \text{ohms}$$

which is the resistance of a circular-mil-foot of copper at 20°C.

19. Variation of Resistance with Volume.—Equation (39), which gives the resistance of a conductor in terms of its dimensions, may be written in the form

$$R = \rho\frac{l}{a} = \rho\frac{al}{a^2} = \rho\frac{l^2}{al}$$

and since al is the volume v,

$$R = \rho\frac{v}{a^2} = \rho\frac{l^2}{v} \tag{41}$$

In other words, for a given volume (or weight) of material, the resistance varies directly as the square of the length or inversely as the square of its cross-section. This relation is useful if it is necessary to compare the resistances of conductors of the same weight but having different lengths and cross-sections.

For example, 1000 ft. of No. 10 wire, having a diameter of approximately 100 mils, has a resistance very nearly equal to 1 ohm at 25°C. (It is easy to remember these figures, for all of them except the temperature are integral powers of 10.) It is desired to find the resistance of an equal weight of wire of which the diameter is 460 mils (No. 0000 wire). Since the ratio of the diameters is 4.6, the ratio of the cross-sections is $(4.6)^2$; and since the resistances are inversely as the squares of the areas, the result is $1/(4.6)^4 = 0.00223$ ohm.

20. Effect of Temperature upon the Resistance of Copper.—It is an experimentally determined fact that the resistance (and the resistivity) of most of the unalloyed metals, including copper, increases with increasing temperature. When measurements of resistance are made over an extended temperature range and the results are plotted as in Fig. 24, it is found that within the temperature limits commonly experienced the graph is very nearly a

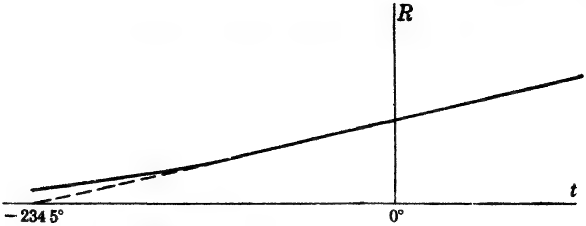


FIG. 24.—Variation of resistance with temperature.

straight line, so that resistance is substantially a linear function of temperature; expressed in mathematical form,

$$R_t = R_0(1 + \alpha t) \tag{42}$$

where R_t = resistance at temperature t°C.
R_0 = resistance at 0°C.
α = *temperature coefficient of resistance.*

Actually, the graph departs from linear form when the temperature is very low; but if the straight portion is extended backward, as indicated by the dashed line, it intersects the axis of temperature at −234.5°C. in the case of copper. Thus, if Eq. (42) held throughout the entire range, R_t would be zero* when $t =$ −234.5°C., or

$$1 + \alpha(-234.5) = 0$$

and

$$\alpha = \frac{1}{234.5} = 0.00427 \text{ (for copper)}$$

For annealed copper conductors, therefore, within the usual limits of temperature encountered in ordinary practice,

$$R_t = R_0(1 + 0.00427t) \tag{43}$$

* Kamerlingh Onnes, in 1911, made the experiment of inducing a current in a ring of mercury cooled to a temperature close to absolute zero. The current persisted for several hours without any impressed e.m.f. whatsoever.

which may be written in the alternative form

$$R_t = R_0 \frac{234.5 + t}{234.5} \tag{44}$$

These relations lead to the conclusion that if the resistance of a copper conductor is R_{t_1} ohms at temperature t_1 and R_{t_2} ohms at temperature t_2, we may write

$$R_{t_2} = R_0(1 + \alpha t_2)$$
$$R_{t_1} = R_0(1 + \alpha t_1)$$

from which

$$\frac{R_{t_2}}{R_{t_1}} = \frac{1 + \alpha t_2}{1 + \alpha t_1} \tag{45}$$

and

$$\frac{R_{t_2} - R_{t_1}}{R_{t_1}} = \frac{\alpha(t_2 - t_1)}{1 + \alpha t_1} = \frac{t_2 - t_1}{234.5 + t_1} \tag{46}$$

so that

$$t_2 - t_1 = \frac{R_{t_2} - R_{t_1}}{R_{t_1}}(234.5 + t_1) \tag{47}$$

Thus, if the resistances R_{t_2} and R_{t_1} are measured at two temperatures t_2 and t_1 (the former assumed to be the higher), the *rise of temperature* can be determined from the observed increase of resistance provided that the initial temperature t_1 is known. In practice, this condition is realized by keeping the resistor inert for a sufficient length of time to let it acquire the temperature of its surroundings. Equation (47) thus provides a convenient and much-used means for determining the temperatures attained by machine windings after they have been operating under load conditions.

The derivation of Eq. (45) relates both R_{t_2} and R_{t_1} to R_0. It is possible, however, to relate R_{t_2} directly to R_{t_1}, without the actual or implied use of R_0. Thus, Eq. (45) may be written

$$R_{t_2} = R_{t_1}\frac{1 + \alpha t_2}{1 + \alpha t_1} = R_{t_1}[1 + \alpha_1(t_2 - t_1)] \tag{48}$$

where α_1 is a new temperature coefficient, with t_1 as a base, which must satisfy the condition

$$\frac{1 + \alpha t_2}{1 + \alpha t_1} = 1 + \alpha_1(t_2 - t_1)$$

whence

$$\alpha_1 = \frac{\alpha}{1 + \alpha t_1} = \frac{1}{(1/\alpha) + t_1} = \frac{1}{234.5 + t_1} \tag{49}$$

Thus, when $t_1 = 0$, $\alpha_1 = \alpha = 1/234.5$, as previously deduced; and at 20°C., $\alpha_1 = 1/254.5 = 0.00393$.

If we exclude from consideration the minor effects of temperature upon the length and cross-section of conductors, resistivity changes with temperature in accordance with the same relations that hold for resistance. For example, the resistivity of copper has been found to be 10.37 ohms per cir.-mil-ft. at 20°C. This value corresponds to 9.55 ohms per cir.-mil-ft. at 0°C. and to 12.161 ohms at 75°C., which is in the neighborhood of the usual working temperature of the windings of electrical machinery. Thus we have the approximate rule that at normal (hot) working temperature the resistivity of copper is nearly 1 *ohm per cir.-mil-in.*

The values of resistivity that have been discussed in the preceding paragraphs pertain to annealed copper of standard purity. Not all commercial copper conforms to these figures, and in the case of hard-drawn copper the resistivity may be about 2.7 per cent higher than the values that have been given. In the Standards of the American Institute of Electrical Engineers* (A.I.E.E.), it is recommended that the resistivity shall not exceed 891.58 ohms per lb.-mile at 20°C. (that is, the conductor is 1 mile in length, with a cross-section that makes the total weight equal to 1 lb.). This figure corresponds to 1.7561 microhms per cm.-cube, or 1.88 per cent higher than the international standard of 1.7241 microhms for annealed copper.

21. American Wire Gage.—The American Wire Gage (A.W.G.), formerly called the Brown and Sharpe (B. & S.) Gage, is given in detail as Appendix A, page 705. It is arbitrary but is so constructed that the diameters constitute a geometrical series, the ratio of any one diameter to the next in order being a constant. The largest size (No. 0000 or 4/0) is assigned an arbitrary diameter of 460 mils, and No. 36 a diameter of 5 mils; between these two sizes there are 39 intervals, and hence the ratio of diameters, taken in ascending order of size (descending order of gage num-

* A.I.E.E. Standards, No. 61, Specifications for Soft or Annealed Copper Wire, September, 1928.

bers) is

$$(460/5)^{1/39} = 1.123$$

Since cross-sections vary as the squares of the diameters, the ratio of cross-sections of successive sizes, in ascending order of dimensions, is $(1.123)^2 = 1.261$. Sizes that are 2 gage numbers apart will be in the ratio $(1.261)^2 = 1.590$; sizes that are 3 gage numbers apart will be in the ratio $(1.261)^3 = 2.005$, or practically 2; and sizes that are 10 gage numbers apart will be in the ratio $(1.261)^{10} = 10.164$, or very nearly 10.

If calculations have to be made in the absence of such a table as Appendix A, the following empirical rules will give a fair approximation:

1. Number 10 wire has a diameter of 100 mils (actually 102 mils), a cross-section of 10,000 cir. mils, and a resistance of 1 ohm per 1000 ft. at 25°C.

2. The cross-section doubles every three numbers in the direction of increasing size of wire and halves every three numbers in the opposite direction.

3. The cross-section increases tenfold every 10 numbers in the direction of increasing size.

Number 0000 wire, having a diameter of nearly $\frac{1}{2}$ in. and a cross-section of nearly $\frac{1}{6}$ sq. in., is the largest solid wire of circular section commonly used. Solid copper conductors of larger section and greater current-carrying capacity, such as are required for bus bars, are usually of rectangular cross-section, because of the greater ease with which they may be bent, and bolted or soldered together; and the greater ratio of lateral surface area to the cross-sectional area of such rectangular conductors is a distinct advantage in radiating the I^2R losses and so holding down the temperature. When circular conductors of larger size than No. 0000 are required, stranded cables are always used because of their greater flexibility; but in such cases the size is expressed directly in terms of circular mils, as in the table given in Appendix B, page 706. Stranded cables of smaller cross-section than No. 0000 are also available for purposes where flexibility is essential, as in the case of lamp cord.

22. Other Conducting Materials. Alloys. Resistors.—The fact that copper is used as a conducting material to a greater extent than any other material is accounted for not only by its

high conductivity and comparatively low cost but also by the excellence of its physical characteristics in general; it has high ultimate tensile strength (49,000 to 67,000 lb. per sq. in. for hard-drawn copper), ductility of the same order as that of steel, freedom from atmospheric corrosion, and it is easily soldered.

Silver, which has the highest conductivity of all the metals, is too expensive except in delicate instruments where cost is not an important consideration. Aluminum is the principal competitor of copper in high-voltage transmission lines. It has a resistivity of 2.828 microhms per cm.-cube at 20°C. as compared with 1.7241 microhms for copper, and so its conductivity is 61 per cent of that of copper; but its density is 2.67, or only 30 per cent that of copper. With reference to Eq. (41), where it was shown that

$$R = \rho\frac{l^2}{v}$$

it is seen that this value of R is equivalent to

$$R = \rho\delta\frac{l^2}{\delta v} = \rho\delta\frac{l^2}{w} \tag{50}$$

where δ is the density of the material and w is the weight of the entire conductor. Using the subscript c to indicate copper and a to indicate aluminum and assuming the same length and weight in both cases, we have the result from Eq. (50) that

$$\frac{R_a}{R_c} = \frac{\rho_a}{\rho_c}\frac{\delta_a}{\delta_c} = \frac{0.30}{0.61} = 0.49$$

In other words, aluminum has about twice the conductance of an equal length and weight of copper. If two transmission lines of the same voltage, one of copper and the other of aluminum, are to transmit the same amount of power with the same loss in transmission, the aluminum line will weigh about half the copper line, and the total cost of the two installations will be the same if the price of aluminum per pound is twice that of copper. For the same loss of power in both lines, the diameter of the aluminum conductor will be 28 per cent larger than that of the copper wire, a feature that is advantageous in reducing corona loss in high-tension transmission lines.

Iron or steel wires have been occasionally used for telephone circuits of minor importance. But it must be remembered that the effect of the current is to magnetize the iron and this in turn

has an important effect when the current varies in strength; in any case, the resistivity of steel is nearly ten times that of copper, a fact which rules it out of consideration except where massive conductors must be used, as in the third rail of electric-railway installations. In long spans or where great tensile strength is essential, copper-clad wire is used; as made by the Copperweld Company, this consists of a steel core surrounded by a concentric layer of copper welded to the steel.

Metallic alloys exhibit numerous interesting characteristics, especially with respect to temperature coefficient and resistivity. *Manganin*, for example, which is an alloy of 84 per cent copper, 12 per cent manganese, and 4 per cent nickel, has a very low temperature coefficient (0.000006) which makes it very useful in the construction of measuring instruments and their accessories in which constancy of resistance, independent of the heating effect of current, is important. *Constantan* (60 per cent copper, 40 per cent nickel) is a similar material, having a temperature coefficient of 0.000005.

In all the preceding discussion, emphasis has been placed on conductors of low resistivity that are primarily intended to convey energy with minimum loss from a source of e.m.f. to the particular device by which the energy is to be utilized. But there is a group of materials which, although they are conductors as distinguished from insulators, have such high resistivity that their principal function is to convert electrical energy into heat energy, as in rheostats and heating elements for miscellaneous purposes. Such appliances are called *resistors*. Cast-iron grids are used, for example, for the starting rheostats of industrial and railway motors.

Among the materials commonly used in building resistors is *German silver*, composed of copper, nickel, and zinc in varying proportions, the resistivity increasing with the percentage of nickel; German silver has a high temperature coefficient. Copper-nickel alloys and iron-nickel alloys have high resistivity, that of the former running from 10 to 30 times that of copper; but nickel-chromium alloys (nichrome) have from 60 to 70 times the resistivity of copper. These alloys are especially useful in high-temperature devices, such as electric furnaces.

A few substances have very remarkable resistance characteristics. Carbon, for example, has a resistivity ranging from

400 to 2400 times that of copper, depending upon the physical state of the material; and with this high resistivity it combines the characteristic of possessing a negative temperature coefficient. Carbon is extensively employed as brush material for generators and motors, and successful commutation is in most cases dependent upon its use. Selenium, one of the nonmetallic elements, possesses the property of decreasing its resistance when light falls upon it; and the resistance of bismuth is dependent upon the intensity of the magnetic field in which it lies.

The resistance characteristics of a number of substances are summarized in Appendix C.

CHAPTER II

MAGNETISM

1. Historical.—The term magnet was originally applied to the loadstone, an oxide of iron, Fe_3O_4, pieces of which are sometimes found to have the property of attracting small bits of iron or steel. It is fairly certain that this phenomenon was known in Greece between 1200 and 1100 B.C., and perhaps earlier; but the first authentic reference appears in works quoted from Thales of Miletus in Asia Minor (585 B.C.), who also mentions the electrification of amber. The word magnet probably derives from a town called Magnesia in Asia Minor near which was a large deposit of magnetite where iron was smelted.

The attractive force exerted by the loadstone, or natural magnet, upon bits of iron remained for many centuries all that was known about magnetism in general. Then it was discovered that a pivoted or freely suspended magnet possessed the property of orienting itself in a particular direction relative to the earth's surface, a fact which led to the development of the compass. It has been claimed that this fact was known and used by the Chinese in remote antiquity; but the most reliable information fixes the date at approximately A.D. 400, and by A.D. 900 this directive property had been accurately studied. This information found its way into Europe probably through the Finns, who were of Mongolian origin; and between A.D. 1000 and 1250 the compass was in general use for navigation purposes by all the seafaring nations of that time.

The first genuinely scientific experiments with magnets appear to have been made by Peter Peregrinus, who in 1269 recorded the results of this early work in a text entitled "De Magnete"; but since this work considerably antedated the invention of printing and occurred, moreover, at a time when the Scholastic influence was bitterly opposed to physical experimentation, the real beginning of an understanding of the subject may be said to date from 1600, when Sir William Gilbert published the book already referred to in Chap. I. Both Peregrinus and Gilbert

experimented with large spherical loadstones and found by means of small pivoted compasses that there were two points, or poles, at opposite ends of a particular diameter of the sphere, from which the directive forces seemed to emanate. One of these poles, say A, always attracted the north-seeking pole of the small testing compass and repelled its south-seeking pole; the other pole of the sphere, say B, always attracted the south-seeking pole of the compass and repelled its north-seeking pole. At other points on the surface of the sphere the small test compass oriented itself on lines that were substantially meridians running from pole to pole. The analogy between these results and the behavior of a compass on the earth's surface led both men to the conclusion that the earth is itself a huge magnet with magnetic poles located in the vicinity of the geographical poles. Gilbert's work had in fact been undertaken to prove the correctness of the Copernican theory and to account, in terms of magnetic forces, for the very great interplanetary forces which that theory requires.

Peregrinus likewise showed that iron in the vicinity of a magnet becomes *magnetized by induction* and that like poles of two magnets repel, whereas unlike poles attract, each other. He demonstrated further that, if a magnet is cut into smaller parts, each part remains a magnet. Gilbert repeated these experiments and made numerous others; he showed that steel bars can be made permanently magnetic by stroking them with the poles of a loadstone; that an iron or steel bar held in the direction of the earth's magnetic field can be magnetized by pounding it; that iron or steel, when heated in a magnetic field and then allowed to cool slowly, becomes magnetic; and that such a bar magnet, if heated, loses its magnetism partly or wholly, depending upon the degree of heating.

The observations made by Gilbert were exploratory and necessarily of a qualitative nature because the basis for quantitative measurements had not yet been laid. Quantitative calculations of any kind were not possible until Newton laid the foundations with the publication of his "Principia" in 1687; and it was not until 1785 that Coulomb, through his invention of the torsion balance, was able to determine the basic relations between the forces exerted by magnetized bodies.

During all this time the phenomena of magnetism remained completely isolated from other fields of scientific inquiry, though

about 1708 it was suspected that there was some relation between electricity and magnetism because of the effects of lightning discharges in changing the strength of permanent magnets in the vicinity. However, it was not until the publication of Oersted's work in 1820 that the phenomena of electricity and magnetism were definitely linked. One of the results of this discovery was that, whereas permanent magnets had previously been made by stroking steel with a loadstone or another magnet, they were thereafter made much more readily by surrounding the steel with a coil of wire carrying a heavy current; this possibility was discovered by Gay-Lussac and Arago just after Oersted's work became known.

2. Molecular Theory of Magnetization.—The simplest form of magnet is a straight rod, in which form it is called a bar magnet. If such a magnet is dipped into a mass of soft-iron filings, the filings will cling to the two ends of the magnet in the form of tufts, while the middle portion of the magnet will be left comparatively free from filings. The notion thus originates that the magnetic properties apparently reside at or near the ends, which are called the *poles* of the magnet; but the magnetized condition is in reality distributed fairly uniformly throughout the mass of the magnet, provided that it is homogeneous, as may be shown by cutting the magnet into short lengths. No matter how far the subdivision is carried, each part remains a complete magnet with two equal poles. This fact is the basis of the theory, admittedly crude and unsatisfactory, but nevertheless useful, in accordance with which a magnet may be looked upon as an aggregate of large numbers of infinitesimal magnets lying side by side, each with its polar axis oriented more or less completely in the same direction as all the others.

This molecular theory, originally advanced by Weber and later amplified by Ewing, assumes that in the original unmagnetized condition the constituent molecular magnets have a completely random distribution or are otherwise so arranged that their joint effect at any external point is nil. On subjecting the material to a magnetizing effect, as by stroking the bar with two magnets as in Fig. 1, the molecular magnets tend to become oriented in the same direction, but they are restrained from instantaneous or complete response by the friction of the matrix in which they are embedded. The theory accordingly accounts

for the known fact that there is a limit beyond which the material cannot be magnetized, in other words, that it becomes magnetically *saturated*. It likewise accounts for the fact that if the material is subjected to a rapidly alternating magnetizing force there is an evolution of heat, as would be expected because of the internal molecular friction as the elementary magnets tend to align themselves first in one direction and then in the other. The theory is furthermore consistent with the fact that the harder the steel the better does it retain any magnetization once imparted, whereas soft iron or steel quickly loses most of its magnetization when the magnetizing action is removed or when the material is jarred.

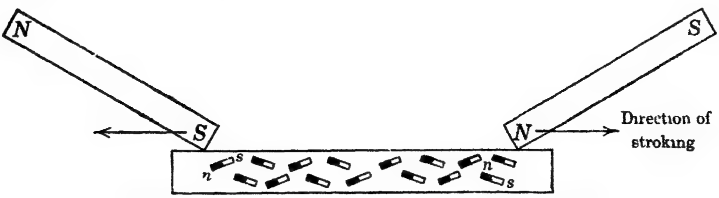


FIG. 1.—Partial orientation of molecular magnets.

The fact remains, however, that this theory explains nothing about the nature of magnetism itself and merely describes the properties of large magnets in terms of smaller ones; the theory is, moreover, inadequate to explain the effect of alloying iron with such elements as nickel, cobalt, chromium, tungsten, silicon, and aluminum. More recent theory* is based upon the idea that the magnetic properties of substances like iron are due to the orbital motions of the electrons around the nucleus of the atom and to the spinning of the electrons themselves. According to this view all magnetic phenomena are to be regarded as mere subjective aspects of the motions of electric charges relative to the observer, the electric charges alone possessing objective reality. The concept of magnetic poles becomes nothing more than a convenient abstraction, useful to the extent to which it leads to conclusions consistent with experiment, but not to be taken seriously as a picture of a real entity. This point of view is the one that has been adopted in presenting the subject in this book, and as the development proceeds it will be pointed

* K. K. DARROW, Spinning Atoms and Spinning Electrons, *Elec. Eng.*, October, 1937, p. 1228.

out in appropriate sections how these confessedly artificial concepts can be discarded in favor of a more direct approach.

Although iron and steel possess magnetic properties to a much greater degree than other substances do, there are materials, of which nickel and cobalt are examples, that have these properties to a lesser extent. Substances in this group are attracted by a magnet and are called *paramagnetic* because they tend to orient themselves in a magnetic field with their longer axes in the direction of, or parallel to, the field. But there are substances like bismuth, phosphorus, and zinc that are repelled by both poles of a magnet and tend to set themselves with their longer axes across the field, for which reason they are called *diamagnetic* substances. Since iron and its alloys are the most prominent examples of the paramagnetic group, substances belonging to this group are frequently called *ferromagnetic* or, for the sake of brevity, merely magnetic. It is interesting to note that certain alloys, called Heusler alloys, which contain no iron at all, have been found to have marked magnetic properties equivalent to that of a low grade of cast iron; these alloys are made of copper, aluminum, and manganese, no one of which is itself magnetic.

Other alloys of iron and steel that have particularly interesting properties and important applications are discussed in Art. 18 of this chapter.

3. Magnetic Field.—The region surrounding a magnet (including under that classification the earth itself) is called the *magnetic field* of the magnet. In such a region other magnets experience a force action, and soft iron or steel and some of their alloys become magnetized by induction. Oersted's discovery that a magnet is deflected when suitably placed near a current-carrying conductor led to the corollary, which he reported in his original paper, that a movable coil, if it carries a current, is itself deflected when suitably placed with respect to a fixed magnet; this is the basic fact underlying the operation of all electric motors.

In 1831 Faraday discovered that if a conductor forming part of a closed electric circuit is moved across a magnetic field, no matter whether that field is produced by a magnet or by a current in another conductor, a flow of current will be induced in the circuit of which the moving conductor is a part; this phenomenon, known as *electromagnetic induction*, led to the

development of the electric generator, the transformer, and the whole array of electrical machinery involved in the generation, transmission, and utilization of electrical energy.

Quite generally, therefore, a magnetic field may be defined as a region within which:

1. A magnet is acted upon by a force.
2. Magnetic substances may become magnetized by induction.
3. A conductor carrying an electric current may* be acted upon by a force.
4. A moving conductor, if part of a closed circuit, may* become the seat of an induced current.

4. Unit Magnet Pole.—Every magnetized body exhibits the phenomenon of polarity, that is, the simultaneous existence of equal poles of opposite sign. One polarity cannot exist without the other. The magnetized condition obtains throughout the entire mass of the magnet, but its intensity generally varies from point to point. In speaking of the pole of a magnet it should be understood that there is no one point at which the magnetism is actually concentrated, but the conception of concentrated point poles is useful for purposes of computation even though the idea is artificial. In the case of a long, slim magnet, like a knitting needle, the magnetism acts as though it were mostly concentrated at or near the ends, so that such a magnet approximates fairly well the condition of concentrated point poles. In particular, if one pole of such a magnet is placed in a magnetic field, its other pole being so far removed as to be acted upon with little or no force, the magnet will behave as though it consisted of a single isolated pole, and the forces acting upon it can be studied.

The first quantitative measurements of the forces between magnet poles (in air) were made by Coulomb, who used two long magnets, one suspended by a fiber in a horizontal position, the other adjustably placed with respect to the first so that the adjacent poles of the two magnets would not be appreciably affected by the two remote poles. The fiber of the torsion balance having been suitably calibrated by using ordinary forces, coulomb was enabled to measure the forces between the

* Under certain conditions, to be explained later, there may be no force upon a current-carrying conductor in a magnetic field, and no induced effect in a conductor moving in a particular manner.

adjacent magnet poles, to each of which he ascribed a definite *magnetic strength.* In this manner he showed that *the force between two magnet poles varies in direct proportion to the product of their strengths and in inverse proportion to the square of the distance between them.* If m and m' are the strengths of the two poles and r is the distance between them, the force with which each pole acts upon the other, when both are in free space, is given by

$$f = \frac{mm'}{\mu_0 r^2} \tag{1}$$

where μ_0 is a proportionality constant whose magnitude and dimensions depend upon the units in which m, r, and f are expressed. The quantity μ_0 is called the *permeability* of free space.

In the *absolute* (*c.g.s.*) *e.m. system of units* the assumption is *arbitrarily* made that $\mu_0 = 1$, but it must not therefore be assumed that μ_0 is a mere numeric (that is, an abstract number); its "dimensions" will be dependent upon those assigned to f, m, and r, such that the two sides of Eq. (1) shall have identical resultant dimensions. But so far as concerns the *numerical* values of f, m, and r, the assumption $\mu_0 = 1$ leads to the relation

$$f = \frac{mm'}{r^2} \tag{2}$$

so that if m and m' are equal

$$m = \sqrt{fr^2}$$

Accordingly, if $f = 1$ dyne and $r = 1$ cm., m and m' will each have unit value; thus, we have the definition: *A unit magnet pole is a point pole of such strength that it will exert a force of one dyne upon an equal point pole placed one centimeter distant, both poles being in free space.*

In general, if the poles are immersed in some medium other than free space, the force between them will be

$$f = \frac{mm'}{\mu_a r^2}$$

where μ_a is a quantity of the same nature as μ_0, but differing in magnitude from the latter to a degree characteristic of the

medium concerned. The quantity μ_a is called the ***absolute permeability*** of the medium that it characterizes. The ratio $\mu_a/\mu_0 = \mu$, called the *relative permeability* of the medium, is a pure numeric. It is very nearly equal to unity in most nonmagnetic substances; in the case of air, for example, $\mu = 1.000004$. In general, μ is greater than unity in paramagnetic materials and less than unity (but only slightly less) in diamagnetic substances. There is no material for which μ is zero.

5. Force Due to Distributed Pole.—The definition of a unit magnet pole is necessarily restricted to the case of *point poles* in order that the distance between any two of them [r in Eqs. (1) and (2)] may have definite meaning. The poles actually encountered in magnets are distributed over finite areas, and so if the effect of a distributed pole is to be determined it is necessary to consider it as the aggregate of infinitesimal elements, each of which may be considered as a point pole.

As an example, consider a bar magnet having a circular cross-section, and let it be assumed that the end face indicated in Fig. 2 is uniformly magnetized over its entire area. It is desired to find the force due to this pole face upon a unit point pole located at the point P on the axis of the bar magnet, distant p cm. from the face of the bar magnet.

Let the total strength of the pole face of the magnet be m units, distributed uniformly over the area $A = \pi r^2$, where r (in centimeters) is the radius of the cross-section; the pole strength per unit area is

$$\sigma = \frac{m}{A} = \frac{m}{\pi r^2} \tag{3}$$

and σ is called the intensity of magnetization.

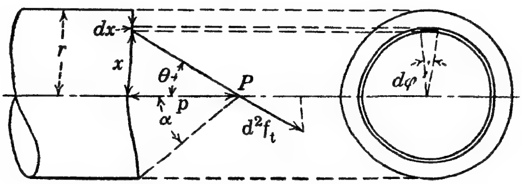


FIG. 2.—Force on unit pole on axis of bar magnet.

Consider an annular element of the pole face bounded by the concentric circles of radii x and $x + dx$, and take a small sector of this ring subtending the angle $d\varphi$ at the center. The area

of this small sector is $x\,d\varphi \cdot dx$, and its pole strength is $\sigma x\,d\varphi\,dx$. The force exerted by this elementary magnet upon the unit pole at P is

$$d^2f_t = \frac{\sigma x\,d\varphi\,dx}{\mu_0(x^2 + p^2)}$$

the direction of this force being along the line joining them. The component of this force acting at right angles to the axis will obviously be balanced by the corresponding component of the force due to an elementary sector diametrically opposite the one that has been sketched, so that only the axial component need be considered; this component is

$$d^2f = \frac{\sigma x\,d\varphi\,dx}{\mu_0(x^2 + p^2)}\cos\theta = \frac{\sigma x\,d\varphi\,dx}{\mu_0(x^2 + p^2)} \cdot \frac{p}{\sqrt{x^2 + p^2}}$$

and the axially directed force due to the entire annulus is

$$df = \frac{\sigma px\,dx}{\mu_0(x^2 + p^2)^{3/2}}\int_0^{2\pi} d\varphi = \frac{2\pi\sigma px\,dx}{\mu_0(x^2 + p^2)^{3/2}} \tag{4}$$

The force in the axial direction due to the entire pole face is given by

$$f = \frac{2\pi\sigma p}{\mu_0}\int_0^r \frac{x\,dx}{(x^2 + p^2)^{3/2}} = \frac{2\pi\sigma}{\mu_0}\left(1 - \frac{p}{\sqrt{r^2 + p^2}}\right)$$
$$= \frac{2\pi\sigma}{\mu_0}(1 - \cos\alpha) \tag{5}$$

The expression

$$\omega = 2\pi(1 - \cos\alpha) \tag{6}$$

is equal to the solid angle* subtended at the point P by the right cone whose base is the circular pole face. In general, therefore, the force on a unit point pole due to a circular pole face is

$$f = \frac{\omega\sigma}{\mu_0} \quad \text{dynes}$$

the force being directed on a line perpendicular to *the pole face.*

* In general, the solid angle at the vertex of any cone is equal, by definition, to the area intercepted by the cone on a spherical surface of unit radius of which the center coincides with the vertex of the cone.

If the distance p is small in comparison with the radius of the pole face, the angle α approaches 90 deg. and ω approaches 2π, in which case

$$f_{p\to 0} = \frac{2\pi\sigma}{\mu_0} \tag{7}$$

But if p approaches zero, a condition which means that the point pole at P is very close to the face of the bar magnet, the shape of the latter, whether circular or not, makes no difference in the result. Accordingly, Eq. (7) represents in general the force on a unit point pole placed so close to a distributed pole face that the latter subtends at the point pole a solid angle substantially equal to 2π, in which case the area is to all intents and purposes indefinitely large.

Equation (7) leads at once to a relation that is useful in computing the lifting power of magnets; for suppose that two parallel pole faces of opposite polarity, each having an area A sq. cm. and an intensity of magnetization σ, are separated in free space by a distance that is small compared with the linear dimensions of the neighboring pole faces. An elementary area dA on one pole, having a pole strength $\sigma\, dA$, is equivalent to a point pole in the presence of a practically indefinite magnetized plane, so that the force of attraction is

$$df = \frac{2\pi\sigma}{\mu_0} \times \sigma\, dA = \frac{2\pi\sigma^2}{\mu_0} dA$$

and the entire force of attraction between the two faces is

$$f = \frac{2\pi\sigma^2}{\mu_0} A \tag{8}$$

Equation (8) makes it possible to determine the intensity of magnetization of a pole face by direct experiment. For example, let two equal and opposite pole faces be brought into contact, and let the force required to separate them be measured. Substitution of the value of f (in dynes) and of A (in square centimeters) will give $\sigma = m/A$, since μ_0 is taken as unity.

6. Magnetic Field Intensity.—When two magnet poles m and m' are immersed in a medium characterized by the constant μ_a, the force with which each acts upon the other

$$f = \frac{mm'}{\mu_a r^2}$$

may be attributed to the fact that m', for example, lies in the magnetic field produced by m; or, conversely, that m lies in the field due to m'. The *intensity* of the magnetic field thus acting upon a point pole may be defined as the *force (in dynes) with which a unit pole is acted upon* when placed at the point of the field in question. The field intensity due to m at a distance r cm. is

$$H_m = \frac{m}{\mu_a r^2}$$

and that due to m' at a like distance is

$$H_{m'} = \frac{m'}{\mu_a r^2}$$

so that

$$f = m'H_m = mH_{m'} \tag{9}$$

In air or other nonmagnetic substances in which μ_a is substantially equal to $\mu_0 = 1$, the *numerical* value of the field intensity at a distance r cm. from a point pole m is given by $H = m/r^2$; but in magnetic materials where $\mu_a > 1$ the field intensity has the smaller value $m/\mu_a r^2$.

The c.g.s. unit in which magnetic field intensity is expressed is called the *oersted.** A magnetic field has an intensity of 1 oersted if a unit magnetic point pole placed therein experiences a force of 1 dyne.

If a magnetic field is so distributed that a test (point) pole is everywhere acted upon by the same force in the same direction, the field is said to be a *uniform field.* It is possible to have magnetic fields in which the intensity is uniform within definite boundaries but in which the direction varies from point to point. Such fields have *uniform intensity.* In general, however, the intensity of the field will vary both in magnitude and direction from point to point, the field then being nonuniform.

7. Lines of Magnetic Force.—If a unit magnet pole were to be moved about in a magnetic field, the force acting upon it would in general vary both in magnitude and in direction from point to point. At each point in the field the force can be represented by a vector of which the length is proportional to the magnitude

* This name was adopted at the 1930 meeting of the I.E.C. Prior to that time the oersted had been used in the United States as the unit of magnetic reluctance (see Chap. III) and is referred to as such in works published before 1930.

of the force and the direction coincides with that of the force. If curves are then drawn in such manner that the tangent at any point is in the direction of the force at that point, the curves are *lines of magnetic force.*

This definition of lines of magnetic force is similar in all respects to that of lines of electric force in Art. 4, Chap. I, and all the deductions there presented apply equally here. That is, lines of magnetic force cannot intersect, and a bundle of such lines, within the boundaries of a given homogeneous medium, everywhere includes the same total number of lines no matter how the bundle may converge or diverge.

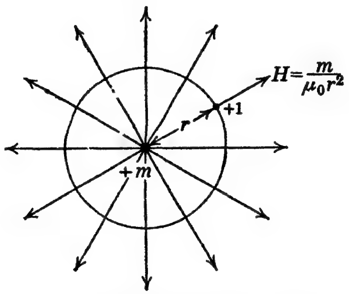


FIG. 3.—Magnetic field intensity surrounding point pole m.

The definition of a line of magnetic force implies that there exists an infinite number of them in any given field, but it is customary to replace this infinite set by a finite number in accordance with the same convention described in Art. 4, Chap. I; that is, if the intensity of the field at a given point is H oersteds, this fact is symbolized by assuming that there are H conventional lines of force per square centimeter of cross-section taken at right angles to the direction of the field.

For example, a point pole of strength m units, isolated in free space, is surrounded by a magnetic field whose lines of force radiate outward from m in all directions. At any distance r cm. from m the field intensity is $H = m/\mu_0 r^2$, as in Fig. 3. The field intensity therefore has the same magnitude at all points of the spherical surface of radius r centered at m, though its direction changes from point to point of the sphere. In terms of the c.g.s. system of units this fact is expressed by saying that $H = m/\mu_0 r^2$ lines of magnetic force cross each square centimeter of the surface of the sphere. Incidentally, it is to be noted that in this example the magnitude of the field intensity varies from point to point along any given line of force, a fact that is always true except in the special case of uniform fields.

The arrowheads on the lines of force in Fig. 3 indicate the direction in which a positive (north) pole would tend to move, and

this is said to be the positive direction (or simply the direction) of a line of force.*

8. Magnetic Flux.—Assume that a point pole of strength m units, originally isolated in free space, is enclosed within a hollow spherical shell the absolute permeability of which is $\mu_a(>\mu_0)$ (Fig. 4). Because of the symmetry of the system, the lines of

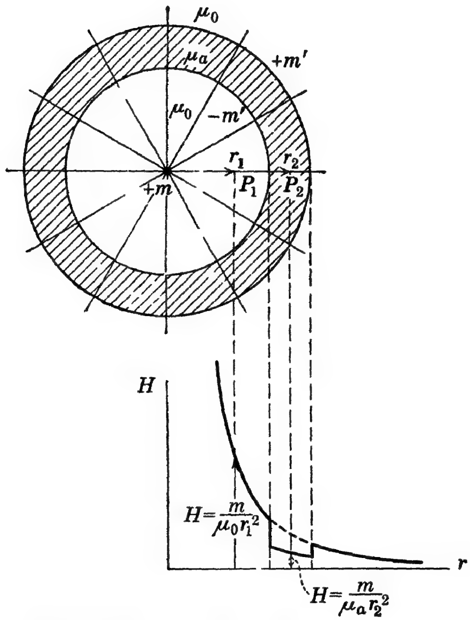


FIG. 4.—Discontinuities in field intensity.

force emanating from m are radial, exactly as in Fig. 3; but whereas at points such as P_1, inside the shell cavity, the field intensity is given by $H = m/\mu_0 r_1^2$, at points such as P_2, inside the material of the shell itself, the field intensity is given by $H = m/\mu_a r_2^2$. The variation of H along a radius is shown by the full line in the lower part of Fig. 4, discontinuities appearing at the shell boundaries. Outside the shell, the original function $H = m/\mu_0 r^2$ again prevails.

From the physical point of view, these conditions may be accounted for by considering that the paramagnetic shell ($\mu_a >$

* Lines of force have no objective existence; they are purely subjective ideas, created mentally because of their convenience.

μ_0) becomes magnetized by induction, its inner and outer surfaces being equivalent to uniformly distributed spherical poles of strengths $-m'$ and $+m'$ units. The field intensity *inside* such a spherical pole is zero, whereas at points *outside* it the field intensity is the same as though the entire distributed pole were concentrated at its center.*

* The proof of these statements is as follows:

Let P, Fig. A, be any point inside the spherical pole of strength m' units, the radius of the sphere being R cm. The pole strength per unit area of the sphere is $\sigma = m'/4\pi R^2$. Through P as a vertex pass a cone of two sheets subtending the infinitesimal solid angle $d\omega$ at P and intersecting the sphere in the elementary areas dA_1 and dA_2, r_1 and r_2 cm. from P. The field intensity (that is, the force on a unit magnet pole) at P is $\sigma dA_1/\mu_0 r_1^2$ due to one elementary pole, and $\sigma dA_2/\mu_0 r_2^2$ due to the other, and these forces are oppositely directed at point P. But since, from the geometry of the figure, $dA_1/r_1^2 = dA_2/r_2^2$, the resultant field intensity at P due to the two elements is zero. The same conclusion holds for all elementary cones with vertices at P. Hence, the resultant field intensity at P due to the entire sphere is zero; and since P was *any* point inside the sphere, the field intensity is zero at *all* points within the sphere.

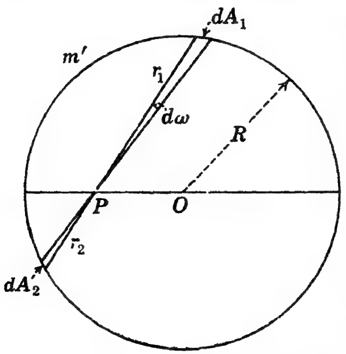


FIG. A.—Field intensity inside a spherically distributed pole.

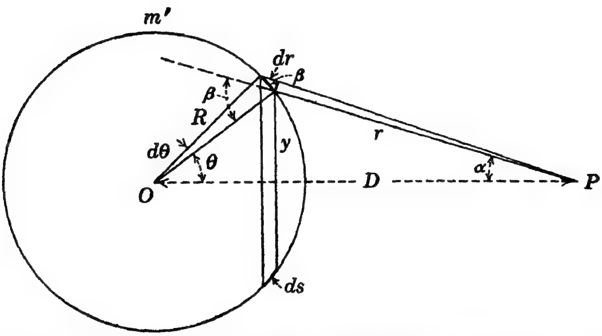


FIG. B.—Field intensity outside a spherically distributed pole.

At points external to the sphere, such as P, Fig. B, consider the effect of the annular element that subtends the angles θ and $\theta + d\theta$ at the center of the sphere. The area of the annulus is $2\pi y\, ds$, and the pole strength of the annulus is $2\pi\sigma y\, ds$. The field intensity it produces at P is

Thus, within the cavity inside the shell of Fig. 4, the field intensity is due to m alone; on passing the inner boundary of the shell, the resultant field intensity is zero so far as $+m'$ is concerned, and the joint effect of m and $-m'$ is the same as though there were a point pole of strength $(m - m')$ at the center. Beyond the outer boundary of the shell the effects of the distributed poles $+m'$ and $-m'$ are the same as though both were at the center where they neutralize each other, the resultant effect being due to m alone.

At points like P_2, Fig. 4, within the material of the shell, the field intensity is given by $H = m/\mu_a r^2$; but since it has now been shown that at such points the effect is the same as though a pole

$$dH = \frac{2\pi\sigma y\, ds}{\mu_0 r^2} \cos \alpha$$

But, from the geometry of Fig. B,

$$y = r \sin \alpha$$
$$dr = ds \sin \beta$$

whence

$$y\, ds = r\, dr \frac{\sin \alpha}{\sin \beta}$$

and since

$$\frac{\sin \alpha}{\sin \beta} = \frac{R}{D}$$

it follows that

$$yds = \frac{R}{D} r\, dr$$

Also, from the figure,

$$\cos \alpha = \frac{r^2 + D^2 - R^2}{2rD}$$

whence

$$dH = \frac{\pi\sigma R}{\mu_0 D^2}\left(1 + \frac{D^2 - R^2}{r^2}\right) dr$$

Integrating this expression with respect to r between the limits $r = D - R$ and $r = D + R$, we have the result

$$H = \frac{4\pi\sigma R^2}{\mu_0 D^2} = \frac{m'}{\mu_0 D^2}$$

or the same as though pole m' were concentrated at the center of the sphere.

of strength $(m - m')$ were concentrated at the center, in free space, it follows that

$$\frac{m}{\mu_a r_2^2} = \frac{m - m'}{\mu_0 r_2^2}$$

or

$$m' = m\frac{\mu_a - \mu_0}{\mu_a}$$

a result that establishes the relation between the magnitudes of the induced and inducing poles at the surface of separation between media of permeabilities μ_a and μ_0.

The conclusion to be drawn from this analysis is that within the mass of the shell of permeability μ_a (where $\mu_a > \mu_0$) there are fewer lines of force, in the conventional sense, than would have been considered to exist in the same region had μ_a been equal to μ_0. The reduction in the value of H that occurs inside the paramagnetic medium may be considered to be due to the demagnetizing effect of the induced poles at the boundary surface. The discontinuities indicated in Fig. 4 arise because of the difference between the permeabilities μ_a and μ_0. It is therefore apparent that, if, in expressions like $H = m/\mu_0 r^2$ and $H = m/\mu_a r^2$, attention is concentrated on the function m/r^2, it is possible to deal with quantities that are *independent of the medium*. Inasmuch as $\mu_0 = 1$ in free space, the expression m/r^2 is *numerically equal* to the field intensity at a distance r cm. from a point pole m in free space. But since μ_0 must in general be assumed to have dimensions that distinguish it from a mere abstract number, it is plain that the quantity m/r^2 is intrinsically different in its nature from H; hence, it is distinguished from H by assigning to m/r^2 the symbol B. In other words, in terms of Figs. 3 and 4, if the field intensity in free space is

$$H = \frac{m}{\mu_0 r^2}$$

the quantity B is given by

$$B = \frac{m}{r^2} = \mu_0 H \tag{10}$$

Now H, expressed in oersteds, is the force in dynes experienced by a unit point pole, and this fact is conventionally indicated by

saying that the field intensity is represented by H lines of force per square centimeter. In the same way, since $\mu_0 = 1$ in free space, B can likewise be thought of in terms of lines per square centimeter; but because of the distinction between the natures of H and B these lines are called *lines of induction*, and their number per square centimeter, represented by B, is called the *flux density*. Unit flux density, by which is implied one line of induction per square centimeter, is called the *gauss*. Equation (10), which relates to conditions in free space, may be written in the form

$$B \text{ (in gausses)} = \mu_0 \times H \text{ (in oersteds)}$$

The conclusion may be drawn that in free space, where $\mu_0 = 1$, the number of lines of force per unit area at any point in the field is the same as the corresponding number of lines of induction per unit area; or any line of induction is at the same time a line of force. In air or other nonmagnetic substance, the absolute permeability of which is very nearly equal to unity, the same thing is true so far as practical results are concerned; but in the interior of magnetic substances, the number of lines of force per unit area is materially less than the number of lines of induction per unit area at the same point, as may be seen from Fig. 4.

With reference to either Fig. 3 or Fig. 4, the flux density at any distance r cm. from a pole m is $B = m/r^2$ and this is interpreted as denoting a flux density of B lines of induction per square centimeter of a spherical surface of radius r centered at m. Since such a sphere has an area $4\pi r^2$ sq. cm., the *total flux* of induction issuing from m is

$$\Phi = \frac{m}{r^2} \times 4\pi r^2 = 4\pi m \tag{11}$$

This result is independent of r, indicating that the same flux crosses all spheres centered at m. Flux is expressed in terms of a unit called the *maxwell;* 1 maxwell denotes a total flux of one line of induction, and Eq. (11) is therefore to be read

$$\Phi = 4\pi m \quad \text{maxwells}$$

It was proved originally by C. F. Gauss that the total flux of induction from a pole m is independent of the size or shape of the enclosing surface crossed by the flux. Thus, in Fig. 5, the flux density at the elementary area dA is m/r^2, and the compo-

nent of flux normal to the elementary surface is $(m/r^2)dA \cos \theta$. The total flux crossing the entire surface is

$$\Phi = \int \frac{m}{r^2} dA \cos \theta = \int B \, dA \cos \theta$$

the integration being extended over the entire surface. Since in this expression the part $dA \cos \theta / r^2$ is by definition the solid angle $d\omega$ subtended at pole m by the element dA, it follows that

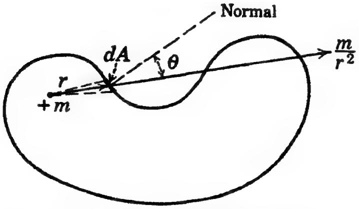


FIG. 5.—Flux crossing general enclosing surface.

$$\Phi = \int m \, d\omega = 4\pi m$$

which, though the same as Eq. (11), has been arrived at in a perfectly general manner.

In the case of the irregularly shaped surface of Fig. 5, any line of induction issuing from m must cross the surface an odd number of times, so that except for the final emergence the other intersections occur in pairs, one outwardly and the other inwardly, which thus annul each other.

The result expressed by Eq. (11) is known as *Gauss's theorem*. The coefficient 4π that appears in Eq. (11) arises from the geometrical properties of space. It appears in many of the formulas that are based upon the system of units here under consideration, namely, the c.g.s. system. The incommensurable nature of π has given rise to an effort to so modify the definitions of the fundamental units that π and 4π will not appear in those formulas which are most commonly used. This procedure is usually referred to as rationalization of the units, and units thus modified are called rationalized units. For example, by appropriately selecting the unit in which pole strength is measured, it would be possible to make $\Phi = m$, but in that case the definition of the unit pole would not be the same as that originally given. This entire subject of units is considered more at length in Chap. V; but it may be said at this point that, though 4π can be made to disappear from some formulas by suitably defining the units, it will be bound to reappear in others, since the properties of space cannot be denied their appropriate expression.

9. Magnetic Field Due to Bar Magnets.—If a homogeneous, uniformly magnetized bar magnet is placed on a horizontal table

and a small compass needle is moved about in its vicinity, the compass will at each point set itself in a definite direction tangent to the line of force through the point. By marking the direction of the needle at a considerable number of points in the plane of the table, the lines of force may thereafter be sketched in, and the resulting pattern will look like Fig. 6. The map of the field may

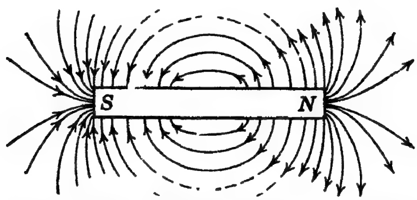


FIG. 6.—Lines of force outside a bar magnet.

also be made by covering the magnet with a thin sheet of glass, cellophane, or ordinary paper and dusting upon it sharp iron filings, each of which will align itself like a tiny compass; this simple procedure is useful in determining the distribution of the field produced by irregularly shaped magnets and by coils of wire carrying steady current.

If two or more bar magnets are placed end to end in the manner shown in Fig. 7, with like poles in contact, the resultant field will consist of a number of parts each of which will have a general

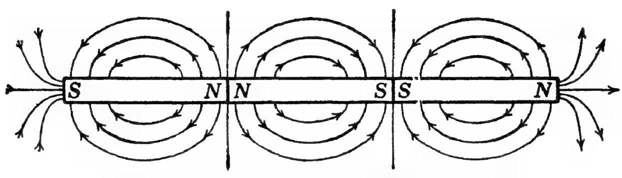


FIG. 7.—Development of consequent poles.

resemblance to Fig. 6 except that the tufting at the points of contact will have a modified configuration. The intermediate poles in Fig. 7 are called *consequent poles.* Consequent poles are occasionally found in a single bar magnet and are usually due to a lack of homogeneity in the steel or to some physical shock.

10. Lines of Force Due to Pair of Point Poles.—The determination of the distribution of the magnetic field, and of the shape of the lines of force, produced by two neighboring point poles (in air or free space), presents several features of theoretical and practical importance, particularly because the solution of

the problem is the same as that which arises in determining the electric field surrounding two point charges. For example, let it be required to map the lines of force due to two point poles of strengths m amd m', at any arbitrary distance, say p cm., apart. Two distinct cases arise, first when m and m' have unlike signs, second when they have like signs.

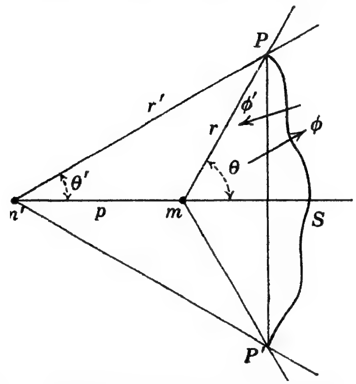


FIG. 8.—Flux relations due to two point poles.

1. *Point Poles of Unlike Sign.* Let the poles be $+m$ and $-m'$, where $|m| > |m'|$, as in Fig. 8.

Within the right cone mPP', having the semiangle θ at its vertex, there issues from pole m a flux ϕ which bears to the entire flux $4\pi m$ the same ratio that the solid angle at m, namely, $2\pi(1 - \cos\theta)$, bears to the entire solid angle 4π at m, so that $\phi = 2\pi m(1 - \cos\theta)$. In the same way, the flux due to m' that converges within the cone $m'PP$ is $\phi' = 2\pi m'(1 - \cos\theta')$. Accordingly, the net flux that crosses the plane of the circle of intersection PP' (or that crosses any surface S which has the circle PP' as its boundary) is

$$\phi - \phi' = 2\pi m(1 - \cos\theta) - 2\pi m'(1 - \cos\theta') \qquad (12)$$

If θ and θ' are altered, points P and P' will move to new positions. But if the angles are constrained to change in such a way that the net flux $\phi - \phi'$ remains constant, the locus of P (and of P') will be a line of force in the plane of the diagram; for since the flux inside this line remains constant, the line is not intersected by any lines of force and by definition must itself be a line of force. Consequently, the equation

$$2\pi m(1 - \cos\theta) - 2\pi m'(1 - \cos\theta') = \text{a constant} \qquad (13)$$

is one form of the general equation of a line of force for the case under consideration. It can be put into more convenient form by observing that when $\theta' = 0$, $\cos\theta' = 1$, and so it is possible to write

$$2\pi m(1 - \cos\theta_{\theta'=0}) = \text{a constant} = 2\pi m(1 - \cos\alpha) \qquad (14)$$

where α is the particular value of θ that corresponds to the

condition $\theta' = 0$. Substituting the value of the constant determined by Eq. (14) in Eq. (13), we have the result

$$m(1 - \cos \theta) - m'(1 - \cos \theta') = m(1 - \cos \alpha) \qquad (15)$$

and this is the general equation of a line of force in a field due to two point poles of opposite sign.

In accordance with Gauss's theorem, the total flux issuing from m is $4\pi m$, and the total flux converging upon m' is $4\pi m'$. The total flux from m may therefore be divided into two parts, one of them equal to $4\pi m'$ which finds its way to m', the other equal to $4\pi(m - m')$ which passes off to infinity; and there must be a particular line of force issuing from m that is the dividing line between these two parts of the flux $4\pi m$. In general, the part of the flux from m that does not return to m' is given by Eq. (12), and if this is made equal to $4\pi(m - m')$ the resulting relation will define the boundary or critical line. Consequently, when

$$2\pi m(1 - \cos \theta) - 2\pi m'(1 - \cos \theta') = 4\pi(m - m') = 2\pi m(1 - \cos \alpha)$$

or

$$\cos \alpha = \cos \alpha_0 = \frac{2m' - m}{m} \qquad (16)$$

the substitution of this particular value of $\cos \alpha$ in Eq. (15) will give the *equation of the critical line of force*. The result is

$$m(1 - \cos \theta) - m'(1 - \cos \theta') = 2(m - m')$$

or

$$m' \cos \theta' - m \cos \theta = m - m' \qquad (17)$$

Assume, for example, that the absolute magnitudes of m and m' are 20 and 5 (or 4 and 1); then, $\cos \alpha_0 = -\frac{1}{2}$, or $\alpha_0 = \theta_{\theta'=0} = 120°$, and therefore the critical line, shown in Fig. 9, starts out from m at an angle of 120° from the axis. Other points on this curve may be found from the relation

$$\cos \theta' - 4 \cos \theta = 3$$

obtained from Eq. (17) by inserting the given values of m and m'; by assuming various values of θ', the corresponding values of θ may be found, and the curve constructed. The point P in Fig. 9 has the property that the field intensity is there exactly

equal to zero, or, in other words, the repulsion due to m on a magnet pole at P is just balanced by the attraction due to m'; for the particular values of m and m' used in this example, the point P lies just as far to the left of m' as m lies to the right of m'. Moreover, the straight line $m'PQ$, extended indefinitely to the left, is a part of the critical line, for it satisfies Eq. (17) inasmuch as at all points on it $\theta = \pi$ and $\theta' = \pi$.

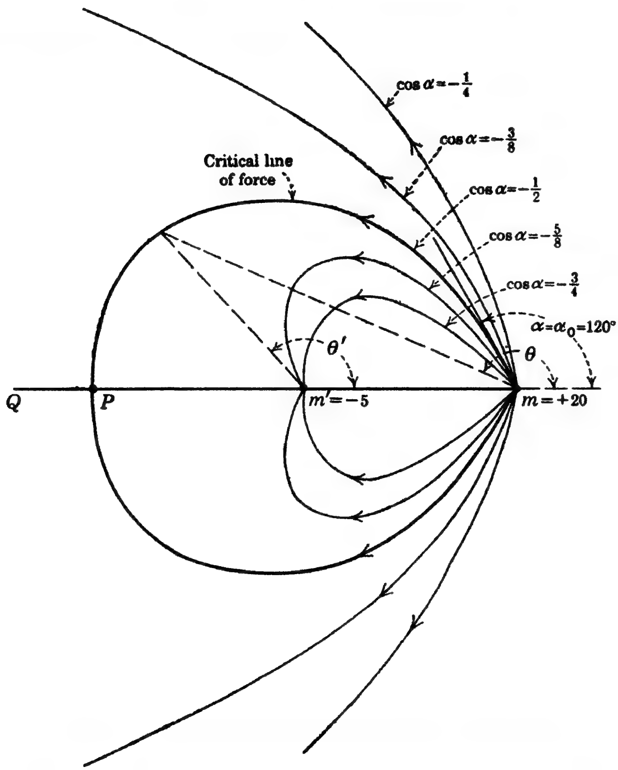


FIG. 9.—Lines of force, unequal poles of unlike sign.

The remaining curves in Fig. 9 are constructed by assuming values of $\cos\alpha$, in Eq. (15), as indicated in the diagram, and then substituting in Eq. (15). It is interesting to note that in this diagram the total flux emanating from m is exactly the same $(4\pi m)$ as would have issued from m had m been isolated in space; but whereas in that case the lines of force would all have been straight, radiating from m to infinity in all directions, they are now bent, owing to the disturbing effect of $-m'$ which draws

$4\pi m'$ of them into itself, leaving the remainder to proceed to infinity, but kinked toward m'. The attraction between m and $-m'$ may be accounted for in terms of a tension acting along the lines of force that connect them.

It is to be noted further that the distribution shown in Fig. 9 represents the lines of force only in the plane of the diagram. In all other planes that include the line mm' the distribution is the same; in other words, if the diagram is rotated about mm' as an axis, the lines of force in Fig. 9 will generate surfaces of revolution which are the boundaries of tubes of force.

2. *Point Poles of Like Sign*.—Let the poles be $+m$ and $+m'$, where $m > m'$. The difference between this case and that indicated in Fig. 8 is that now the flux crossing the plane of circle PP' is

$$\phi + \phi' = 2\pi m(1 - \cos\theta) + 2\pi m'(1 - \cos\theta') \qquad (18)$$

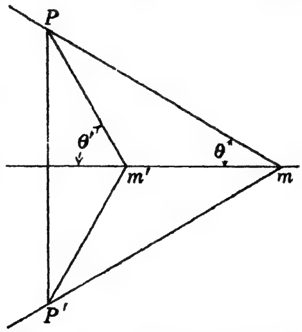


FIG. 10.—Interpretation of angles for flux from m'.

If the condition is imposed that this flux shall remain constant, the locus of P (and of P') will again be a line of force; but in the case under consideration there are obviously two sets of lines of force, one emanating from m, the other from m', as shown in Fig. 11, and there is a different equation for each set, which may be determined as follows:

a. Lines Issuing from m.—By using the diagram Fig. 8, except that $-m'$ is replaced by $+m'$, $\phi + \phi'$ in Eq. (18) can be set equal to $2\pi m(1 - \cos\alpha)$, where α is the particular value of θ that corresponds to $\theta' = 0$. Hence,

$$m(1 - \cos\theta) + m'(1 - \cos\theta') = m(1 - \cos\alpha) \qquad (19)$$

is the equation of the lines of force issuing from m.

b. Lines Issuing from m'.—It is now necessary to use the diagram Fig. 10, where the flux crossing PP' is again expressed by Eq. (18); but when $\theta = 0$, θ' will have a particular value, say α', so that

$$m(1 - \cos\theta) + m'(1 - \cos\theta') = m'(1 - \cos\alpha') \qquad (20)$$

is the equation of the lines emanating from m'.

Figure 11 is constructed by assuming $m = 20$, $m' = 5$. The sets of radiating straight lines that would issue from each pole if it were alone in space, or at an infinite distance from the other, suffer a mutual repulsion when the two poles are brought together. The distortion of the flux issuing from the weaker pole is clearly greater than that of the flux issuing from the stronger pole, and if the poles ultimately coincide there will be a single field composed of straight lines radiating from the single resultant pole $m + m'$.

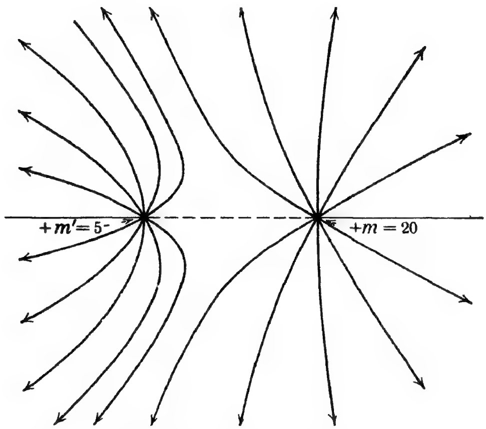


FIG. 11.—Lines of force, unequal poles of like sign.

Finally, it may be remarked that in Fig. 9 the two sets of oppositely directed lines due to m and m' tend to coalesce into a single set that draws m and m' together; whereas in Fig. 11 the two sets of similarly directed lines show a mutual repulsion, accounting for the actual repulsion between m and m'. In fact, the space between two similarly directed lines of force acts as if it were compressed, tending to push the lines apart. In the direction of a line of force there is, on the other hand, the equivalent of a tension.

11. Magnetic Potential. Difference of Magnetic Potential.—When one magnet pole, say m', is moved about in the magnetic field due to another pole m, m' experiences a force that varies in magnitude from point to point in accordance with Coulomb's law. Accordingly, either work must be done by some external agency or else the system itself will cause the motion at the expense of its stored or potential energy. The situation is similar in all

respects to the case of an electrical charge q' moving in the electrical field of another charge q. In fact, the discussion of electrical potential in Art. 5, Chap. I, applies without change to *magnetic potential*, all that is necessary being to replace the charges q and q' by the poles m and m', the word electric by the word magnetic, and the constant ϵ_0 by μ_0. Consequently the magnetic potential at a point in free space distant r cm. from a pole m is $m/\mu_0 r$, and thus $m/\mu_0 r$ ergs of work are required to bring a unit magnet pole from an infinite distance to a point r cm. from m. In the same way,

$$W = \frac{m}{\mu_0 r_1} - \frac{m}{\mu_0 r_2} \tag{21}$$

is the work required to carry a unit pole from a point distant r_2 cm. from m to another point distant r_1 cm. from m, and this work measures the *difference of magnetic potential* between the two points. In general, *the difference of magnetic potential between any two points in a magnetic field is measured by the work in ergs required to carry a unit pole from the one point to the other;* this amount of work is independent of the path followed by the unit pole, and magnetic potential is therefore a scalar quantity like electrical potential. Though both electrical and magnetic potential in the c.g.s. system are expressible in ergs, the nature of the units in which they are measured is different, for the former is *work per unit charge* whereas the latter is *work per unit magnet pole.*

12. Equipotential Lines and Surfaces.—The locus of all points in a magnetic field that have the same magnetic potential is called an *equipotential surface* Linear (or curvilinear) elements of such a surface, connecting points of equal potential, are called *equipotential lines.* No work is required to carry a magnet pole from point to point in an equipotential surface or line. It follows, therefore, that the lines of force must intersect the equipotential surfaces at right angles; for if they did not, there would be a component of force acting along the tangent to the equipotential surface at the point of intersection, and consequently work would be required to move the pole along the surface, a condition that is contrary to the definition.

13. Relation between Potential and Field Intensity.—If we look back at the derivation of the expression $m/\mu_0 r$, which is the

potential in free space at a point distant r cm. from a pole m, we shall see that it comes from the integral

$$W = \int_r^\infty \frac{m}{\mu_0 r^2}(-dr) = \frac{m}{\mu_0 r}$$

where $m/\mu_0 r^2$ is the force on a unit pole (or the field intensity) and $(-dr)$ measures an infinitesimal displacement, toward the pole m, as the unit testing pole is moved in from infinity toward m. Potential, then, is the integral of the force function $m/\mu_0 r^2$. It may therefore be anticipated that the derivative of the potential function will bring us back to the force function. Thus, if $W = m/\mu_0 r$,

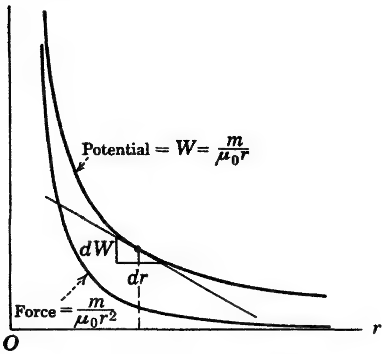


FIG. 12.—Variation of potential and force with distance.

$$\frac{dW}{dr} = -\frac{m}{\mu_0 r^2}$$

so that, although the differentiation does give the force function, there is a change of sign. The meaning of this reversal of sign will be clear from Fig. 12, which shows potential $m/\mu_0 r$ and force or field intensity $m/\mu_0 r^2$ as functions of r. The derivative dW/dr is the slope of the potential curve at the point r, in the direction of positive r, and is obviously negative when dr is in the positive direction, or the force is positive in the direction of diminishing potential.

In general, field intensity H at any point, and in any direction, is equal, with a change of sign, to the derivative of the potential at that point, taken with respect to the direction in question; that is, if the potential at a point is W, the field intensity in the direction ρ is

$$H_\rho = -\frac{\partial W}{\partial \rho} \tag{22}$$

The derivative of a function, say y, with respect to a variable parameter x means, in general, the change in y per unit change in x; it is a measure of the slope of the curve that shows the relation between y and x. Otherwise expressed, the derivative is a

measure of the *gradient* of the function under consideration. Consequently it may be concluded from Eq. (22) that *field intensity H is to be regarded as the gradient of the magnetic potential in the direction of H*, and this statement constitutes a definition of field intensity that is more fundamental than the one previously given.

The usefulness of the relation embodied in Eq. (22) arises from the fact that the potential at any point is easily computed by the simple addition of the separate potentials due to each pole (or electric charge) of a miscellaneous assembly; and the result, when differentiated with respect to any arbitrary direction, is, with a change of sign, the field intensity in that direction. The following example will help to clarify these statements.

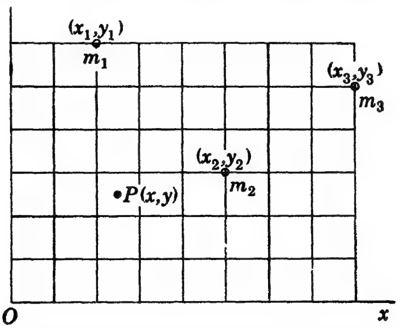


FIG. 13.—Potential and force at a point P.

Example.—Let m_1, m_2, m_3, Fig. 13, be point poles (or electric charges) of which the coordinates are (x_1, y_1), (x_2, y_2), (x_3, y_3). It is required to find the field intensity at the point P, of which the coordinates are (x, y), in the y direction.

The potential at P is

$$W_P = \frac{m_1}{\mu_0\sqrt{(x - x_1)^2 + (y - y_1)^2}} + \frac{m_2}{\mu_0\sqrt{(x - x_2)^2 + (y - y_2)^2}} + \frac{m_3}{\mu_0\sqrt{(x - x_3)^2 + (y - y_3)^2}}$$

Differentiating W_P with respect to y (considering x as constant) and changing the sign, we have, as the y component of the field intensity at P,

$$H_y = -\frac{\partial W_P}{\partial y} = \frac{m_1(y - y_1)}{\mu_0[(x - x_1)^2 + (y - y_1)^2]^{3/2}} + \frac{m_2(y - y_2)}{\mu_0[(x - x_2)^2 + (y - y_2)^2]^{3/2}} + \frac{m_3(y - y_3)}{\mu_0[(x - x_3)^2 + (y - y_3)^2]^{3/2}}$$

The same result can be obtained by finding the forces at P due to m_1, m_2, and m_3 separately and resolving each force into x and y components which can then be separately combined. In this particular type of problem the advantage of using Eq. (22) is not great, but there are numerous other types of problem that would be very difficult to handle in any other way.

This example illustrates the case of two-dimensional distribution of the poles m_1, m_2, ... The method is equally applicable to three-dimensional distributions.

14. Lines of Magnetization and Induction.—Within the substance of a magnet, conditions are not the same as in the surrounding nonmagnetic space (or vacuum). Thus, if a long, slim magnet, assumed to be uniformly magnetized, is cut transversely in the manner shown in Fig. 14, poles appear at the severed ends, whereas nowhere in the surrounding air space do

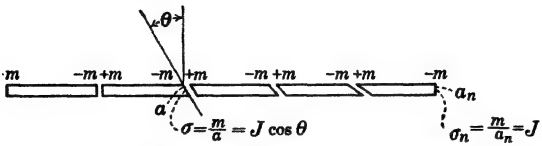


FIG. 14.—Poles at sections of bar magnet.

such free poles exist. In the uniformly magnetized bar all the intermediate poles thus created are equal in magnitude to the two original poles, but their polarities alternate in the manner indicated. *The strength of the pole per unit area* at any section will depend upon the angle θ, becoming smaller as θ approaches 90 deg., when it is zero. If the cross-section of the magnet taken normal to its axis is a_n and the pole strength at the extreme ends is m, the pole strength per unit area at a normal section is

$$\sigma_n = \frac{m}{a_n} = J \tag{23}$$

where $\sigma_n = J$ is called the *intensity of magnetization.* At any other intermediate section, where the area a is inclined to the normal section at angle θ,

$$a_n = a \cos \theta$$

and the pole strength per unit area is

$$\sigma = \frac{m}{a} = \frac{m}{a_n} \cos \theta = J \cos \theta \tag{24}$$

Figure 14 represents an idealized state of affairs in which the poles of the original magnet may be thought of as con-

centrated at the end faces and the direction of the magnetization coincides with the longitudinal axis; but one is led to the conclusion that if the magnetization is not uniform it is still possible to subdivide the entire mass into filaments so shaped that there are no poles along their longitudinal walls, but only on their ends, as in Fig. 15. These filaments or tubes may be made as slender as desired; but however small their sections may be, the bounding walls will be composed of curvilinear elements which are *lines of magnetization.* The number of such lines is infinite in a

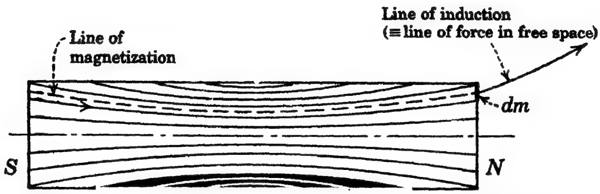


FIG. 15.—Lines of magnetization and lines of induction.

magnet of finite dimensions, but for purposes of conventional representation they are replaced by a finite number in a manner somewhat similar to that used in the case of lines of force. The number is so chosen that each of these conventionalized lines is the axis of a slender tube of magnetization which has at its north end a pole of strength $1/4\pi$ units, from which originates one line of induction. If the magnet is in a vacuum, any line of induction thus issuing from a north pole is at the same time a line of force and finds its way through the surrounding space back to the corresponding south pole on the other end of the tube of magnetization. The *line of induction* in the space *external* to the magnet may thus be thought of as forming a closed curve with the *internal line of magnetization.*

15. Relation between Field Intensity and Flux Density in Iron Core. Relative Permeability.—Suppose that a bar of paramagnetic material (Fig. 16) is introduced lengthwise into a magnetic field (in free space) of which the field intensity, before the introduction of the bar, is everywhere uniform and equal to H oersteds. This supposition is equivalent to saying that originally there are H lines of force per square centimeter of area taken perpendicular to the direction of the field; it is also equivalent to the statement that originally there are $B_0 = \mu_0 H$ lines of induction per square

centimeter of area, likewise taken at right angles to the direction of the field.

Poles of strength $+m$ and $-m$ will be induced in the iron, and a flux of induction equal to $4\pi m$ will issue from the north pole and return through the surrounding space to the south pole. Let it be assumed for the moment that the inducing field retains its original intensity H regardless of the effect of the induced poles. If

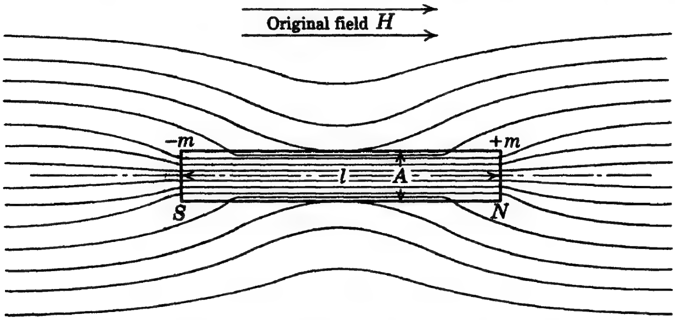


FIG. 16.—Soft-iron core in magnetic field.

the uniform cross-section of the iron bar is A sq. cm., the total flux of induction Φ across its middle section will be the sum of the original flux AB_0 and the induced flux $4\pi m$, whence

$$\Phi = AB_0 + 4\pi m = A\mu_0 H + 4\pi m \tag{25}$$

The resultant flux density across the middle section of the bar is

$$B = \frac{\Phi}{A} = \mu_0 H + 4\pi\frac{m}{A} = \mu_0 H + 4\pi J \tag{26}$$

and this new value of B is in general much greater than the original flux density B_0 that existed in the same region before the iron was introduced. The ratio B/B_0 is called the *relative permeability* μ of the material. It is given by the expression

$$\mu = \frac{B}{B_0} = \frac{B}{\mu_0 H} = 1 + 4\pi\frac{m}{\mu_0 AH} \tag{27}$$

and is clearly a dimensionless quantity. The *absolute permeability* of the magnetic substance is

$$\mu_a = \mu\mu_0 = \mu_0 + 4\pi\frac{m}{AH} \tag{28}$$

The ratio $J/H = m/AH$ is called the *magnetic susceptibility* of the material and is usually represented by the symbol κ, hence, by Eq. (27),

$$\mu = 1 + 4\pi\frac{\kappa}{\mu_0}$$

or

$$\kappa = \frac{(\mu - 1)\mu_0}{4\pi}$$

The susceptibility κ is so named because it is ultimately a measure of the ratio of m, the strength of the induced pole, to H, which is the intensity of the inducing field. There is no way by which susceptibility can be determined by a priori considerations, so that κ, and μ which depends upon it, can be found only by actual measurements.

The configuration of the lines of induction in Fig. 16 may be considered to be the resultant of two component sets of lines: one of them consists of uniformly spaced lines parallel to the lines of force of the original inducing field H; the other consists of a curved set of lines of induction similar to those indicated in Fig. 6. Many more lines of induction pass through the middle section of the iron bar than were there before the iron was introduced, the effect being as though the iron were more permeable to the lines of induction than the empty space the iron replaces. The conditions are somewhat analogous to the seepage of water flowing through a porous medium in which there is a cavity; the flow lines, parallel in the absence of the cavity, converge upon it when it is present, because the cavity presents less resistance or is more permeable than the material that was removed to form it.

The preceding discussion is based upon the assumption that the strength of the inducing field at the middle section of the bar of Fig. 16 is not affected by the induced poles that appear at the ends. This is not strictly true, for the induced poles exert a demagnetizing effect at the central section. Thus, if the length of the magnetized bar is l cm., the pole $+m$ produces a field intensity at the middle point equal to $m/\mu_0(l/2)^2$ directed toward the left, and this will be doubled by the effect of $-m$, the result being a total demagnetizing field equal to $8m/\mu_0 l^2$. In general, this demagnetizing effect is very small if l is sufficiently large. It can be eliminated entirely if the iron bar is bent to form an endless ring, provided that the magnetizing field is also disposed

so that it is everywhere along the axis of the ring core; this result can be accomplished by passing a current through a coil wound spirally around a toroidal ring, in a manner discussed more at length in Chap. III.

16. B-H Curves.—Experimental methods which make possible the simultaneous measurement of B and H show that B and

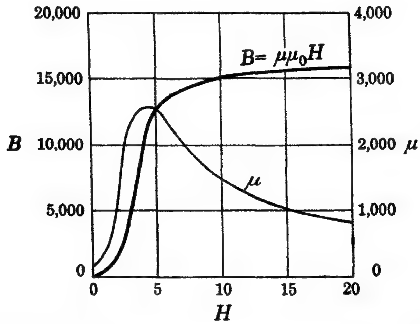


FIG. 17.—*B-H* curve and μ-*H* curve for iron wire. (*Ewing*.)

H are related in the manner shown in Fig. 17, the curve being called the *magnetization*, or *normal B-H, curve*. Different kinds of iron and steel and of steel alloyed with other elements have widely differing *B-H* curves, though all have the same general form; the curve for any given sample changes markedly when the material is subjected to various physical processes such as cold-rolling, heat-treatment, and the like.

From the measured values of B and H the corresponding values of $\mu = B/\mu_0 H$ can be computed, and μ can be plotted as a function of either H or B. There is no known relation between B and H that would enable B to be expressed mathematically as a function of H with any semblance of theoretical accuracy; an empirical relation between them, wholly arbitrary, known as Froelich's equation, sometimes serves a useful purpose;* but when accuracy is required, dependence must be placed upon the *B-H* curves determined by actual measurements made on samples of the material in question.

In most commercial grades of steel used in electrical machinery, the *B-H* curve is concave upward in the vicinity of the origin and then straightens out, though not on a line through the origin; if this nearly straight portion passed through the origin, the ratio

* See Chap. X.

$\mu_a = B/H$ would be constant, which is not the case. Thereafter, as H increases, B continues to increase, but not proportionally, with the result that the curve flattens out and, within the limits of H that are practically attainable, B appears to approach a limit. The portion of the curve where the approximate proportionality between B and H ceases to exist is called the *knee of the curve*, and the subsequent tendency of B to approach a limit is described as a condition of *saturation*.

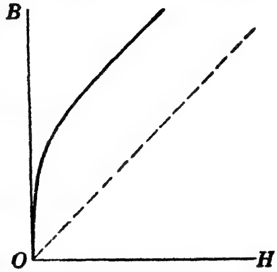


FIG. 18.—Nature of B-H curve for large values of H.

The kind of saturation indicated by the slowly rising portion of the *magnetization* (or B-H) curve is mostly a consequence of the widely different numerical scales used in plotting B and H. If it were physically possible to increase H indefinitely, it would be found that the B-H curve would take the form of Fig. 18; for when H becomes very great the intensity of magnetization J approaches a finite limit (a genuine kind of saturation thus being accounted for) and B becomes more and more nearly equal to H, the relative permeability then approaching unity as a limit. If B and H are plotted to the same scale, it would follow that for very large values of H, B approaches as an asymptote a line inclined at an angle of 45 deg. with the horizontal.

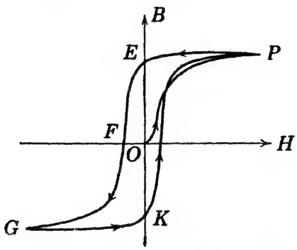


FIG. 19.—Hysteresis loop. (*Ewing.*)

Further discussion of B-H curves and the methods of determining them experimentally will be deferred to the next chapter.

17. Hysteresis.—It is found by experiment that, if an originally unmagnetized sample of iron or steel is subjected to a gradually increasing magnetizing force, the magnetization increases in the manner shown by curve OP, Fig. 19, which is similar to the curve of Fig. 17. Thereafter, when the magnetizing force H is gradually decreased to zero and then again increased in the reverse direction, the magnetization follows the course indicated by curve $PEFG$; decreasing H back again to zero and then increasing it in the original direction yields the curve GKP.

In other words, there may be two or more values of B corresponding to one and the same value of H, any particular value of B depending upon the past history of the specimen under test. The changes in B are seen to lag behind the corresponding changes in H, and for this reason this phenomenon was called *hysteresis* by Ewing who first observed it, the word hysteresis coming from a Greek word which means "to lag behind." Figure 19 is taken from Ewing's original paper,* and the diagram as a whole is called a *hysteresis loop*.

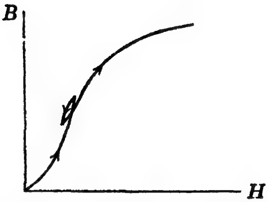


FIG. 20.—Effect of irregular variation of H.

It will be observed that in Fig. 19 the flux density remaining in the sample when H is zero is OE (or OK); this is called the *remanence* of the material. Similarly, if the remanent magnetism is to be reduced to zero, there must be applied a reversing magnetizing force represented by OF, which is called the *coercive force* or *retentivity*. In soft iron the remanence is large and the coercive force is small; moreover, the remanence is not permanent but decreases gradually with time until very little residual magnetism remains. In harder steels the remanence is less than in soft iron, and the coercive force is considerably greater. Obviously what is needed for a powerful permanent magnet is the largest remanence consistent with a large coercive force.

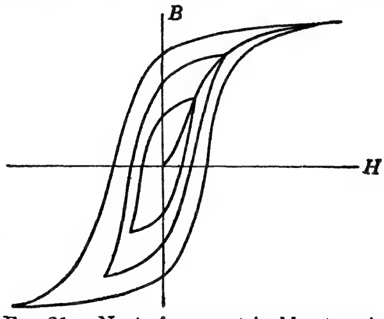


FIG. 21.—Nest of symmetrical hysteresis loops.

Experiment shows that, if a sample of iron is subjected to a magnetizing force which at first increases steadily, then decreases, and thereafter again increases, a small hysteresis loop appears in the curve which relates corresponding values of B and H, as in Fig. 20. It is therefore important that H should be varied steadily in one direction if a continuous portion of a hysteresis loop is to be determined.

* *Phil. Trans. Roy. Soc.*, 1885, p. 524.

It is found that if H is varied steadily, first in one direction and then in the other, between equal but opposite values, as in Fig. 21, the hysteresis loop is symmetrical in form; and if a nest of such loops for a particular sample is constructed by superposing the loops corresponding to various limiting values of H, the cusps of the several loops lie on the normal magnetization curve.

In numerous types of a-c machinery, conditions are such that the flux density B in the iron core of the apparatus varies with time in accordance with a sinusoidal law; that is,

$$B = B_m \sin \omega t$$

where B is the flux density at any time t, ω is a constant, and B_m is the maximum value of B that recurs cyclically whenever

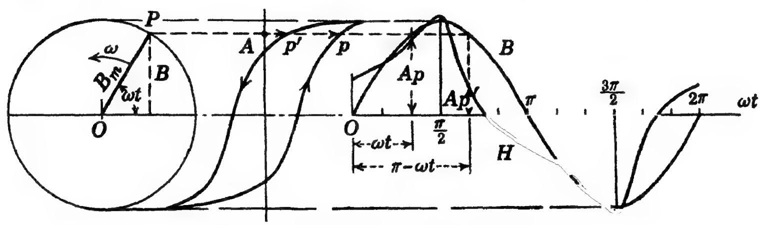


FIG. 22.—B and H as functions of time.

$\omega t = \pi/2$ or any odd multiple of $\pi/2$. The magnitude of B may be represented geometrically as the vertical projection of radius vector $OP = B_m$ (Fig. 22) which rotates counterclockwise at uniform angular velocity ω. For the particular value of ωt shown in the diagram, the value of B will be the same as when the angle is $\pi - \omega t$; but the corresponding values of H, determined by the hysteresis loop, will be Ap and Ap', the former pertaining to the ascending part of the loop, the latter to the descending part. When these values of H are plotted as in the right-hand portion of the diagram, the graph of H as a function of time is seen to differ materially from that of B, though both pass through their maximum values at the same instant.

It will be shown in the next chapter that the area of a hysteresis loop is proportional to the energy lost in carrying the magnetization through a complete cycle; this lost energy, which reappears in the form of heat, must of course be kept as small as possible in the iron or steel cores which are subjected to a cyclically varying flux. To this end, materials are selected in which the ascending and descending parts of the hysteresis loop are as

close together as possible, consistent with the condition that the increased cost of obtaining such material does not exceed the resultant saving in the cost of the energy. But, even if the hysteresis loss were reduced to zero, the curvature of the *B-H* curve due to saturation would still remain.

18. Magnetization Curves of Ferromagnetic Alloys.—Consideration of the *B-H* curves of the types of iron and steel used in the heavier electrical machines, such as generators and motors, will be deferred to a later chapter for the reason that they are most conveniently expressed in terms of units and measurements that have not been developed as yet in this discussion. There are, however, several alloys the properties of which are of special value for certain purposes, and these will now be considered.

Permalloy.—The Bell Telephone Laboratories* have developed a series of nickel-iron alloys, to which the name *permalloys* has been given, that have remarkable magnetic properties, especially in very weak magnetic fields. Permalloy containing 78.5 per cent nickel and 21.5 per cent iron, when properly heat-treated, is so highly susceptible to magnetization in a weak field that a bar of it, pointed toward the earth's magnetic pole in the direction of the dipping compass, will become magnetized sufficiently to pick up a light piece of iron; but it will promptly lose its magnetization on turning it east and west. The heat-treatment is particularly important, and close control is essential, for very small variations produce disproportionately great effects.

The particular alloy containing 78.5 per cent nickel and 21.5 per cent iron, designated as 78.5 permalloy, has magnetic properties shown in Figs. 23 and 24. The annealed alloy was heat-treated by raising its temperature to between 900° and 1000°C. in an electrical resistance furnace, holding its temperature at that value for 1 hr., and then, after turning off the furnace current, allowing it to cool in the furnace to room temperature. The air-quenched sample received the same annealing treatment and, in addition, after reheating for 15 min. in a furnace held at 600°C., was cooled quickly by withdrawing it from the furnace and placing it on a copper plate in the open air.

* G. W. ELMEN, Magnetic Alloys of Iron, Nickel, and Cobalt, *Bell System Tech. Jour.*, **8**, (No. 3), p. 435, July, 1929.

See also H. D. ARNOLD, and G. W. ELMEN, *Jour. Franklin Inst.*, May, 1923, p. 621.

The high permeability of this alloy, which reaches a maximum of about 87,000, greatly exceeded that of any material known at

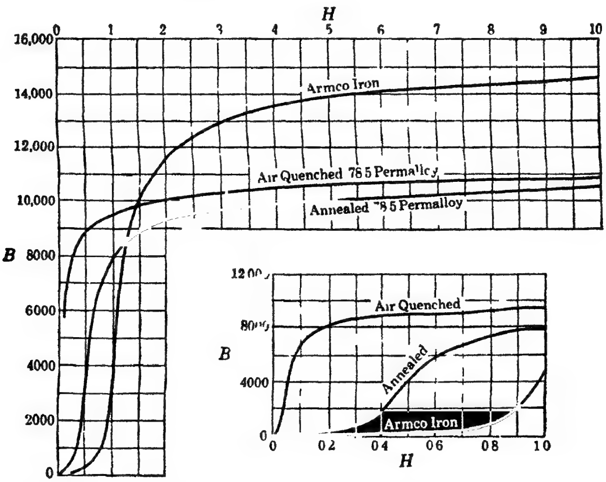


FIG. 23.—Magnetization curves for 78.5 permalloy and armco iron.

the time it was developed; but the feature that gives to this alloy its particular value is the extremely high flux density produced by very feeble magnetizing forces, as may be seen in the small inset in Fig. 23, where the B-H curves for both air-quenched and annealed permalloy are shown together with that of armco iron (which is almost pure iron). At the larger values of H that occur in heavy machinery, *the properties of armco iron are seen to be superior to those of the alloy.*

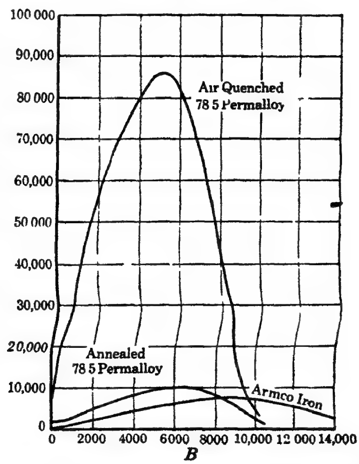


FIG. 24.—Permeability curves, 78.5 permalloy and armco iron.

Permalloy, in the form of thin strips, is used in the construction of submarine telephone cables; for land use, in loading coils for long telephone circuits, the material in the form of a fine powder is mixed with a resinous binder and compressed in the shape of toroidal rings which serve as the cores of the surrounding coils.

Recent experiments* made at the Bell Telephone Laboratories have yielded a relative permeability as high as 1,330,000. The material used was a single crystal of permalloy which contained 66 per cent nickel. This was purified by high-temperature heat-treatment in hydrogen, cut so that its permeability could be measured accurately in a chosen direction in the crystal, then again heat-treated at a high temperature and finally at a relatively low temperature in a magnetic field.

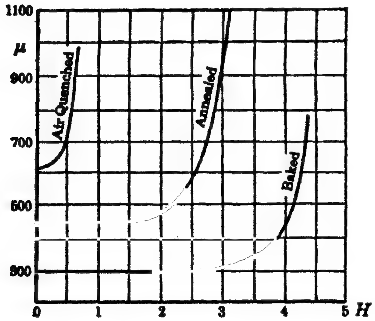


Fig. 25.—Permeability curves, 45 per cent nickel, 25 per cent cobalt, 30 per cent iron.

The *perminvar* series of alloys of iron, nickel, and cobalt is characterized by nearly constant permeability and extremely low hysteresis loss at low magnetizing forces. The perminvars have compositions varying from 10 to 40 per cent iron, 10 to

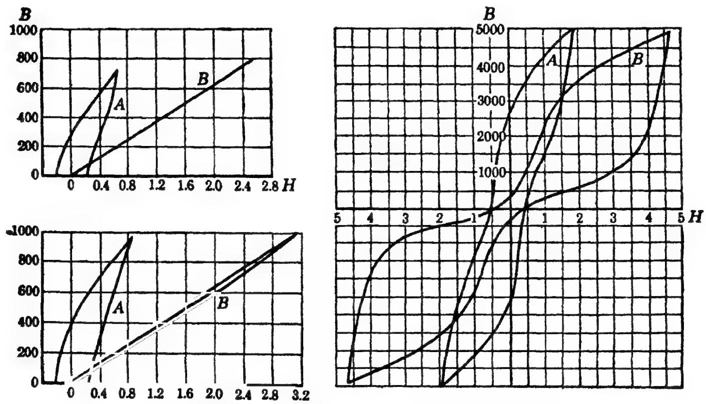


Fig. 26.—Hysteresis loops, perminvar, 45 per cent nickel, 25 per cent cobalt 30 per cent iron. A = air quenched; B = baked.

80 per cent nickel, and 10 to 80 per cent cobalt. The particular composition 45 per cent nickel, 25 per cent cobalt, and 30 per cent iron yields the permeability curves of Fig. 25 and the hysteresis loops of Fig. 26. The sample marked "baked" was first annealed and then held for a long time at 425°C.

* R. M. Bozorth, *Bell Lab. Record*, **16** (No. 7), 229, 1938.

Hypernik.—This material, developed by T. D. Yensen of the Westinghouse Research Laboratory, is an iron-nickel alloy containing 40 to 60 per cent nickel, highly refined to remove all traces of impurities, which, if present, greatly impair its characteristics. The permeability is very sensitive to heat-treatment, and values as high as 167.000 have been reached,* though ordinarily the maximum is in the neighborhood of 57,000. The hysteresis loss is very much lower than in the silicon steel usually employed in transformers, and the properties as a whole particularly adapt this material for the cores of current transformers.†

Coupernik.—The composition of this material is the same as that of hypernik, but a different heat-treatment imparts the property of constant permeability over a considerable range of flux density.

Alloys for Permanent Magnets.—Reference has been made in Art. 15 to the fact that there is a demagnetizing effect within the body of a magnet which is so shaped that poles exist at its open ends. This effect is not important in large electromagnets, but in relatively small permanent magnets it becomes significant. It would be entirely eliminated if the permanent magnet were of completely closed ring shape; but such a form is not adapted to practical use, for an airgap must be present if the flux of induction is to be available. It is because of the demagnetizing effect of exposed poles that bar magnets, for example, when not in use, should be kept in pairs, with the north and south poles of the one in contact with the south and north poles of the other; and a horseshoe magnet should have an armature, or keeper, of soft iron bridging the gap between the adjacent poles.

It has also been pointed out that the material for permanent magnets should ideally have large remanence combined with large coercive force. In general, both of these increase with increasing hardness, but, for a given composition of the metal, only within definite limits. For this reason, chilled cast iron and hardened (carbon) tool steel were originally used but both were found to be inferior to *tungsten* steel. *Cobalt-chromium* steel has somewhat smaller remanence than tungsten steel, but its retentivity is considerably greater, giving it greater stability

* *Elec. Jour.*, June, 1931, p. 386.

† See A. S. Langsdorf, "*Theory of Alternating-current Machinery,*" p. 217.

when the airgap is of considerable size. More recently* the General Electric Company has developed an iron alloy containing aluminum, nickel, and cobalt, called *alnico* which has a retentivity more than double that of cobalt-chromium steel.

19. Measurement of Pole Strength and Field Intensity.—The definitions of pole strength and of field intensity given in the preceding discussions make use of idealized conceptions of point poles that cannot be realized in actual practice. Isolated point poles of a single polarity do not exist, and the two poles of an actual magnet are so irregularly distributed that it is impossible to assign fixed points in or on the magnet as the locale of *actual* point poles separated from each other by a uniquely defined

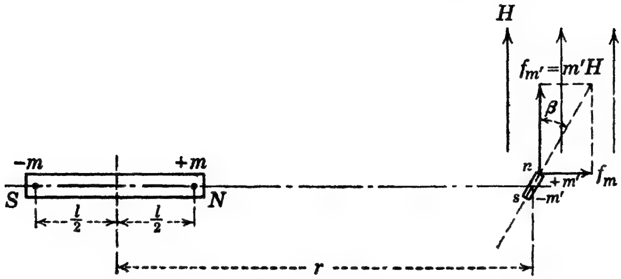


FIG. 27.—Magnetometer measurement.

distance. It is nevertheless possible to assign to a given magnet *equivalent* point poles of strength m, separated by a distance l cm., such that the *product*

$$M = ml$$

is accurately measurable, M being called the *magnetic moment* of the magnet. The meaning of this term is to be interpreted as the sum of all products of the form $\Delta m \cdot x$, where Δm represents equal elementary (point) poles of opposite sign separated by a distance x. It will be noted that, in Art. 14, Eq. (23) defines the intensity of magnetization as $J = m/a_n$, which is the same as $J = ml/a_nl$, or the *magnetic moment per unit volume* of the magnet. The method originally devised for the measurement of magnetic moment employed the following procedure:

1. Let NS (Fig. 27) represent the bar magnet the magnetic moment of which is to be determined, and let ns be a small

* R. F. EDGAR, Permanent Magnets, *Gen. Elec. Rev.*, October, 1935, p. 466.

compass needle which, in the absence of NS, points in the direction of the horizontal component of the earth's magnetic field which has, say, an intensity H oersteds. The magnitude of H is known to vary from point to point on the earth's surface, and for a given point its direction changes with time; but in the limited space occupied by the apparatus and for the limited time required to make the observations, H is uniform and constant.

The magnet NS is accurately aligned with its axis perpendicular to the lines of force of the field H and on a line through the center of the needle ns, the distance from center to center of the two magnets being r. If $m'l'$ is the magnetic moment of the compass needle, it will come to rest in a position such that its axis makes an angle β with the lines of force due to H, where

$$\tan \beta = \frac{f_m}{f_{m'}}$$

f_m being the force on m' due to m, and $f_{m'}$ the force on m' due to H. It is clear that

$$f_m = \frac{mm'}{\mu_0\left(r - \dfrac{l}{2}\right)^2} - \frac{mm'}{\mu_0\left(r + \dfrac{l}{2}\right)^2} \tag{29}$$

and, if r is sufficiently large in comparison with l, f_m becomes

$$f_m = \frac{2m'ml}{\mu_0 r^3} = \frac{2m'M}{\mu_0 r^3}$$

It is also seen that $f_{m'} = m'H$; hence,

$$\tan \beta = \frac{2M}{\mu_0 H r^3}$$

or

$$\frac{M}{H} = \frac{\mu_0}{2} r^3 \tan \beta \tag{30}$$

This establishes a relation between M and H, but since both are unknown an additional independent equation between them must be established to permit their separate evaluation.

2. Let the magnet NS be suspended in a supporting stirrup from a fiber that has such small cross-section that the torque required to twist it is negligibly small; magnet NS, thus suspended, will be in static equilibrium when its axis lies in the

magnetic meridian through the point of support, and if it is angularly displaced from this position it will oscillate to and fro as a torsional pendulum with a period that can readily be measured.

Consider the system composed of the magnet and its supporting stirrup when it is in the instantaneous position shown in Fig. 28, swinging, say counterclockwise. Both the north pole and the south pole of the magnet are acted upon by forces equal to mH, and the total moment of these forces about the axis of suspension is $mH \cdot l \sin \theta = MH \sin \theta$.

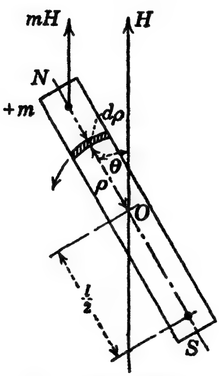


FIG. 28.—Magnet as torsional pendulum.

The only other force acting upon the system, the torsion of the suspending fiber and the retarding effect of air friction being ignored, is due to the inertia of the material in the oscillating system. To evaluate this force, consider an element of volume dV, all parts of which are at a distance ρ from the axis. If the density of the material is δ, the mass of the element is $\delta\, dV$. In the position indicated, the angle θ will change by an amount $d\theta$ in the time dt; so the distance traversed by the element in time dt is $\rho\, d\theta$, and its linear velocity is $v = \rho(d\theta/dt)$. Accordingly, the linear acceleration of the element is $a = dv/dt = \rho(d^2\theta/dt^2)$, and by Newton's law (force = mass × acceleration) the force acting on the element is $(\delta\, dV)\rho(d^2\theta/dt^2)$. As the direction of this force is perpendicular to ρ, the moment of the force about O is $(\delta\, dV \rho^2)(d^2\theta/dt^2)$, and for the entire system the moment is

$$\frac{d^2\theta}{dt^2}\int \rho^2\delta\, dV$$

the integral being taken within limits that will include the entire suspended mass; but the integral

$$\int \rho^2\delta\, dV = J$$

is known as the polar moment of inertia;* hence, the moment of the entire inertial force is $J(d^2\theta/dt^2)$.

* The symbol J is used also to denote intensity of magnetization.

For equilibrium, the sum of the two moments must be zero; that is,

$$J\frac{d^2\theta}{dt^2} + MH \sin \theta = 0 \tag{31}$$

In the actual use of such an oscillating system, the angle θ is kept sufficiently small to warrant the approximation $\sin \theta = \theta$, whence Eq. (31) can be replaced by

$$J\frac{d^2\theta}{dt^2} + MH\theta = 0 \tag{32}$$

which is a simple type of differential equation. of which the integral is in general

$$\theta = C \sin \left(\sqrt{\frac{MH}{J}}t + \alpha\right) \tag{33}$$

where C and α are constants defined by the physical conditions of the problem. In this case, if we assume that time is counted from the moment when the magnet passes through its mid-position, we shall have $\theta = 0$ when $t = 0$, in which case $\alpha = 0$, so that

$$\theta = C \sin \sqrt{\frac{MH}{J}}t \tag{34}$$

When in this equation the quantity $\sin\sqrt{(MH/J)}t$ is unity (its maximum value), $\theta = C$ is likewise a maximum; thus the magnet is at the end of its swing, and the time t is then one-fourth of the time of a complete period, or $T/4$. Therefore,

$$\sqrt{\frac{MH}{J}}\frac{T}{4} = \frac{\pi}{2}$$

or

$$T = 2\pi\sqrt{\frac{J}{MH}} \tag{35}$$

The period T can easily be measured by timing the swings of the magnet, and J can be computed from the mass and dimensions of the magnet; hence,

$$MH = \frac{4\pi^2 J}{T^2} \tag{36}$$

and this relation, together with Eq. (30), provides two equations from which M and H can be computed.

It may be that the forms of the magnet and of its supporting stirrup are too complicated to permit the simple calculation of their combined moment of inertia J; in that case it is possible to attach to the magnet two equal and symmetrically placed masses of such simple shape that their moment of inertia J_1 can be computed with precision. Their effect will be to change the time of a complete period from T to T_1, hence, in accordance with Eq. (36),

$$MH = \frac{4\pi^2(J + J_1)}{T_1^2} \tag{37}$$

If the interval between the measurements of T and T_1 is sufficiently small to warrant the assumption that M and H have not changed in the meantime, it is possible to equate (36) and (37), and then

$$J = \frac{T^2}{T_1^2 - T^2} J_1 \tag{38}$$

This method, which as a whole is known as the *magnetometer method*, is extensively used to measure the horizontal component of the earth's magnetic field; accompanied by information concerning the declination of the compass (the angular departure between H and the true north) and the angle of dip of the earth's field, the data are recorded on maps which are of great importance in navigation.

20. Refraction of Lines of Induction.—It has been established in Art. 14 that lines of induction are closed curves, the number of them within the substance of a magnet being equal to the number outside its boundary surface. It is conceivable that the magnet may be immersed in two or more concentric layers of materials, one or more of which may be nonmagnetic like air, the others more or less magnetic; but in any case, at an element of area dA in the boundary surface between two different media, the number of lines of induction that approach the element from one side must be equal to the number of lines that leave the other side. Thus, in Fig. 29, let it be assumed that the induction which has flux density B_1 on the left of element dA has flux density B_2 on the right of this element and that the directions of B_1 and B_2 are

inclined at angles θ_1 and θ_2 to the normal NN' at element dA. Continuity of the flux crossing the element requires that

$$B_1 \cos \theta_1 \, dA = B_2 \cos \theta_2 \, dA$$

or

$$B_1 \cos \theta_1 = B_2 \cos \theta_2 \tag{39}$$

If it is assumed for the moment that the shaded portion to the left of the boundary in Fig. 29 represents magnetic material like a solution of chloride of iron and that the unshaded part to the right of the boundary represents air, the element dA will be the equivalent of a point pole $\sigma \, dA$, from which *lines of force* originate if it is a positive or north pole or on which they terminate if it is a south pole; but the number of such lines of force per unit area within the chloride of iron (and therefore the field intensity H_1) is less than in the outside air where $H_2 = B_2/(\mu_0 = 1)$.

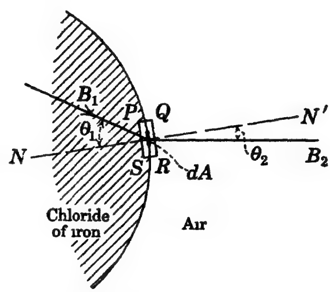


FIG. 29.—Refraction of lines of induction.

In general, therefore, when the two materials have different permeabilities, the field intensities on the two sides of the bounding surface do *not* have equal *normal* components. It is, however, true that the *tangential* components of the field intensities H_1 and H_2 are equal, as may be proved in the following manner.

Let $PQRS$ be an exceedingly small rectangular path surrounding the element dA, the side PS being just inside the magnetic material and the side QR being in the air just outside the boundary. The sides PQ and RS which cross the boundary surface are parallel to the normal NN' and are of infinitesimal length. Suppose, now, that a minute point pole dm is carried once around the closed path $PQRS$, starting at any point such as P, so that the magnetic potential is the same at the initial and terminal points of the path followed. The total amount of work done in making the complete circuit is therefore zero.

Since the rectangle $PQRS$ may be made as small as we please, the force on the moving test pole dm along the sides PQ and RS will be due to the elementary pole $\sigma \, dA$, and the force will be $(2\pi\sigma \cdot dm)/\mu_0$ in the air and $(2\pi\sigma \cdot dm)/\mu_a$ in the magnetic material; but the travel along PQ and RS takes place in opposite

directions, and so the net amount of work in traversing PQ and RS is zero. It follows that the work done in traversing QR and SP must also be zero, which can be true only if the tangential components of the field intensities along these two sides of the rectangle are equal. Consequently,

$$H_1 \sin \theta_1 = H_2 \sin \theta_2 \tag{40}$$

Dividing Eq. (39) by Eq. (40),

$$\frac{B_1/H_1}{\tan \theta_1} = \frac{B_2/H_2}{\tan \theta_2}$$

or

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\mu_1}{\mu_2} \tag{41}$$

where $\mu_1 = B_1/\mu_0 H_1$ and $\mu_2 = B_2/\mu_0 H_2$ are the relative permeabilities of the two media.

Equation (41) defines what may be called the law of refraction for lines of magnetic induction when they pass from one medium to another, or from layer to layer of the same medium in which the permeability is not constant throughout. It is seen that the lines are bent toward the normal on passing from a region where the permeability is large to another where it is small. For example, if lines of induction emerge from iron or steel having a relative permeability of 2000 (or more) into an airgap, Eq. (41) shows that

$$\tan \theta_{air} = \tfrac{1}{2000} \tan \theta_{iron}$$

so that even if θ_{iron} is as large as 89 deg., $\tan \theta_{air} = 57.29/2000 = 0.02865$, and $\theta_{air} = 1°38'$. Thus, lines of induction (or lines of force) entering or leaving an iron pole face that is surrounded by air or other nonmagnetic material are practically perpendicular to the boundary curve. This fact is very useful in mapping the magnetic field in the vicinity of irregularly shaped permanent magnets and electromagnets, particularly when it is applied in connection with the fundamental principle that *a magnetic field distributes itself in such a manner that for a given set of magnetizing forces the resultant magnetic flux has maximum value.*

CHAPTER III

ELECTROMAGNETISM

1. Ampère's Rule.—The fact that there is an interrelation between electricity and magnetism was first discovered by Oersted, who in 1819 observed that a compass needle placed just below a current-carrying wire was deflected from its normal position. The article in which this discovery was announced, in July, 1820, also disclosed the reciprocal fact that a movable current-carrying conductor is deflected by a neighboring fixed magnet. When this publication came to his attention, Ampère carried the experiments much further and before the end of the year 1820 had found that there is an attraction between parallel currents flowing in the same direction and a repulsion when the currents are oppositely directed; also that a current flowing in a helix acts exactly as if it were a bar magnet. In the same period Ampère formulated a definite rule fixing the direction of deflection of a compass needle in terms of the direction of the deflecting current; since the north pole of the compass moves in the direction of the magnetic field produced by the current, the rule fixes the direction of the magnetic field set up by a given current.

The form in which the rule was first stated is as follows: *Ampère's rule:* If an observer imagines himself in the electric circuit, with the current flow in the direction from his feet to his head, then on facing the compass needle the north pole will be deflected toward his left.

The awkwardness of the original rule has led to simpler, but equivalent, rules, one of which is known as the *right-hand screw rule* or the *corkscrew rule:* The direction of the lines of magnetic force surrounding a current bears the same relation to the direction of the current as the direction of rotation of a right-hand screw bears to the direction of advance of its point.

Another rule, much used because of its simplicity, is the *hand rule:* If the conductor is held in the *right* hand with the extended thumb pointing in the direction of the current, the fingers will encircle the wire in the direction of the magnetic field.

These rules read upon Figs. 1 and 2, where the directions of the lines of force and of the currents are indicated. In the case of Fig. 2, which represents a helix, or solenoid, having an air core, the distribution of the lines of force in a plane passing through the axis will have the general form indicated in the sketch, as may be proved experimentally by sprinkling iron filings on a sheet of paper or glass through which the turns of the solenoid are wound; there is a similar distribution of lines of force in every other plane through the axis of the coil. The similarity between Fig. 2

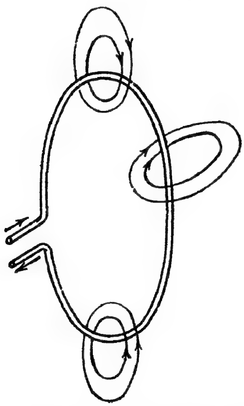


Fig. 1.—Lines of magnetic force due to current in a single turn.

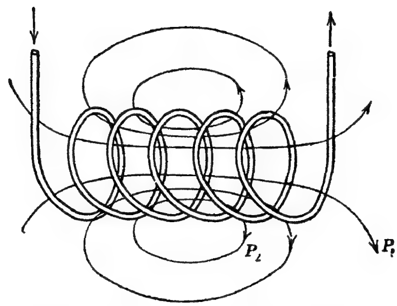


Fig. 2.—Lines of magnetic force due to current in a solenoid

and Fig. 6, Chap. II, is evident. The lines of force in Fig. 2 are also lines of induction and hence are closed curves or loops which thread through one or more turns of the coil in which the current is flowing. The closed lines of induction and the loop of wire bear to each other somewhat the same relation as do the successive links of a chain, as is clearly seen in Fig. 1, and it is because of this fact that the flux is said to link with the coil.

Figure 3 represents a cross-section of a pair of parallel wires, both carrying current flowing out of the plane of the paper, as indicated by the central dots, which are the conventional representations of the points of arrows coming toward the observer. Figure 4 shows similarly a pair of wires carrying current in opposite directions, the symbol for a current flowing into the plane of the drawing being a cross which symbolizes the feathered end of a retreating arrow. In each figure the lines of force are

indicated without any attempt at mathematical accuracy. The oppositely directed sets of lines shown in Fig. 3 tend to coalesce into a single system, in somewhat the same manner as do the separate fields of m and m', in Fig. 9, Chap. II, the attrac-

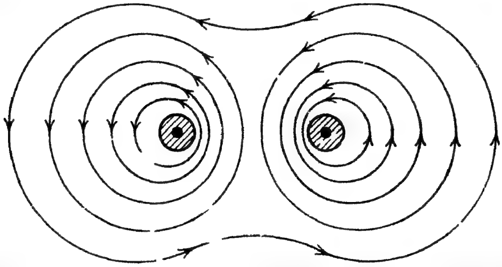


FIG. 3.—Distribution of lines of force parallel currents in the same direction.

tion between the wires being thus accounted for; whereas in Fig. 4 the similarly directed sets of lines exert a mutual repulsion in much the same way as in Fig. 11, Chap. II, the repulsion between conductors carrying oppositely directed currents being thus accounted for.

2. Ampère's Law. Absolute Electromagnetic Unit of Current.—The initial experiments that led to the formulation of the

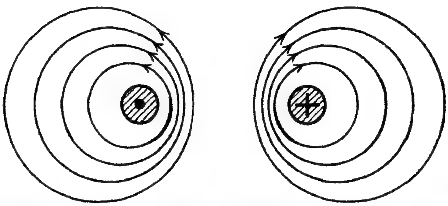


FIG. 4.—Distribution of lines of force—parallel currents in opposite directions.

rule given in the preceding article enabled Ampère in 1822 to state the fundamental law for the interaction between a current and a magnet and between two current-carrying conductors. In its simplest form the law states that, if a current of $\bar{I}$ units flows in a wire of length dl, it will exert a force upon a point pole m, located on a line perpendicular to dl, and distant r from dl, (Fig. 5), of

$$df = \frac{\bar{I}\,dl\,m}{r^2} \tag{1}$$

and this force is independent of the surrounding medium.

The force on the pole m is perpendicular to the plane through dl and m and in the direction required by Ampère's rule; thus in Fig. 5, with the current flowing vertically upward, the pole $+m$ will tend to move into the plane of the paper; conversely, if m is fixed in position, the element of wire will tend to move out of the plane of the paper.

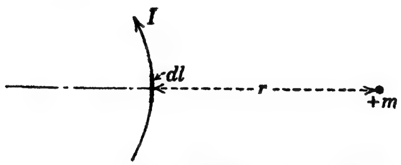


FIG. 5.—Current element and magnet pole, Ampère's law.

It follows from Eq. (1) that if the element dl is a portion of a circle of radius r, with the pole m at the center of the circle, every similar element of the circle will exert the same force, in the same direction, upon m. The total force upon m due to the current in the entire circle will be, therefore,

$$f = \frac{\bar{I}m}{r^2}\int_0^{2\pi r} dl = \frac{2\pi m\bar{I}}{r} \qquad (2)$$

The unit in which the force is expressed will depend upon the units used to measure m, $\bar{I}$, and r; but if, as in the c.g.s. system, force is measured in dynes, distance in centimeters, and the strength of a magnet pole in terms of the unit already defined, the unit of current in which $\bar{I}$ must be expressed becomes definitely fixed. This unit of current has been given the name *abampere* (a contraction of absolute ampere); and since its magnitude depends upon electromagnetic force, it is also referred to as the absolute *electromagnetic unit of current* (e.m.u.).

Thus, if the radius of the circle is 1 cm. and the point pole at its center has the magnitude $m = 1$, Eq. (2) states that the force will be 2π dynes if $\bar{I} = 1$. From these relations the abampere has the following definition:

The absolute e.m. unit of current, or the abampere, is a current of such strength that if it flows in a circular loop of 1 *cm. radius, it will exert a force of* 2π *dynes upon a unit point pole placed at the center of the circle.*

Inasmuch as equal elements of the circle contribute equally to the resultant force, as shown by Eq. (2), it follows that an arc 1 cm. long, subtending at the center an angle of 1 radian, as in Fig. 6, will exert a force of 1 dyne upon the unit pole at the center of curvature; hence, an alternative definition of the abampere is as follows:

The abampere is a current of such strength that if it flows in a wire 1 cm. long, bent into a circular arc of 1 cm. radius, it will exert a force of 1 dyne upon a unit point pole placed at the center of curvature.

The importance of Ampère's law is that it provided for the first time a quantitative basis for the direct measurement of current strength in terms of the force upon magnets and thereby laid the foundation for the *absolute electromagnetic system of units* (the e.m. system).

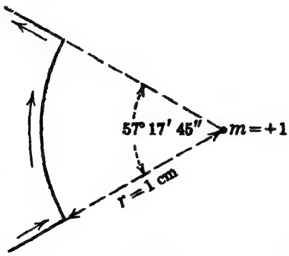


FIG. 6.—Definition of abampere.

3. Electromagnetic Unit of Quantity. The absolute e.m. unit of current, or the abampere, having been thus defined, the corresponding unit of quantity, called the *abcoulomb*, follows as a matter of course. *The abcoulomb is the quantity of electricity that will pass a given cross-section of a conductor in* 1 *sec. if the conductor carries an unvarying current of* 1 *abamp.*

For reasons that will appear later (see Chap. V), the abcoulomb is 3×10^{10} times as large as the statcoulomb defined in Art. 2, Chap. I, the factor 3×10^{10} being equal to the velocity of light (in centimeters per second). Since 3×10^9 statcoulombs have been taken as the equivalent of 1 coulomb, it follows that

$$1 \text{ abcoulomb} \equiv 10 \text{ coulombs}$$
$$1 \text{ abamp.} \equiv 10 \text{ amp.}$$

The symbol $\bar{I}$ has been used in Eqs. (1) and (2) to distinguish current in abamperes from current in amperes, the symbol for the latter being I. Hereafter, whenever quantities are expressed in e.m. units, they will be designated by a bar over the corresponding symbol.

4. Direction of Force Due to Current in Magnetic Field.—The form of the law given in Eq. (1) is a special case of a more general expression, also due to Ampère,

$$df = \frac{m}{r^2}\bar{I}\,dl \sin \theta \tag{3}$$

where θ (Fig. 7) is the angle between the radius vector r and the tangent T at the element dl. Equation (2) follows from (3) by substituting $\theta = 90$ deg. When $\theta = 0$, the force on the pole

m due to element dl is zero, and therefore the end-connection wires in Fig. 6 are drawn along radial lines through m, so that the entire effect of the current may be due to the circular arc only. In Fig. 7 the direction of the force on the element dl, m being assumed fixed in position, will be perpendicular to the plane through r and T and toward the reader.

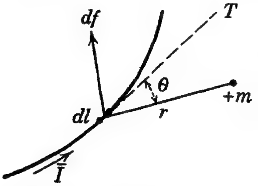


FIG. 7.—Force due to a current in a magnetic field.

In Eq. (3) the term m/r^2 is the flux density B at the element dl due to the pole m (see Art. 8, Chap. II), so that the force on the element is

$$df = B\bar{I}\,dl \sin\theta$$

If r is perpendicular to dl, $\sin\theta = 1$, and

$$df = B\bar{I}\,dl$$

It follows, therefore, that if a straight wire l cm. long, carrying a current of $\bar{I}$ abamp., is placed in a magnetic field having a uniform flux density of B gausses in such a manner that its length is perpendicular to the lines of induction (Fig. 8a) it will be acted upon by a force

$$f = Bl\bar{I} \quad \text{dynes} \qquad (4)$$

but if the wire makes an angle θ with the direction of the field (Fig. 8b), the force is

$$f = Bl\bar{I}\sin\theta \qquad (5)$$

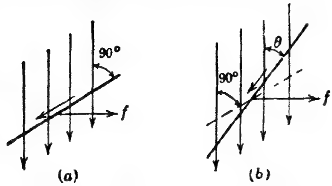


FIG. 8.—Force due to a current in a magnetic field.

Equation (4) provides the basis for another definition of the abampere. For if B, l, and $\bar{I}$ are each equal to unity, f is likewise unity, and it therefore follows that *the abampere is a current of such magnitude that if it flows in a straight wire* 1 *cm. long, placed perpendicular to the lines of induction of a magnetic field having a flux density of* 1 *gauss* (one line per square centimeter), *the wire will experience a side thrust of* 1 *dyne.*

In Fig. 7 the direction of the force on the element dl is determined by first applying Ampère's rule to find the direction in which the pole m is urged. There is, however, a simpler way to determine the direction of the force on the current element in Fig. 7 and on wires such as those represented in Fig. 8, the

method being known as *Fleming's left-hand rule:* Hold the thumb, forefinger, and middle finger of the *left hand* mutually perpendicular to one another, as in Fig. 9. Point the forefinger in the direction of the lines of induction (where they cross the wire) and the middle finger in the direction of the current flow; the thumb will then point in the direction of the force on the wire. This rule is very convenient in determining the direction in which a motor will rotate.

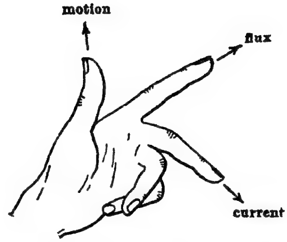


FIG. 9.—Fleming's left-hand rule. Motor action.

5. Field Intensity Produced by Current in Straight Wire. The Biot-Savart Law.—Equation (3) serves to determine the strength of the magnetic field set up in the vicinity of a wire. For if $m = 1$ in equation (3), df is the force acting on a unit pole, and this, by definition, is the field intensity at the point occupied by the unit pole. Hence,

$$dH = \frac{dB}{\mu_a} = \frac{\bar{I}\,dl \sin\theta}{r^2}$$

or

$$dB = \mu_a \frac{\bar{I}\,dl \sin\theta}{r^2} \tag{6}$$

where μ_a, the absolute permeability of the medium surrounding the wire, is assumed to be constant.

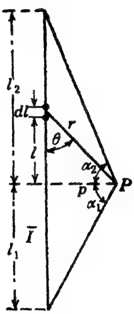


FIG. 10.—Field intensity due to a straight wire.

Let it be required to find the flux density at a point P, Fig. 10, at a perpendicular distance p cm. from a straight wire $(l_1 + l_2)$ cm. long, l_1 and l_2 being the lengths of the portions on either side of the perpendicular line p. If the current in the wire is $\bar{I}$ abamp., the flux density at the point P due to an element dl is, by Eq. (6),

$$dB = \frac{\mu_a \bar{I}\,dl \sin\theta}{l^2 + p^2} = \frac{\mu_a p \bar{I}\,dl}{(l^2 + p^2)^{3/2}}$$

and the flux density due to the entire wire is

$$B = \mu_a p \bar{I} \int_{-l_1}^{+l_2} \frac{dl}{(l^2 + p^2)^{3/2}} = \frac{\mu_a \bar{I}}{p}(\sin\alpha_2 + \sin\alpha_1) \tag{7}$$

If the wire is infinitely long in both directions, $\sin \alpha_2 = \sin \alpha_1 = 1$, and

$$B = \frac{2\mu_a \bar{I}}{p} \tag{8}$$

If the wire is in air where $\mu_a = \mu_0 = 1$, Eq. (8) becomes

$$B_0 = \mu_0 \frac{2\bar{I}}{p} \tag{9}$$

which means that at all points equally distant from the axis of the wire the field intensity is the same as to magnitude, and the direction of the field is tangent to circles that have their centers on the axis of the wire and their planes perpendicular thereto. It follows, therefore, that these circles are lines of magnetic force (as well as lines of magnetic induction). The relation expressed by Eq. (9) is known as the *Biot-Savart law.**

* There is much confusion in current literature in regard to the naming of Eq. (3), $df = (m/r^2)\bar{I}\,dl \sin \theta$, which here has been called Ampère's law, and of Eq. (9), $B_0 = \mu_0(2\bar{I}/p)$, to which the names of Biot and Savart have been attached. There are numerous books in which the fundamental relation given by Eq. (3) is called the law of Biot and Savart and still others in which it is called the law of Laplace.

An examination of a partial list of authoritative references shows, for example, that the following writers attribute Eq. (3) to Biot-Savart: HAAS, "Introduction to Theoretical Physics"; LORENTZ, "Lectures on Theoretical Physics"; WHITTAKER, "History of Aether and Electricity"; SLATER and FRANK, "Introduction to Theoretical Physics."

The following authors refer to Eq. (3) as the law of Laplace: B. O. PEIRCE, "Newtonian Potential Function"; G. CAREY FOSTER and A. W. PORTER, "Electricity and Magnetism"; J. A. FLEMING, Article on Electrokinetics, *Encyclopaedia Britannica.*

The following references attach Ampère's name to Eq. (3): MASCART and JOUBERT, "Electricity and Magnetism"; J. H. JEANS, "Electricity and Magnetism"; L. B. LOEB, "Fundamentals of Electricity and Magnetism"; A. O'RAHILLY, "Electromagnetics"; WÜLLNER, "Experimental Physik."

The facts appear to be that Biot, Savart, and Ampère all began their experimental studies as soon as Oersted's original results were published in 1820. Biot and Savart, immediately thereafter, announced in Vol. XV of "Annales de chimie et de physique," 1820, the discovery of the experimental fact that the field strength in the vicinity of a long, straight wire is inversely proportional to the distance from the wire [this fact being the check on Eq. (9)]. In commenting on this result, Laplace remarked that it could be accounted for mathematically if each infinitesimal element of the wire contributed in inverse proportion to the square of its distance from the point

Example.—A storage battery having a discharge rating of 10,000 amp. is connected to the switchboard by copper bus bars which have a cross-section of 1 by 10 in. and which are spaced 6 in. center to center, the 10-in. faces being in parallel vertical planes. On the assumption that the current may be considered to be concentrated at the center of cross-section, what must be the distance between supporting brackets in order that the bus bars may not deflect more than ¼ in.?

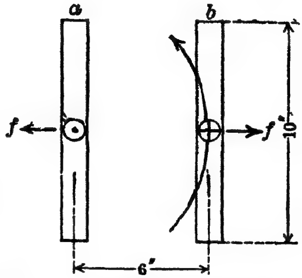


FIG. 11.—Section of bus bars.

Let Fig. 11 be a cross-section of the bus bars, the current flow having the directions indicated. Assume that the bus bars are sufficiently long so that for all practical purposes they may be considered as infinite in length. The flux density at the center of bus bar b due to the current in a is, by Eq. (9),

$$B_0 = \mu_0 \frac{2\bar{I}}{p} = \frac{2 \times 10{,}000}{6 \times 2.54 \times 10} = 131.2 \quad \text{gausses}$$

and the lines of induction have the direction shown by the curved arrow. By Fleming's left-hand rule, the direction of the force on b will be to the right; similarly, the force on a will act toward the left, the direction of these two forces agreeing with the known fact that parallel conductors carrying currents in opposite directions repel each other.

The force per inch of length on bus bar b is, by Eq. (4),

$$f = B_0 l\bar{I} = 131.2 \times 2.54 \times \frac{10{,}000}{10} = 333{,}333 \quad \text{dynes per in.}$$
$$\equiv 0.749 \quad \text{lb. per in.}$$

Each bus bar is equivalent to a uniformly loaded beam, which, for simplicity, may be considered to be like a beam fixed at the ends, the ends being the points of attachment to the supporting brackets. It is proved in books on mechanics that the deflection of such a beam is given by the formula

$$\delta = \frac{wl^4}{384EI}$$

where w is the load per unit length, l is the length between supports, E is the modulus of elasticity of the material of the beam, and $I = \frac{1}{12} bh^3$ is the

pole (see MASCART and JOUBERT, "Electricity and Magnetism," Vol. I, p. 443). The complete analysis of the whole subject is unquestionably the work of Ampère, Eq. (3) being only one of a number of important relations that he developed by mathematical methods on the basis of his numerous, carefully conducted experiments. His results are to be found in his book "Théorie des phénomènes électrodynamiques uniquement déduite de l'expérience," 1836.

moment of inertia of the cross-section, where b is the width of the beam (10 in. in this problem) and h is the depth (1 in.). Substituting $\delta = \frac{1}{4}$ in., $w = 0.749$ lb. per in., $E = 15 \times 10^6$ (the value for hard-drawn copper), and $I = \frac{1}{12}bh^3 = \frac{5}{6}$, l is found to be 200 in., or 16 ft. 8 in.

The fact that the lines of magnetic induction surrounding a long, straight conductor are circles, in the manner indicated in Fig. 12, is readily shown by passing such a wire vertically through a hole in a horizontal sheet of glass or paper and sprinkling iron filings on the latter. If such a wire, initially without current, is placed between the poles of a magnet N, S, as in Fig. 13*a*, the lines of induction being originally straight from pole to pole, on

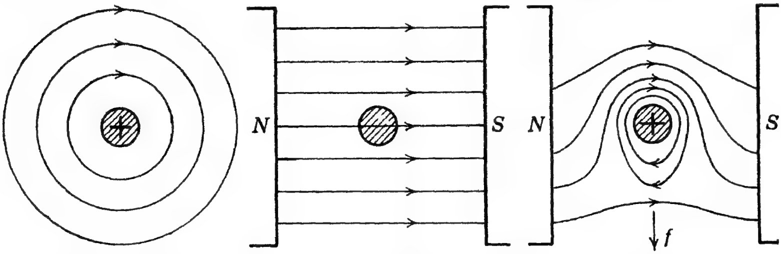


Fig. 12. Fig. 13*a*. Fig. 13*b*.
Figs. 12 and 13.—Distortion of magnetic field produced by current in a conductor.

establishing the current the field becomes distorted as indicated in Fig. 13*b*. Above the wire, the resultant field is more intense than that due to N, S, alone, for the field due to the current in the wire adds vectorially to the N, S field; below the wire the two component fields are in opposition, and so the resultant field is weakened. There are more lines per unit area above the wire than below it, and the tension along the distorted lines of force, which act as though they were stretched elastic bands, may be considered to be the cause of the downward force on the wire; the direction of the force checks with Fleming's left-hand rule.

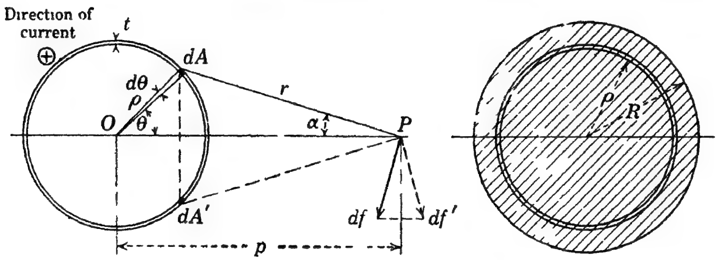
The preceding analysis of the strength of the magnetic field produced by a current in a straight wire was based upon the implicit assumption that the wire is a geometrical line. But if the wire has finite circular cross-section, the results are still applicable so far as points outside its boundary are concerned, provided that the current is uniformly distributed over the cross-section. The proof proceeds in the following manner:

The actual conductor (assumed to be nonmagnetic) may be considered to be built up of a series of concentric cylinders each

of infinitesimal thickness t, one of them being shown in Fig 14a. If the total current in the annular ring is $\bar{I}_0$ abamp., the current per unit area is $\bar{I}_0/2\pi\rho t$, and in the elementary area $dA = t\rho\, d\theta$ the current is $(\bar{I}_0/2\pi)d\theta$. At a point P, p cm. from the center O and r cm. from dA, the force on a unit magnet pole, or the field intensity, is

$$df = dH = \frac{2(\bar{I}_0\, d\theta/2\pi)}{r}$$

acting at right angles to r. It is necessary to consider only the component of dH at right angles to p because the component



(a) (b)

FIG 14—Magnetic field outside a solid conductor.

along p is balanced by the equal and opposite component of the field produced by the symmetrically located element dA'. The total field intensity at P, perpendicular to p, is

$$H = 2\int_0^{\pi} \frac{\bar{I}_0\, d\theta}{\pi r} \cos\alpha$$

From the geometry of the figure,

$$\rho^2 = p^2 + r^2 - 2pr\cos\alpha$$

and

$$r^2 = p^2 + \rho^2 - 2p\rho\cos\theta$$

whence $\cos\alpha$ and r may be found in terms of θ, and, on substituting their values,

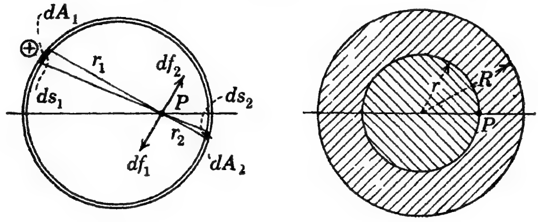
$$H = \frac{2\bar{I}_0}{\pi}\int_0^{\pi} \frac{p - \rho\cos\theta}{p^2 + \rho^2 - 2p\rho\cos\theta}\, d\theta$$

The definite integral in this expression can be shown* to be π/p if $p > \rho$ and zero if $p < \rho$. Hence, the force on the unit

* B. O. PEIRCE, "A Short Table of Integrals."

pole at P, and therefore the field intensity at P, is $H = (2\bar{I}_0/p)$, or the same as though the current were concentrated at the center. This result being true for the tubular element of Fig. 14*a*, it is true for all such elements in a solid conductor of circular section, as in Fig. 14*b*, and hence also for the solid conductor as a whole.

Inside the solid conductor the conditions are different, for the internal point is surrounded by annular elements like the one shown in Fig. 14*a*; this corresponds to the condition $p < \rho$, in which case the effect of the external annular element is nil. This conclusion can be verified in another way by means of Fig.



(*a*) (*b*)

FIG. 15.—Magnetic field inside a solid conductor.

15*a*, where a unit magnet pole is supposed to be at point P inside the circular annulus there indicated; if the current per unit area of the annulus is $\bar{I}_0$, the forces due to the elements dA_1 and dA_2 are

$$df_1 = \frac{2\bar{I}_0\,dA_1}{r_1} \qquad \text{and} \qquad df_2 = \frac{2\bar{I}_0\,dA_2}{r_2}$$

and as these are oppositely directed, the resultant force is

$$df = 2\bar{I}_0\left(\frac{dA_1}{r_1} - \frac{dA_2}{r_2}\right)$$

But from the geometry of Fig. 15*a*,

$$\frac{ds_1}{r_1} = \frac{ds_2}{r_2}$$

and therefore

$$\frac{t\,ds_1}{r_1} = \frac{dA_1}{r_1} = \frac{t\,ds_2}{r_2} = \frac{dA_2}{r_2}$$

so that df is zero at all points inside the annulus. Consequently, at a point P, Fig. 15*b*, inside a solid conductor, the field intensity

due to the annulus outside P is zero, and the intensity due to the solid portion inside P is, if $\bar{I}$ is the total current in the conductor,

$$H = \frac{2\left(\frac{\pi r^2}{\pi R^2}\bar{I}\right)}{r} = \frac{2\bar{I}r}{R^2} \tag{10}$$

or directly proportional to the distance from the center. The variation of H with distance from the center of the wire is shown in Fig. 16.

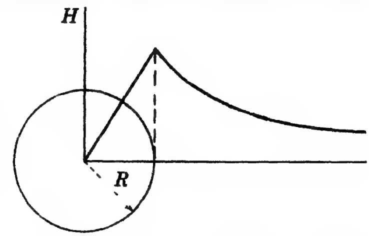


FIG. 16.—Variation of field intensity at right angles to long straight wire.

6. Field Intensity on Axis of Circular Coil.—Let P, Fig. 17, represent a unit magnet pole on the axis of a plane circular coil of N turns and radius R cm. It is assumed that the N turns are so closely compacted that all of them may be considered to be in the same plane and to have the same radius R. Let P be x cm. from the plane of the coil, and let the current in the coil be $\bar{I}$ abamp. The force acting on an element dl of the coil is, by Eq. (1),

$$df = \frac{1}{R^2 + x^2}N\bar{I}\,dl$$

acting in the direction indicated in the figure; and the pole is acted upon by an equal force in the opposite direction. If df is resolved into components parallel to, and perpendicular to, the axis of the coil, the sum of all the perpendicular components is zero, since for each element of the coil that gives rise to a perpendicular component of force in one direction, there is a diametrically opposite element that produces an equal and opposite component.

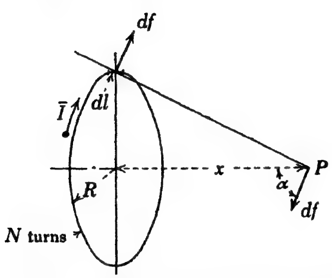


FIG. 17.—Field intensity on axis of circular coil.

The axial component of df is

$$dH = \frac{N\bar{I}\,dl}{R^2 + x^2}\cos\alpha = \frac{NR\bar{I}\,dl}{(R^2 + x^2)^{3/2}}$$

Hence.

$$H = \frac{N\bar{I}R}{(R^2 + x^2)^{3/2}} \int_0^{2\pi R} dl = \frac{2\pi N\bar{I}R^2}{(R^2 + x^2)^{3/2}} = \frac{2\pi NIR^2}{10(R^2 + x^2)^{3/2}} \tag{11}$$

where I is the current in amperes. At the center of the coil, where $x = 0$,

$$H = H_0 = \frac{2\pi N\bar{I}}{R} = \frac{2\pi NI}{10R} \tag{12}$$

which checks with Eq. (2) if $m = 1$ and $N = 1$.

7. Field Intensity in Plane of Circular Coil but not on Axis.—In Fig. 18, let P be a general point in the plane of the coil of which the radius is R cm. and which carries the current $\bar{I}$ abamp. The position of P is defined by the distance $D = KR$, where K is a fraction that may have any value between 0 and 1. The field intensity at P will be equal to the force in dynes exerted by the entire coil on a unit pole placed at P; and, by Eq. (3), this is determined by

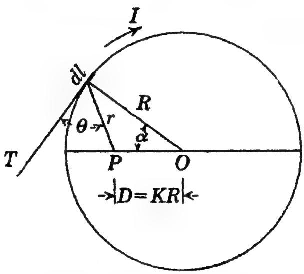


FIG. 18.—Field intensity in the plane of a circular coil.

$$df = \frac{\bar{I}\,dl}{r^2}\sin\theta \tag{13}$$

As dl takes all possible positions around the circle, both r and θ change in magnitude, so that integration of Eq. (13) requires the substitution of a single variable in place of the three there present. The most convenient single variable is α, and it follows from the geometry of the diagram that

$$dl = R\,d\alpha$$

$$\sin\theta = \frac{r^2 + R^2 - K^2R^2}{2rR}$$

$$r^2 = R^2 + K^2R^2 - 2KR^2\cos\alpha$$

and when these relations are substituted in Eq. (13), with appropriate reductions,

$$df = \frac{\bar{I}(1 - K\cos\alpha)d\alpha}{R(1 + K^2 - 2K\cos\alpha)^{3/2}} \tag{14}$$

The total force at P, which is the desired value of the field

intensity H, is obtained by integrating Eq. (14) between the limits $\alpha = 0$ and $\alpha = \pi$ and doubling the result; that is

$$H_P = \frac{2\bar{I}}{R}\int_0^\pi \frac{1 - K\cos\alpha}{(1 + K^2 - 2K\cos\alpha)^{3/2}}d\alpha \tag{15}$$

the ratio K being treated as a constant.

This expression is an elliptic integral that cannot be integrated by any of the simple formulas ordinarily used, except for the limiting cases when $K = 0$ and $K = 1$. When $K = 0$, corresponding to point P at the center of the circle, $H_P = 2\pi\bar{I}/R$, agreeing with Eq. (2); and when $K = 1$,

$$(H_P)_{K=1} = \frac{2\bar{I}}{R}\int_0^\pi \frac{d\alpha}{2^{3/2}(1 - \cos\alpha)^{1/2}} = \frac{\bar{I}}{R}\log\tan\frac{\alpha}{4}\bigg]_0^\pi = \infty$$

The last expression means that, if the wire is so fine that it is a geometrical line, the field intensity at the wire approaches infinity as a limit, a result that might be anticipated from Eq. (9), where the same condition would arise if $p = 0$. Actually the wire has a finite radius, and P must not be taken inside the boundary of the wire. In Eq. (15) the integral can be written

$$\int_0^\pi \frac{(1 - K\cos\alpha)d\alpha}{(1 + K^2 - 2K\cos\alpha)^{3/2}} = \int_0^\pi f(\alpha)d\alpha \tag{16}$$

where $f(\alpha)$ is finite and continuous when a particular value is assigned to K, and values of the function may be computed for a series of values of α between the limits 0 and π. The curves in Fig. 19 show the result of such computations for $K = 0.1$, 0 3, 0.8. The area under any one of these curves is proportional to the integral in Eq. (16) corresponding to the value of K used in plotting the curve. In this way the integral can be evaluated* to a degree of accuracy that depends upon the care exercised in making the drawing and upon the scale used.

The results can be used to compare the field intensity H_P at any distance from the center with the value $H_0 = 2\pi\bar{I}/R$

* In the original drawing from which Fig. 19 was reproduced, the area of the crosshatched rectangle was 5/3 sq. in. This corresponds to an area, to the scale of the drawing, of $1 \times \pi/6$ units; hence, each square inch of the measured area under the curve is equivalent to $3/5 \times \pi/6 = \pi/10$ units as fixed by Eq. (16).

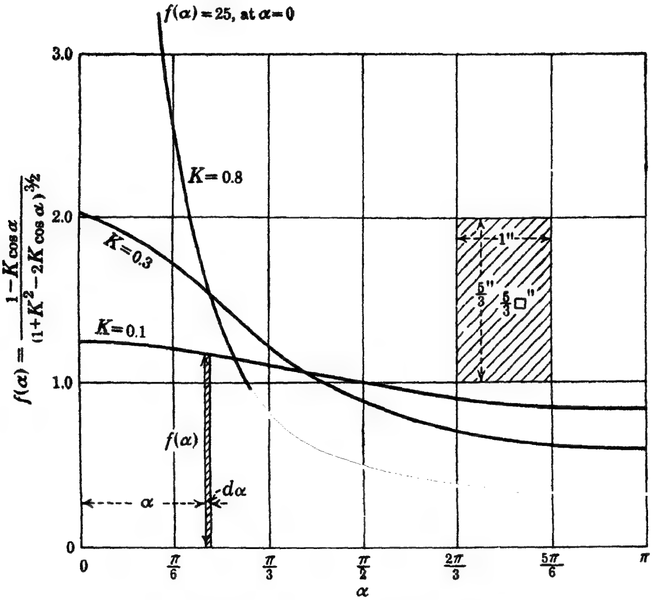


FIG. 19.—Plot of $f(\alpha)$ for various values of K.

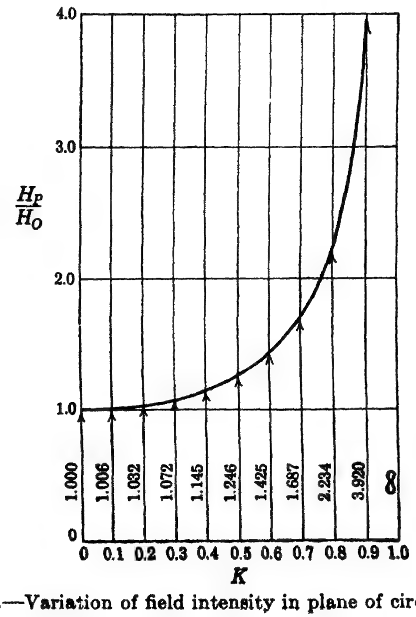


FIG. 20.—Variation of field intensity in plane of circular coil.

at the center. The ratio H_P/H_0 for various values of K is plotted in Fig. 20.* This diagram shows that near the center the field intensity is fairly constant up to about $K = 0.2$; beyond this central zone the field intensity increases rapidly as the wire is approached.

8. Field Intensity on Axis of Solenoid.—Let Fig. 21 represent a solenoid of radius R, carrying a current of I amp., having N turns uniformly distributed over the length l cm. It is desired to find the field inten ty at a point P on the axis, distant D cm. from the center of the solenoid.

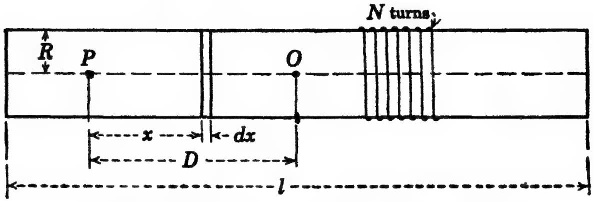


FIG 21.—Field intensity on axis of solenoid

Consider an elementary section dx of the solenoid, distant x cm. from P The element may be considered as a plane circular coil of $(N/l)dx$ turns; the field intensity due to this element at the point P is, by Eq. (11),

$$dH = \frac{2\pi\left(\frac{N}{l}dx\right)IR^2}{10(R^2 + x^2)^{3/2}}$$

and the total field intensity is

$$H = \frac{2\pi}{10}\frac{NIR^2}{l}\int_{-\left(\frac{l}{2}-D\right)}^{\frac{l}{2}+D}\frac{dx}{(R^2 + x^2)^{3/2}}$$

$$= \frac{2\pi}{10}\frac{NI}{l}\left[\frac{\frac{l}{2} + D}{\sqrt{R^2 + \left(\frac{l}{2} + D\right)^2}} + \frac{\frac{l}{2} - D}{\sqrt{R^2 + \left(\frac{l}{2} - D\right)^2}}\right] \qquad (17)$$

At the center of the solenoid, where $D = 0$,

$$H_0 = \frac{2\pi NI}{10\sqrt{R^2 + \frac{l^2}{4}}}$$

* F. E. NIPHER, "Electricity and Magnetism."

which reduces to

$$H_0 = \frac{4\pi NI}{10l} \tag{18}$$

if l is large compared with R.

At the ends of the solenoid, where $D = l/2$,

$$H = H_e = \frac{2\pi NI}{10l}$$

or half as great as at the center, provided that l is large compared with R.

Figure 22 shows the variation of H along the axis of a solenoid the length of which is twenty-five times its radius, that is, $R/l = 0.04$. The value of H_0 is a trifle less than $4\pi NI/10l$, namely. $3.9872\pi NI/10l$; and $H_e = 1.9984\pi NI/10l$ instead of $2\pi NI/10l$. It will be observed that H is very nearly constant over the greater part of the axis and that it falls off abruptly near the ends.

The physical interpretation of these facts concerning the variation of H along the axis is as follows: For some distance on either side of the middle section of the solenoid, the lines of force inside the winding are nearly parallel, and hence the field is nearly uniform and H is practically constant; near the ends of the solenoid the lines diverge in the manner indicated in Fig. 2, and the greater the divergence the more rapidly will H decrease.

9. Electromagnetic Unit of Potential.—The absolute e.m. units of current and quantity having been defined and some of the magnetic effects of a current having been outlined, it is now convenient to turn attention to other deductions that follow from the fundamental definitions already considered.

In the first place, it is evident that two given electric charges will experience definite mutual forces of attraction or repulsion, depending upon their signs, independent of the magnitude of the units in which they are expressed, and that in moving one charge in the presence of the other, or in an electric field of force in general, a definite amount of work will be required—unless the motion happens to be on an equipotential line, in which case the work is zero. Accordingly, just as difference of potential in the *e.s. system* is defined in terms of the work in ergs required to move a statcoulomb from point to point, so, in the *e.m. system* of units, *difference of potential, measured in abvolts, is measured*

by the work in ergs required to move 1 *abcoulomb from the one point to the other;* otherwise stated, *unit difference of potential, the abvolt, exists between two points if* 1 *erg of work is required to move* 1 *abcoulomb from the one point to the other.*

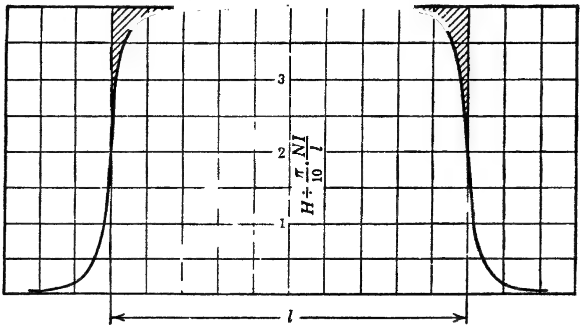


FIG. 22.—Variation of field intensity along axis of solenoid.

For example, let P_1 and P_2, Fig. 23, be two points between which there is a difference of potential of V_s statvolts, and let Q be a charge which is moved from P_2 to P_1 to measure the work done. If the magnitude of Q, expressed in statcoulombs, is Q_s, the work done is

$$W = V_s Q_s \quad \text{ergs} \tag{19}$$

It is important to remember that the quantities to be inserted in this formula in order to compute the work are the *number* of statvolts and the *number* of statcoulombs involved.

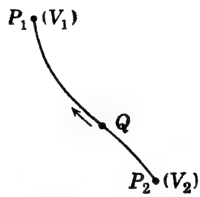


FIG. 23.—Work done in moving a charge in electric field.

If the *same* potential difference is expressed as an equivalent number of abvolts $\bar{V}$ and the *same* testing charge is expressed as an equivalent number of abcoulombs $\bar{Q}$, the work done is the same as before, or

$$W = \bar{V}\bar{Q} \quad \text{ergs} \tag{20}$$

whence

$$\frac{\bar{V}}{V_s} = \frac{Q_s}{\bar{Q}} \tag{21}$$

However, since 3×10^{10} statcoulombs are equivalent to 1 abcoulomb, $\bar{V}/V_s = 3 \times 10^{10}$, and hence the *number* of abvolts required to express a given potential difference is 3×10^{10} times the corresponding *number* of statvolts; this fact in turn

means that the *magnitude* of the abvolt is smaller* than that of the statvolt in the ratio $1/(3 \times 10^{10})$.

Since 1 statvolt has been shown to be the equivalent of 300 volts, it follows that

$$10^8 \text{ abvolts} \equiv 1 \text{ volt}$$

or the magnitude of the abvolt is only 10^{-8} as large as the magnitude of the volt.

Thus, when $\bar{Q}$ abcoulombs are moved between points that have a potential difference of $\bar{V}$ abvolts, the work done is $W = \bar{V}\bar{Q}$ ergs, and if this work is done in t sec. the average power is

$$P = \bar{V}\frac{\bar{Q}}{t} = \bar{V}\bar{I} \quad \text{ergs per sec.}$$

Since $\bar{V} = V \times 10^8$, where V is the equivalent potential difference in volts, and $\bar{I} = I \times 10^{-1}$, where I is the equivalent current in amperes, the average power is

$$\begin{aligned} P &= (V \times 10^8)(I \times 10^{-1}) = VI \times 10^7 \quad \text{ergs per sec.} \\ &\equiv VI \quad \text{watts} \end{aligned}$$

for the watt, by definition, is equivalent to 10^7 ergs per sec. (or 1 joule per sec.).

10. Electromagnetic Induction.—The preceding discussion of the magnetizing effect of an electric current is based upon Ampère's analysis of Oersted's discovery that a current produces a magnetic field. The converse possibility, namely, that a magnetic field might produce a current, long sought by Faraday, was finally established in 1831 in the manner already explained in Art. 1, Chap. I. The phenomenon of electromagnetic induction thus discovered by Faraday may be briefly described in either of two ways:

1. An e.m.f. will be induced in a stationary coil or circuit if it links with a stationary magnetic field the magnitude of which changes with time.

* A clear distinction must always be made between the magnitude of a unit and the number of such units required to express a given quantity. Thus, a certain length may be said to be either 10 ft. or 120 in.; the magnitude of the foot is twelve times that of the inch, and so the number of feet required to express a given length is only one-twelfth as great as the number of inches.

2. A conductor, situated in a magnetic field, will have an e.m.f. induced in it if there is relative motion between the conductor and the magnetic field in such a way either that the conductor cuts across the lines of induction or the lines cut across the conductor.

Case 1 is illustrated in Fig. 24*a*, which represents the type of apparatus used by Faraday in the original experiment described in Art. 1, Chap. I. The magnetic field set up by the current in coil *A* links with the turns of coil *B*, and any change in the magnitude of the field caused by a change of current in *A* immediately induces an e.m.f. in *B*.

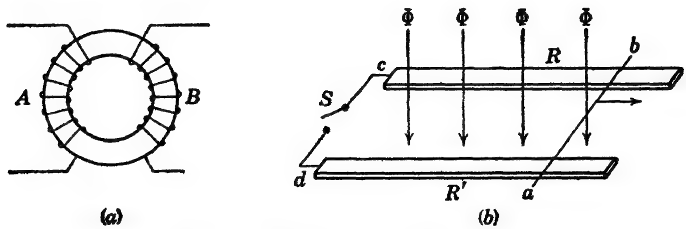


FIG. 24.—Development of e.m.f.

Case 2 is illustrated in Fig. 24*b*, where the conductor *ab* is sliding toward the right along rails *RR'*, thus cutting across the lines of induction Φ directed at right angles to the plane of the rails. With the connecting wire *cd* opened at switch *S*, experiment shows that there will be a displacement of electricity along the wire *ab*, a positive charge appearing at the end *b* (and also on rail *R* and wire *c*), whereas a negative charge develops at end *a* (and on *R'* and *d*). There will be a difference of potential between *c* and *d*, with no current flowing in *ab* except during the brief period while the charges are building up; but on closing the switch, on the assumption that the motion of *ab* continues as before, a current will flow around the circuit in the direction *abcda*.

The experimental facts observable with the aid of such apparatus as is shown in Figs. 24*a* and *b* are not "explained" by ascribing them exclusively to the intervention of the lines of induction of the magnetic field. These lines, as has already been pointed out, are merely convenient abstractions which have no objective existence and can therefore play no actual part in the physical mechanism that is involved in the phenomenon. The obvious

fact that is common to both cases 1 and 2 is that somehow the loosely bound electrons in the atoms of the induced conducting circuit are subjected to forces that cause them to move under the stated conditions. It seems reasonable to believe that the ultimate cause of the action is the force exerted by the moving charges in the inducing circuit upon the free charges in the induced circuit. In Fig. 24*a*, for example, the moving charges in coil *A* exert forces upon the free electrons in coil *B*; in Fig. 24*b* the flux Φ is produced either by moving charges in a neighboring coil (not shown in the drawing) or by the molecular currents in a suitably placed permanent magnet. It is to be noted, however, that in Fig. 24*a* the induced e.m.f. in coil *B* appears only when the current in coil *A* is changing, and not when the current in *A* is steady. Accordingly, it is not sufficient to think of the forces between charges in *A* and in *B* in terms of the static forces that might exist between them at a given instant; there must be something in the nature of a relative acceleration between the charges to account for the actual observed force. In any event the action between the two sets of moving charges is conceived to be transmitted through the intervening space, and the concept of lines of force is useful, therefore, as picturing a state of stress therein.

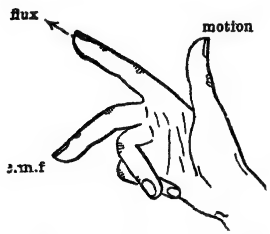


FIG. 25.- Fleming's right-hand rule. Generator action.

The arrangement shown in Fig. 24*a* constitutes an elementary transformer. An alternating current in coil *A* will induce in coil *B* an alternating e.m.f. whose magnitude increases in proportion to the number of turns in *B*.

The device of Fig. 24*b* is an elementary form of electric generator (electron pump). The direction of the induced e.m.f. in the case of generator action (and of the resulting current if the circuit is closed) is conveniently determined by *Fleming's right-hand rule:* Hold the thumb, forefinger, and middle finger of the *right* hand mutually perpendicular to one another, as in Fig. 25; point the forefinger in the direction of the lines of force, the thumb in the direction of motion of the wire relative to the field, and the middle finger will point in the direction of the induced e.m.f.

This right-hand rule for generator action should be compared with the corresponding left-hand rule for motor action, discussed

in Art. 4 and illustrated in Fig. 9. Each rule is a sort of mirror image of the other; in fact if Fig. 9 is observed in a mirror it will be identical with Fig. 25.

11. Magnitude of Induced E.M.F.—Consider the circuit of Fig. 26, and let the wire *ab* of length *l* cm. move to the right with a velocity $v = ds/dt$ cm. per sec., through the magnetic field of which the lines of induction B, B are directed at right angles to the plane of the circuit *abcd*. It is not essential that the flux

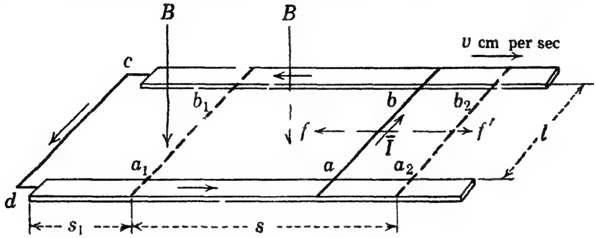


FIG. 26 – Moving conductor in a magnetic field.

density B be uniform or that the velocity v be constant. It need only be assumed that B is uniform across the infinitesimal distance ds traversed in the infinitesimal time dt. There will be generated in the wire an e.m.f. of, say $\bar{E}$ abvolts, in accordance with Faraday's discovery, and in consequence a current of $\bar{I}$ abamp. will be set up. The power developed will be $\overline{EI}$ ergs per sec., and the energy developed during the time dt will be

$$dW = \overline{EI}\, dt \quad \text{ergs}$$

The current $\bar{I}$ in the wire ab produces a thrust of

$$f = Bl\bar{I} \quad \text{dynes}$$

acting toward the left in accordance with Fleming's left-hand rule, and this must be balanced by the equal force f', if the wire is to continue its uniform motion to the right; consequently, the work done in moving the wire through a distance ds is

$$dW = f'\, ds = Bl\bar{I}\, ds$$

and by the principle of the conservation of energy

$$\bar{E}\bar{I}\, dt = Bl\bar{I}\, ds$$

or

$$\bar{E} = Bl\frac{ds}{dt} = Blv \quad \text{abvolts} \tag{22}$$

In Eq. (22), $Bl\,ds$ represents the number of lines of induction cut in the time dt; hence $Bl\,ds/dt$, or Blv, is the *time rate* of cutting lines of induction. We are thus led to the important conclusion, known as the *law of electromagnetic induction*, that:

The e.m.f. (in abvolts) induced in a circuit when it cuts the lines of a magnetic field is at any instant numerically equal to the rate at which magnetic flux is being cut at that instant.

If both B and v in Eq. (22) are constant, it follows that the absolute e.m.f. is equal to the number of lines of induction actually cut in 1 sec.; but, in general, neither B nor v remains the same from point to point and from instant to instant, and so the e.m.f. will vary from instant to instant, but always in such manner that its absolute value is equal to the rate at which lines of induction are being cut at that instant. The mathematical expression for this fact is

$$|\bar{E}| = \left|\frac{d\phi}{dt}\right| \tag{23}$$

where $d\phi = Bl\,ds$ is the flux cut in the time interval dt. Equation (23) is a more general representation of the law of electromagnetic induction than Eq. (22).

Both Eqs. (22) and (23) lead to the conclusion that 1 *abvolt is the e.m.f. induced by cutting magnetic flux at the rate of* 1 *line (that is,* 1 *maxwell) per sec.*

Let the wire start from position a_1b_1, and move to position a_2b_2. The flux originally linked with circuit a_1b_1cd is

$$\phi_1 = Bls_1$$

and in the final position the flux linked is

$$\phi_2 = Bl(s_1 + s)$$

The change of flux during the movement is

$$\phi = \phi_2 - \phi_1 = Bls$$

and if this change occurs in t sec. the average rate of change is

$$\frac{\phi}{t} = Bl\frac{s}{t} = Blv_{\text{aver.}} = \bar{E}_{\text{aver.}}$$

which, in words, states that the *average* induced e.m.f. (in abvolts) is equal to the *average* rate of change of flux linked with the circuit.

The formulas for e.m.f. deduced thus far are not in the most general form, which will be found in Art. 13.

12. Lenz's Law.—It is interesting to observe that in Fig. 26 the initial movement of the wire *ab* through the magnetic field induces or generates an e.m.f. which in turn sets up the current $\bar{I}$ through the closed circuit; the reaction of this current against the magnetic field calls into being the force f, which, if unopposed, would produce motion to the left. The balancing force f', if permanently equal to f, would result in uniform motion to the right, and if f' is greater than f there will be an acceleration which will increase both $\bar{E}$ and $\bar{I}$ and so again tend to restore the balance between f and f'.

Also, it will be observed that as the wire *ab* moves to the right, thus enclosing within the loop *abcd* an increasing flux, the direction of the induced current $\bar{I}$ is such that it would itself produce a magnetic field directed vertically upward, in opposition to the inducing flux, as may be verified by Ampère's rule. In other words, not only does the induced current set up a force that opposes the original driving force, but it also tends to demagnetize, or to annul, the magnetic field responsible for its existence.

These facts are summarized in *Lenz's law: An induced current always has such a direction as to oppose the action that produces it.*

13. General Equation for Induced E.M.F.—Inasmuch as the induced current has the same direction as the induced e.m.f., it may be concluded from the diagram (Fig. 26) that if the flux linked with circuit *abcd increases* with time (that is, $d\phi/dt$ is *positive*), the induced e.m.f. must be considered to be *negative*, for the resultant current that it sets up tends to *decrease* the flux. Accordingly, Eq. (23) must be modified to the form

$$\bar{E} = -\frac{d\phi}{dt} \tag{24}$$

and if the circuit has N turns, all so closely superposed that each turn is linked with the entire flux, the equation becomes

$$\bar{E} = -N\frac{d\phi}{dt} \quad \text{abvolts} \tag{25}$$

or

$$E = -N\frac{d\phi}{dt} \times 10^{-8} \quad \text{volts} \tag{26}$$

which is the most general form of the equation for the induced e.m.f. in any circuit whatsoever.

Equations (25) and (26), which may be called the *flux-linking* equations, should be compared with the corresponding equations

$$\bar{\bar{E}} = Blv \quad \text{abvolts} \tag{27}$$

and

$$E = Blv \times 10^{-8} \quad \text{volts} \tag{28}$$

which may be called the *flux-cutting* equations. Both pairs represent the same fundamental law; but sometimes one form, sometimes the other, is more convenient. The flux-linking equation is applicable to a circuit that links with a magnetic flux whose magnitude is changing with time; the flux-cutting equation is applicable where there is relative motion between a conductor and a constant flux.

In Eqs. (26) and (28) the factor 10^{-8} appears because of the change from abvolts to volts. This factor can be made to disappear from Eq. (26) if the flux ϕ, instead of being expressed in maxwells, is expressed in terms of a unit 10^8 times as large; such a unit, equivalent to 10^8 maxwells, is called the *weber;* that is,

$$1 \text{ weber} \equiv 10^8 \text{ maxwells}$$

so that if flux (ϕ_1) is expressed in webers Eq. (26) becomes

$$E = -N\frac{d\phi_1}{dt} \quad \text{volts} \tag{29}$$

The weber is the unit of flux in the meter-kilogram-second (m.k.s.) system, recently adopted by the I.E.C. and to be discussed in Chap. V. It is, however, of interest at this point to see how the use of the m.k.s. system affects Eq. (28): In the first place, it is obvious that B in that equation (maxwells per square centimeter) must be changed to B_1 (webers per square meter); l (centimeters) must be changed to l_1 (meters); and v (centimeters per second) must be changed to v_1 (meters per second); therefore,

$$\begin{aligned} E = Blv \times 10^{-8} &= \left(\frac{B_1 \times 10^8}{10^4}\right)(l_1 \times 10^2)(v_1 \times 10^2) \times 10^{-8} \\ &= B_1 l_1 v_1 \quad \text{volts} \end{aligned} \tag{30}$$

or the factor 10^{-8} disappears from Eq. (28) as well as from Eq. (26). The m.k.s. system is in fact so designed that the factor 10 and all its multiples and submultiples drop out of all the commonly used formulas; but except for this alteration the formulas remain unchanged.

14. Interrelation among Statvolt, Abvolt, and Volt.—Considering the definitions of the several units in which potential difference (and e.m.f.) may be expressed, it will be seen that they are linked together in several ways:

1. In Art. 5, Chap. I, the *statvolt* is defined in terms of the work in ergs required to move a statcoulomb in an electric field; and the statcoulomb is in turn dependent upon Coulomb's fundamental law defining the force of attraction or repulsion between static charges.

2. Likewise in Art. 5, Chap. I, the *volt* is defined in terms of the work in joules required to carry a coulomb from point to point in a similar electric field, the volt being therefore related to the concepts of static charges and the forces between them.

3. In Art. 9, Chap. III, the *abvolt* is defined in terms of the work in ergs required to move an abcoulomb in an electric field, this unit being thus related also to electrostatic considerations.

4. In Art. 5, Chap. III, the abcoulomb, instead of being defined as in (3) in terms of Coulomb's law, is made to depend upon Ampère's law; for its definition depends upon that of the abampere, and this in turn depends upon the mutual forces between a current and a unit magnet pole. Consequently the abvolt is related to electrodynamic considerations, and with it the volt and the statvolt also.

5. Finally, the abvolt has been found to depend upon the electromagnetic effects involved in Faraday's discovery, and therefore also the volt and the statvolt.

The inference from these relations is that there must be a close relation between electrostatic, magnetic, and electrodynamic phenomena. It was Maxwell who in 1865 first established these intimate relations in mathematical form, and it was Hertz who in 1887 confirmed them by the experiments that led to the development of radio communication.

15. Alternative Approach to Electrodynamic Relations.—It will be observed that the analysis leading up to Eqs. (4), (25), and (27) has been made to depend upon the concept of the unit

magnet pole, the reason being that this procedure parallels the historical development of the subject. It is evident, however, that Eq. (4)

$$f = Bl\bar{I}$$

expresses a fundamental experimental fact concerning the force on a current-carrying conductor in a magnetic field, and that Eqs. (25) and (27)

$$\bar{E} = -N\frac{d\phi}{dt}$$

and

$$\bar{E} = Blv$$

express the basic experimental fact of electromagnetic induction. It is therefore just as legitimate to start with these equations, expressive of incontrovertible physical facts, as it is to start with Ampère's law

$$f = \frac{m\bar{I}\,dl}{r^2}\sin\theta$$

and if we start with Eqs. (4), (25), (27) it is unnecessary to use the concept of the isolated point pole; but in place of the magnet pole there appear the symbol B, flux density, and the related magnitude ϕ, magnetic flux. The flux is itself an abstract concept, for it must be regarded as a vector quantity, having a space distribution and possessing the property that any change in its magnitude will induce an e.m.f. in a closed circuit which links with it.

It has been pointed out already that, in Eq. (4), $f = Bl\bar{I}$, the three quantities f, B, and $\bar{I}$ can be represented by three mutually perpendicular vectors, like the three mutually perpendicular axes of reference in Cartesian geometry; their space relations are given by Fleming's left-hand rule (Fig. 9). The force f (or the motion that it tends to produce) is the counterpart of the velocity v in Eq. (27), $\bar{E} = Blv$; and $\bar{I}$ in Eq. (4), which indicates the electric current, corresponds to $\bar{E}$ in Eq. (27); the flux density, B, plays the same role in both equations. But whereas f, B, and $\bar{I}$ constitute a left-handed set of vectors, as in Fig. 9, their counterparts v, B, and $\bar{E}$ in Eq. (27) form a right-handed set of mutually perpendicular axes, as indicated by Fleming's right-hand rule (Fig. 25).

16. Magnetic Potential on Axis of Circular Coil.—It is shown in Art. 6 that the field intensity on the axis of a circular coil, at a distance x from the plane of the coil, is

$$H = \frac{2\pi N\bar{I}R^2}{(R^2 + x^2)^{3/2}}$$

this expression giving the force in dynes acting upon a unit magnet pole placed at the point. With the current flowing as indicated in Fig. 27, the unit (positive) pole would be urged to the left, or toward the coil. To move the pole to the right over a distance dx there must be expended

$$dW = H\,dx = \frac{2\pi N\bar{I}R^2\,dx}{(R^2 + x^2)^{3/2}} \text{ ergs}$$

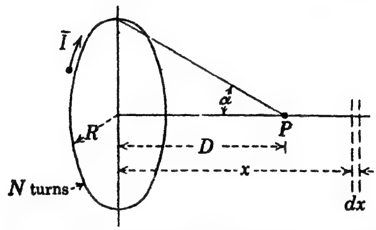


FIG. 27.—Magnetic potential on axis of circular coil.

of work, and the total work required to move the unit pole out to infinity from a point P distant D cm. from the plane of the coil is

$$W = 2\pi N\bar{I}R^2 \int_D^\infty \frac{dx}{(R^2 + x^2)^{3/2}} = 2\pi N\bar{I}\left(1 - \frac{D}{\sqrt{R^2 + D^2}}\right) \quad (31)$$
$$= 2\pi N\bar{I}(1 - \cos\alpha)$$

where α is the semiangle of the right cone subtended at the point P by the coil. But $2\pi(1 - \cos\alpha) = \omega$ is the solid angle at the vertex of the cone; hence,

$$W = \omega N\bar{I} \quad (32)$$

If the test pole had been of strength m units, the work done would have been m times as great as is given by Eq. (32), or

$$W_m = \omega m N\bar{I} \quad (33)$$

The expression for W, Eq. (32), is the magnetic potential at a general point on the axis of the coil; it represents the work required to move a unit pole from the point out to an infinite distance, when the current flow is as indicated. If the current is reversed, W becomes the work required to bring the unit pole from an infinite distance up to the point in question.

17. General Expression for Magnetic Potential Due to Coil of Any Shape at Any Point.—In Fig. 28 a point pole of $+m$ units is indicated at any general point in the vicinity of an irregularly shaped coil of N turns carrying current $\bar{I}$ abamp. The coil may or may not be a plane coil; but whatever its shape, it subtends a solid angle ω at the pole m, the solid angle being by definition equal to the area cut from a sphere of unit radius, having its center at m, by a cone the vertex of which is at m and which has the coil as its directrix.

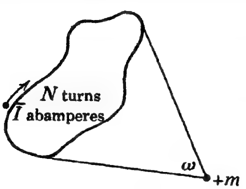


FIG. 28.—Magnetic potential due to coil of any shape.

Since the total flux Φ emitted by the pole m is $4\pi m$, the part φ included within the solid angle ω may be found from the proportion

$$\frac{\varphi}{\Phi} = \frac{\varphi}{4\pi m} = \frac{\omega}{4\pi}$$

whence

$$\varphi = \omega m \quad \text{maxwells}$$

Suppose now that pole m moves through an infinitesimal distance so that the solid angle changes by $d\omega$ in time dt. The flux linked with the coil will change by

$$d\varphi = m\, d\omega$$

and there will be induced in the coil an e.m.f.

$$\bar{E} = -N\frac{d\varphi}{dt} = -Nm\frac{d\omega}{dt} \quad \text{abvolts}$$

If this change of flux occurs while a current of $\bar{I}$ abamp. is flowing in the coil, work will be done to the extent

$$dW = -\bar{E}\bar{I}\, dt = mN\bar{I}\, d\omega \quad \text{ergs}$$

and the total work, required to bring the pole from an infinite distance (or from any other point where $\omega = 0$) to a general point where the solid angle is ω, is

$$W = mN\bar{I}\int_0^\omega d\omega = \omega mN\bar{I} = \varphi N\bar{I} \quad \text{ergs} \tag{34}$$

which is the same as Eq. (33).

Inasmuch as each line of the flux φ links with each of the N turns of the coil, the product φN may be interpreted as the total number of *flux linkages*, symbolized by λ; therefore,

$$W = \lambda \bar{I} = \varphi N \bar{I} \quad \text{ergs} \tag{35}$$

is a general expression for the potential energy of a system comprising a coil of N turns, carrying $\bar{I}$ abamp., when the coil is

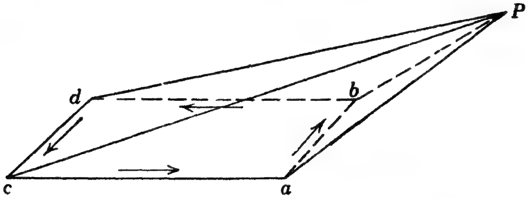


FIG. 29.—Potential due to rectangular coil.

linked with a flux φ set up by *an agency external to, and independent of,* the coil itself.

18. Magnetic Potential in Vicinity of Long, Straight Wire.—Assume that a current $\bar{I}$ abamp. flows in a square or rectangular loop *abcd*, shown in perspective in Fig. 29, and that a unit magnet pole is placed at a point P, not in the plane of the coil. On

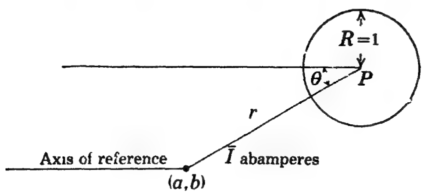


FIG. 30.—Potential adjacent to long straight wire.

joining point P with each of the corners of the rectangle, the resultant figure is a pyramid that has a definite solid angle ω at its vertex; and this solid angle will be equal to the area intercepted by the cone on a sphere of unit radius drawn about P as center. Whatever the actual magnitude of ω, the magnetic potential at P is given by Eq. (33) and is

$$W = \omega \bar{I} \tag{36}$$

since in this case N and m are unity.

Suppose now that the sides *ab* and *cd* of the rectangular loop are lengthened indefinitely in both directions and that at the same

time the sides *ac* and *bd* are also made so large that the fourth side *cd* is at an infinite distance to the left of the diagram. We therefore have the unit pole at P, r cm. distant from an infinitely long filament of current, the return conductor being infinitely removed. The conditions are represented in Fig. 30, which is a sectional view cut by a plane through P perpendicular to the current $\bar{I}$; the horizontal axis of reference is drawn toward the position of the infinitely distant return conductor.

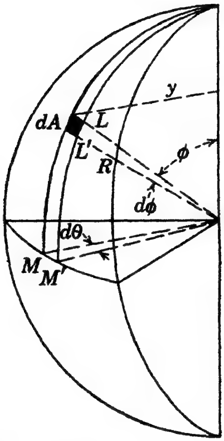


FIG. 31.—Area of spherical lune.

The pyramid of Fig. 29 now becomes the simple dihedral angle θ of Fig. 30, bounded by two planes through point P, one of them including the wire *ab*, the other parallel to the axis of reference. These planes intersect the sphere of unit radius, drawn about P as center, and the spherical area subtended is the solid angle ω of Eq. (36) The shape of this spherical area for a general radius R is indicated in Fig. 31, and its area may be computed as follows:

Consider an elementary area dA bounded by the two meridians M and M', which include the angle $d\theta$, and by the two parallels L and L', which are defined by the angles ϕ and $\phi + d\phi$. Then,

$$dA = y\,d\theta \cdot R\,d\phi$$
$$y = R \sin \phi$$

whence

$$A = R^2 \int_0^{\pi} \sin \phi \, d\phi \int_0^{\theta} d\theta = 2R^2\theta$$

and

$$\omega = \frac{A}{R^2} = 2\theta$$

The potential due to the current in an infinitely long conductor is therefore

$$W = 2\bar{I}\theta$$

In the absence of any other definition of direction, one axis of reference would be as good as another, so that any arbitrary potential can be assigned to a single point; but in any practical *problem there* will be some given condition that fixes, say the

zero of potential, and so defines the axis of reference for the angle θ.

An application of this result can be made to a pair of parallel wires carrying current in opposite directions, as in a power circuit, illustrated in Fig. 32. In this case the rectangle of Fig. 29 becomes infinitely long in the ab direction, and

$$W = 2\bar{I}\theta_1 - 2\bar{I}\theta_2 = 2\bar{I}\theta \tag{37}$$

where θ is the dihedral angle at P between planes passing through the two conductors. The conclusion follows immediately that,

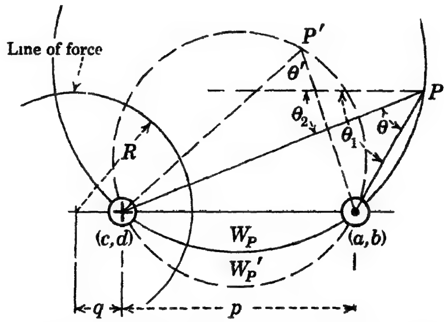


FIG. 32.—Magnetic potential due to a pair of parallel wires.

if θ is constant, W is constant, which is the condition for an equipotential line; but if θ is constant, the locus of point P in Fig. 32 is a circle through P and the points (a,b) and (c,d), therefore having its center on the line perpendicular to the plane of the wires and midway between them. Two such equipotential circles are shown in Fig. 32.

The equipotential lines always intersect the lines of force at right angles. In Fig. 32, the lines of force due to the two parallel but oppositely directed currents are also circles, as illustrated in Fig. 4. The relation between the radius of any of these circular lines of force and the position of its center is derived in Arts. 14 and 18, Chap. IV; for the present it will be sufficient to state that the condition is $(p + q)q = R^2$, the meaning of p, q, and R being shown in Fig. 32.

19. Magnetomotive Force.—Let Fig. 33 represent the side and front elevations of a plane coil of any configuration, having N turns and carrying $\bar{I}$ abamp.; and let a unit magnet pole be placed at *any* point P in the plane of the coil but outside its boundary. Since the solid angle subtended at P by the coil is zero, none of

the flux emitted from the pole will pass through the coil, and therefore the magnetic potential at P is zero. Now let the unit pole be moved along any path, not linking the coil, to a point Q infinitely close to the plane of the coil. The solid angle subtended at Q by the coil is 2π, the magnetic potential at Q is by Eq. (33) equal to $2\pi N\bar{I}$, and the difference of magnetic potential between P and Q is also $2\pi N\bar{I}$. This is the amount of work required to move the unit pole from P to Q.

Similar reasoning shows that the work required to carry the pole from P to Q', which is infinitely close to the plane of the coil on the other side, is $-2\pi N\bar{I}$; or, what amounts to the same thing, the work required to carry the unit pole from Q' to P is $+2\pi N\bar{I}$. It follows, therefore, that *to move a unit magnet pole once around a closed path that links with the coil requires the expenditure of* $4\pi N\bar{I}$ *ergs of work.*

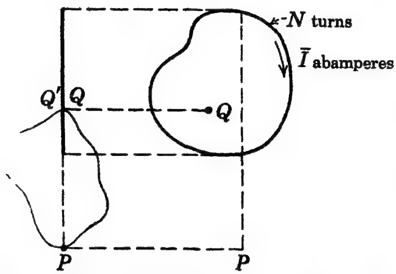


FIG. 33.—Closed path linking with coil

The derivation of this expression $4\pi N\bar{I}$, which is called the *magnetomotive force* (m.m.f.) of the coil, has been made to depend upon the assumption that the coil is a plane coil. However, this assumption was made merely for the sake of simplicity, for it is not necessary that the coil should be plane. Whatever its shape, there will always be at least one point on any closed curve that threads through the coil where the solid angle subtended by the coil is zero and where the magnetic potential is accordingly zero; and there will be two other points, infinitely close together, one of which subtends the solid angle 2π, the other the solid angle -2π. Hence, the m.m.f. of any coil whatever is given by the expression $4\pi N\bar{I}$.

A helix (Fig. 2) is equivalent to N turns all connected in series, so that the work required to move a unit pole once around a closed path linking all the N turns is

$$4\pi N\bar{I} = \frac{4\pi}{10}NI \quad \text{ergs}$$

whereas for a closed path such as P_2 the work required is $4\pi N'\bar{I}$, where N' is the smaller number of turns linked with path P_2.

Magnetomotive force is expressed in terms of a unit called the *gilbert;* in other words, the m.m.f. of a coil of N turns carrying I amp. is said to be $0.4\pi NI$ gilberts. But, since 0.4π is a pure number, it is usually more convenient to deal with the product NI, called the *ampere-turns* of the coil.

It is shown in Art. 8 that the field intensity at the middle point of the axis of a long solenoid is very nearly

$$H_0 = \frac{4\pi NI}{10\ l} = \frac{\text{number of gilberts}}{\text{number of centimeters}}$$

from which it follows that field intensity is expressible not only in oersteds but also in gilberts per centimeter.

The last equation can be written in the form

$$H_0 l = \frac{4\pi}{10}NI = \text{m.m.f.}$$

which means that, if the field intensity were constant from end to end of the solenoid and equal to H_0, the work required to move the unit pole from one end of the solenoid to the other would be

$$\text{Force} \times \text{distance} = H_0 \cdot l = \frac{4\pi}{10}NI$$

Actually, however, the field intensity is not constant all along the axis but varies in the manner indicated in Fig. 22, so that H is a function of position. In any case,

$$\int H\,dl = \frac{4\pi}{10}NI \tag{38}$$

if the integration is made between limits that define a closed path linking with the coil; therefore Eq. (38) expresses the fact that *m.m.f. is the line integral of the magnetic force.*

It is interesting to note that the area under the curve of Fig. 22, within the limits that define l, represents the actual work required to carry a unit magnet pole from end to end of the solenoid. At the same time the area of the rectangle that has the base l and an altitude equal to the maximum ordinate $(4\pi/10)(NI/l)$ is equal to $(4\pi/10)NI$. But the last expression is the total amount of work required to carry the unit pole once around a path *completely linking the entire solenoid;* it follows, therefore, that the crosshatched area in Fig. 22 represents to scale the work

required to carry the unit magnet pole along any path from end to end of the solenoid but *outside* its windings.

The fact that the lines of magnetic induction associated with a magnet or with a coil of current-carrying wire are closed curves gives rise to the concept of the *magnetic circuit*. The term "magnetic flux" is used as though there were something flowing in the magnetic circuit; actually, there is nothing more tangible than a state of stress in the space that includes the circuit, this stress manifesting itself through the force exerted on magnet poles or current-carrying conductors placed therein.

Magnetomotive force bears the same relation to a magnetic circuit that electromotive force bears to an electric circuit; neither is a force, for m.m.f. has been defined as of the nature work per unit magnet pole, whereas e.m.f. is of the nature work per unit electric charge. Otherwise, the analogy between the two is complete; for though m.m.f. is the work required to move a unit pole once around a closed magnetic circuit, e.m.f. is the work required to move a unit electric charge once around a closed electric circuit. The latter fact, not yet proved, is easily seen from the following considerations:

In Art. 9, Chap. I, it is seen that when a battery having e.m.f. E volts and internal resistance r ohms discharges through an external resistance of R ohms,

$$E = Ir + IR = v + V$$

where v is the drop of potential in the battery and V is the drop through the external circuit. If this equation is multiplied through by Q, the quantity of electricity flowing in a time t, we have

$$EQ = vQ + VQ$$

Here, vQ is the work done when Q is moved through the battery itself, and VQ is the work done in the external circuit between the terminals that have the potential difference V. Consequently, EQ is the work done in moving Q once around the circuit; and if $Q = 1$, this work is equal to E, the result thus agreeing with the statement that was to be proved.

20. Law of Magnetic Circuit.—When a long helical winding such as that indicated in Fig. 21 is provided with an iron core. the field intensity

$$H = \frac{4\pi}{10}\frac{NI}{l}$$

is substantially constant over the greater part of its length. Poles are induced at the ends of the core, and the resultant flux density at the central section of the iron core is

$$B = \mu\mu_0 H$$

Actually, the induced poles exert a demagnetizing action, so that the resultant field intensity at the middle of the solenoid is slightly reduced; but if the iron core and its magnetizing winding are bent to form a completely closed ring, as in Fig. 44, the free poles disappear, the demagnetizing action being thus eliminated. The value of H then becomes uniform all along the curved axis of the coil, and in computing its magnitude from the formula $H = 0.4\pi(NI/l)$ the value of l is taken as the circumference of a circle that passes through the centroid of the section of the ring.

The total flux through the ring, if A is its cross-section, is

$$\Phi = AB = A\mu\mu_0 H = \frac{4\pi}{10}\frac{NI}{l}\mu\mu_0 A \tag{39}$$

which may be changed to the form

$$\Phi = \frac{(4\pi/10)NI}{l/\mu\mu_0 A} \tag{40}$$

The form of the denominator, $l/\mu\mu_0 A = l/\mu_a A$, immediately calls to mind the formula for the resistance of a conductor, namely, $\rho l/a$ or $l/\gamma a$, as discussed in Art. 18, Chap. I. In both expressions, lengths and cross-sections agree term by term; and μ_a, which is a measure of magnetic conductivity, agrees with γ, which represents electrical conductivity. Accordingly, $l/\mu_a A$ is a kind of magnetic resistance analogous to electrical resistance; but to distinguish it from resistance it is called* the *reluctance of the magnetic circuit*. Equation (40) therefore has the form

$$\text{Flux (in maxwells)} = \frac{\text{m.m.f. (in gilberts)}}{\text{reluctance}}$$

* Prior to the 1930 (Oslo) meeting of the I.E.C., the unit of reluctance was called the oersted; but at that meeting the term oersted was transferred to the unit of field intensity, so that at present there is no official name for the unit of reluctance.

which corresponds term by term with Ohm's law of the electric circuit,

$$\text{Current (in amperes)} = \frac{\text{e.m.f. (in volts)}}{\text{resistance (in ohms)}}$$

and for this reason Eq. (40) is sometimes called the law of the magnetic circuit.

The concept of the magnetic circuit as analogous to the electric circuit was developed largely by Henry A. Rowland in 1873. It was he also who conceived the notion of m.m.f.

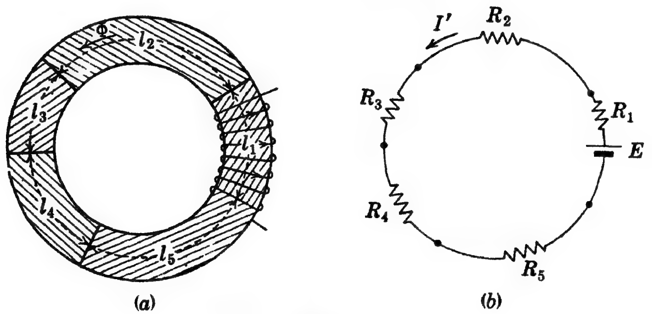


FIG. 34.—Series circuit.

21. Connections of Magnetic Circuits.—The obvious similarity between magnetic and electric circuits indicates that magnetic circuits may be built up of individual parts connected in series, in parallel, and in series-parallel. It is also obvious that rules entirely analogous to Kirchhoff's laws for electrical circuits will apply to them; thus,

1. The algebraic sum of the fluxes at a junction must be equal to zero; or the sum of all fluxes entering a junction must be equal to the sum of all fluxes leaving the same junction.

2. The algebraic sum of all m.m.fs. and drops of magnetic potential in a closed magnetic circuit must be equal to zero.

The following examples will serve to clarify these concepts.

1. *Series Circuits.*—Let any number of segments of magnetic material be assembled to form a closed circuit as in Fig. 34*a*, and let a magnetizing coil of N turns be wound completely around the circuit, only a portion of the winding being shown in the diagram. The sectors will in general have different lengths $l_1, l_2, l_3, \ldots$, different cross-sections $A_1, A_2, A_3, \ldots$, and different absolute permeabilities $\mu_1, \mu_2, \mu_3, \ldots$. When the

coil carries a current of I amp., it will produce a total m.m.f. of $(4\pi/10)NI$ gilberts, and a certain flux Φ will be set up in the circuit. The analogous electrical circuit is shown in Fig. 34*b*. In accordance with Eq. (40),

$$\Phi = \frac{(4\pi/10)NI}{(l_1/\mu_1 A_1) + (l_2/\mu_2 A_2) + \cdots + (l_n/\mu_n A_n)} \tag{41}$$

which is similar to the corresponding expression for the electrical circuit

$$I = \frac{E}{R_1 + R_2 + \cdots + R_n}$$

Transforming Eq. (41) to

$$\frac{4\pi}{10}NI = \Phi\frac{l_1}{\mu_1 A_1} + \Phi\frac{l_2}{\mu_2 A_2} + \cdots + \Phi\frac{l_n}{\mu_n A_n} \tag{42}$$

which corresponds with

$$E = IR_1 + IR_2 + \cdots + IR_n$$

it is obvious that individual terms like $\Phi(l/\mu A)$ represent the fall, or drop, of magnetic potential in the corresponding part of the circuit just as terms like IR represent the drop of electrical potential in an individual resistor. Equation (42) thus checks the second of the two Kirchhoff laws. The relations between the total m.m.f. and the individual drops of magnetic potential are indicated in Fig. 35 which should be compared with Fig. 9, Chap. I.

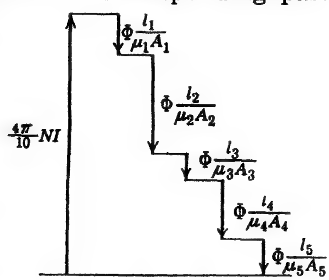


FIG. 35.—Magnetic potential diagram.

If the constituent parts of the series circuit consist of different materials, the (absolute) permeabilities μ_1, μ_2, . . . will have different values even if the flux densities $B_1 = \Phi/A_1$, $B_2 = \Phi/A_2$, . . . are all equal. Consequently, for a given value of the m.m.f., $(4\pi/10)NI$, not only is Φ unknown, but also unknown are all the reluctances of the type $l/\mu A$, and it is therefore impossible to solve either Eq. (41) or Eq. (42) by ordinary algebraic processes. Moreover, if all parts of the circuit were made of the same material, but with cross-sections A_1, A_2, . . . that differ

from part to part, the permeabilities would again all be different, a simple solution being thus precluded. And finally, if all parts are of the same material and have the same section, so that

$$\Phi = \frac{(4\pi/10)NI}{l/\mu_a A}$$

there still remain two unknown quantities, Φ and μ_a, with only one equation relating them. Clearly, therefore, though Eqs. (41) and (42) are entirely correct, they are not sufficient to solve

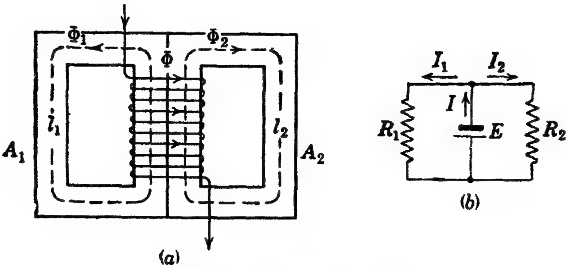


FIG. 36.—Parallel circuits.

for the flux. The procedure to be followed in such cases will be considered in Art. 24.

2. *Parallel Circuits.*—Figure 36*a* represents two independent circuits, of which the mean lengths are l_1 and l_2 and the sections A_1 and A_2, both magnetized by a single coil that produces the m.m.f. $(4\pi/10)NI$ gilberts. In this case,

$$\Phi_1 = \frac{(4\pi/10)NI}{l_1/\mu_1 A_1} = \frac{4\pi}{10}NI\frac{\mu_1 A_1}{l_1}$$

and

$$\Phi_2 = \frac{(4\pi/10)NI}{l_2/\mu_2 A_2} = \frac{4\pi}{10}NI\frac{\mu_2 A_2}{l_2}$$

The sum of these two expressions will give the total flux in the central core, or

$$\Phi = \Phi_1 + \Phi_2 = \frac{4\pi}{10}NI\left(\frac{\mu_1 A_1}{l_1} + \frac{\mu_2 A_2}{l_2}\right) \tag{43}$$

just as in the electrical analogue (Fig. 36*b*) the battery current is given by

$$I = I_1 + I_2 = E\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Terms like $\mu_1 A_1/l_1$, which are the reciprocals of reluctance, are called *permeances* and correspond to electrical conductance. It is seen that permeances of circuits in parallel are additive, just as in the analogous electrical case the conductances are additive.

In this case, as in case (1), μ_1 and μ_2 are not known until Φ and the corresponding flux densities $B_1 = \Phi_1/A_1$ and $B_2 = \Phi_2/A_2$ are known; so Eq. (43) is again not sufficient for a complete solution.

3. *Series-parallel Circuits.*—The analogy between magnetic circuits that present combinations of series and parallel connections, and the corresponding electrical circuits, is so readily seen that further consideration will be left to develop as the analysis proceeds.

22. Modified Form of Magnetization (B-H) Curves.—In Eq. (42) each of the terms on the right-hand side is of the form

$$\Phi \frac{l_x}{\mu_x A_x} = \frac{\Phi}{A_x} \cdot \frac{l_x}{\mu_x} = \frac{B_x}{\mu_x} l_x = H_x l_x$$

where H_x is the magnetizing force, in oersteds, operative in the corresponding part of the circuit. That is, the field intensity changes abruptly from part to part of the circuit whenever there is a change due to either a change in material or a change in the flux density B. Each value of H_x may be regarded as the result of an excitation $(4\pi/10)(NI)_x$ such that

$$H_x = \frac{4\pi}{10} \frac{(NI)_x}{l_x} \tag{44}$$

from which it follows that Eq. (42) can be written

$$NI = (NI)_1 + (NI)_2 + \cdot \cdot \cdot + (NI)_n \tag{45}$$

The product NI, which represents the total number of *ampere-turns* applied to the circuit, may therefore be considered to be subdivided into as many parts as there are separate parts in the circuit of Fig. 34, each part consuming a portion of the whole number of ampere-turns in proportion to its reluctance.

In Eq. (44) the ratio $(NI)_x/l_x$ may be described as the number of *ampere-turns per centimeter*, (amp.-turns/cm.), and as this value is proportional to H_x it is a simple matter to replace H in a normal B-H curve by the equivalent amp.-turns/cm. Thus,

$$H = \frac{4\pi}{10} \frac{\text{amp.-turns}}{\text{cm.}} = 1.256 \frac{\text{amp.-turns}}{\text{cm.}}$$

or

$$\frac{\text{Amp.-turns}}{\text{Cm.}} = 0.8H$$

In English and American practice, where lengths are commonly expressed in inch units,

$$H = \frac{1.256}{2.54}\,\frac{\text{amp.-turns}}{\text{in.}} = 0.495\,\frac{\text{amp.-turns}}{\text{in.}}$$

or

$$\frac{\text{Amp.-turns}}{\text{In.}} = 2.02H$$

It follows that if in a series circuit like Fig. 34*a* the values of H in the successive parts correspond to (amp.-turns/in.)$_1$, (amp.-turns/in.)$_2$ . . . , Eq. (45) takes the form

$$NI = \left(\frac{\text{a.-t.}}{\text{in.}}\right)_1 l_1'' + \left(\frac{\text{a.-t.}}{\text{in.}}\right)_2 l_2'' + \cdots + \left(\frac{\text{a.-t.}}{\text{in.}}\right)_n l_n'' \qquad (46)$$

To facilitate calculations based upon this analysis, the magnetization curves used in practice are plotted with *ampere-turns per inch* as abscissas and flux density in *lines* (or kilolines) *per square inch* as ordinates, in the manner shown in Fig. 37. In this diagram the corresponding scales for B and H (in metric units) are also indicated. From these curves it is possible to compute $\mu_a = \mu\mu_0 = B/H$ and its reciprocal $1/\mu_a$, called the *reluctivity*, which when plotted against excitation gives the curves of Fig. 38.

23. Experimental Determination of B-H Curves.—The ring method for the experimental determination of B-H curves, which exemplifies the basic principles of other procedures, is illustrated in Fig. 39. Here T is a sample of the magnetic material to be tested, made in the form of a toroid with circular cross-section. It is wound with a uniformly distributed magnetizing coil of N_1 turns which receives current from a storage battery B_1 through the reversing switch S_1, regulating rheostat R_1, and ammeter I_1. Wound around the test ring T there is also an exploring, or search, coil having n_1 turns, which is connected to a ballistic galvanometer in series with another search coil of n_2 turns, wound around the middle of the calibrating solenoid. The latter is a long helix, having N_2 turns uniformly

distributed along its length l_2, which may be energized from battery B_2 by way of the reversing switch S_2, regulating rheostat

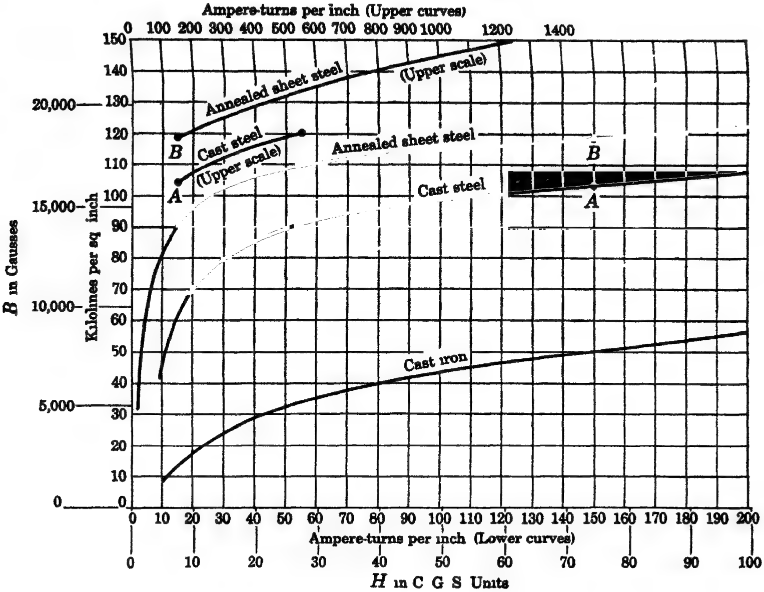


FIG. 37.—*B-H* curves.

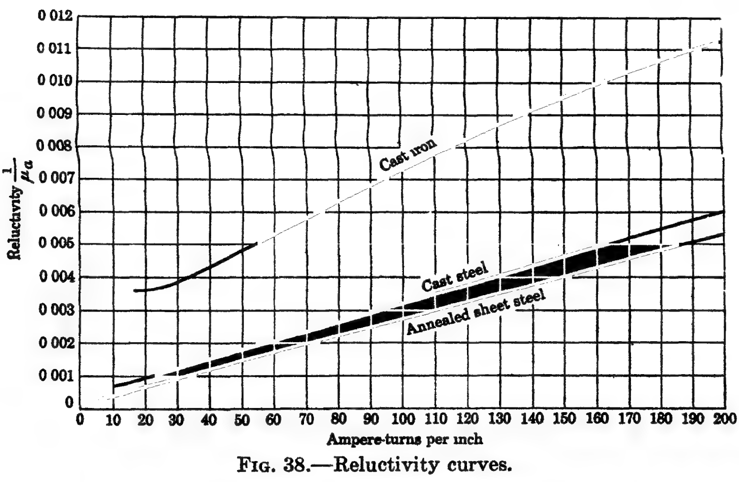


FIG. 38.—Reluctivity curves.

R_2, and ammeter I_2. The nonmagnetic core of the calibrating solenoid must be made of material that will not warp, expand, or contract with changing temperature and humidity

If a current of I_2 amp. is made to flow in the N_2 turns of the calibrating solenoid, the field intensity at its middle cross-section will be $H = (4\pi/10)(N_2I_2/l_2)$, and the number of flux linkages with the search coil will be $\lambda_2 = (4\pi/10)(N_2I_2/l_2)\mu_0A_2n_2$, where A_2 sq. cm. is the cross-section of the solenoid. While the current I_2 is being adjusted to the desired value, the ballistic galvanometer should be short-circuited; when the current has the steady value I_2, the galvanometer short circuit is removed and switch S_2 is suddenly reversed. An e.m.f. is thus induced in coil n_2 due to the change in flux linkages $2\lambda_2$, and the galvanometer

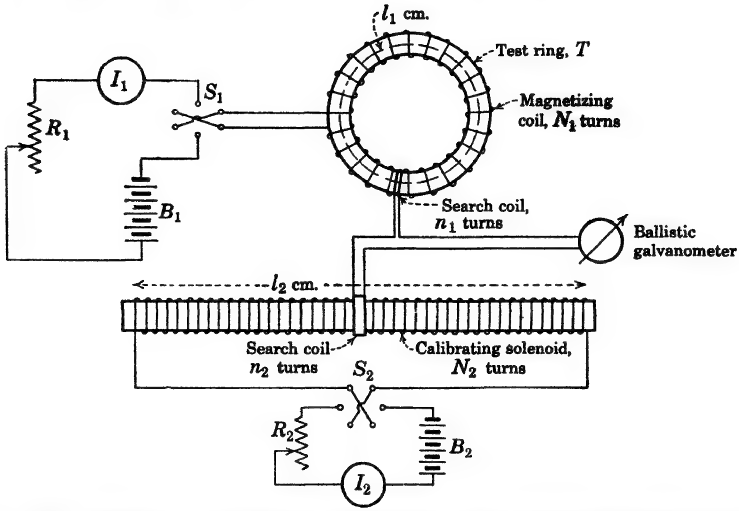


FIG. 39.—Diagram of connections, magnetization and hysteresis measurements.

will give a deflection of D_2 scale divisions. The ratio $K = \lambda_2/D_2$ fixes the number of flux linkages per unit scale deflection, and this ratio will remain unchanged so long as the resistance of the galvanometer circuit remains unaltered. The object of the rheostat R_2 is to permit variation of the current I_2 through a considerable range, the value of K being thus established as the average of a large number of observations.

The galvanometer having been thus calibrated, the test ring is magnetized by adjusting the exciting current I_1 to any desired value by means of R_1, the galvanometer being meanwhile short-circuited. Upon removing the short circuit, switch S_1 is suddenly reversed, whereupon the galvanometer deflection D_1 will be a measure of the change of flux linkages for the search coil n_1;

the flux itself will be $\Phi = KD_1/n_1$, and on dividing this by A_1 (the cross-section of the toroid) there is obtained the flux density B that corresponds to the field intensity $H = (4\pi/10)(N_1I_1/l_1)$, where l_1 is the mean length of the ring. Proceeding in this manner, varying the exciting current I_1 through a sufficient range, we can fix any desired number of points on the B-H curve.

The ring method has the disadvantage of being slow and expensive when large numbers of specimens are to be tested, since each sample must first be machined to ring form and then be wound by hand. The *Koepsel permeameter* (Fig. 40) permits the use of simple rod-shaped specimens R, 6 mm. in diameter, or 6 mm. square, which are held in place in the massive iron yokes Y by means of clamps and setscrews. The flux produced by the adjustable current in the exciting winding W passes through the test specimen and the yokes and crosses the small airgap in which the pivoted coil C is mounted as in a Weston ammeter. Coil C is supplied with a current of 0.015 to 0.018 amp. from a 4-volt source, the current being held constant by a regulating rheostat. The torque acting upon coil C is proportional to the product of the flux and the current I_c; and since I_c is constant, the resulting deflection of the coil is proportional to the airgap flux.

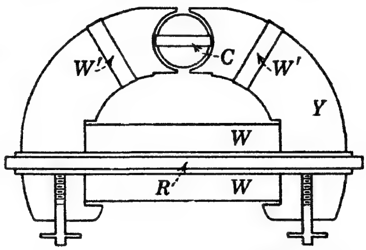


FIG. 40.—Koepsel permeameter.

If all the flux produced by winding W reached the airgap, the deflection would be proportional to the flux in the specimen, and therefore proportional to the flux density B therein. However, some of the flux leaks across between the yokes; so for accurate work it is necessary to calibrate the apparatus by determining experimentally the percentage of the flux in the specimen that reaches the airgap, a requirement that can be met once for all by means of search coils. The percentage thus determined will vary somewhat as the excitation of coil W is changed, but the results can be summarized in the form of charts which give the correction factor to be applied to the instrument readings.

Current in the winding W would produce some deflection of coil C even if there were no specimen in the yokes. For this reason the compensating coils W', in series with, but opposing, coil W,

are wound around the yokes close to the airgap; they are adjusted until the deflection is zero when current flows through W, with the test specimen removed.

The value of H in the test specimen is approximately equal to $(4\pi/10)/(NI/l)$, where N is the number of turns in coil W, I is the current in W, and l is the length of the specimen in centimeters. To facilitate calculations, the quantity $(4\pi/10)(N/l)$, is made equal to 100, a value which means that N/l (that is, the turns per centimeter in coil W) is equal to 79.6. Actually, the value of H in the specimen is less than $100I$, since part of the excitation is consumed in the yokes and the airgap, and for accurate work it is necessary to make appropriate corrections.

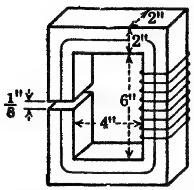


FIG. 41.—Example of series circuit.

24. Examples of Magnetic Calculations.—1. Let it be required to find the number of ampere-turns necessary to produce a total flux $\Phi = 160{,}000$ maxwells in the cast-iron core illustrated in Fig. 41. The magnetic circuit is made up of a cast-iron part in series with an airgap. For present purposes, it is sufficiently accurate to assume that the mean path of the lines of induction in the iron follows the center of gravity of the cross-section and that at the corners the mean path follows quadrants of circles; also, that the flux in the airgap passes straight across from one pole face to the other without fringing at the edges. It follows that in both the iron and the air the flux density, in lines per square inch, is

$$B'' = \frac{160{,}000}{4} = 40{,}000$$

and the lengths of the paths in the iron and the air are

$$l_1'' = 2(6 + 4) + 2\pi - 0.125 = 26.15 \text{ in.}$$
$$l_2'' = 0.125 \text{ in.}$$

Ampere-turns required by the iron: From the curve for cast iron in Fig. 37 it is found that a flux density of 40,000 lines per sq in. requires an excitation of 79 amp.-turns per in. of core. Therefore,

Ampere-turns required by the iron $= 79 \times 26.15 = 2060$

Ampere-turns required by the airgap: In air, where $\mu = 1$,

$$B = \mu_0 H = \mu_0 \frac{4\pi}{10}\frac{NI}{l}$$

where all quantities are expressed in metric units; and since $\mu_0 = 1$,

$$NI = \frac{10}{4\pi}Bl = 0.8\left(\frac{B''}{(2.54)^2}\right)(l'' \times 2.54) = 0.3133B''l''$$

where B'' = flux density in lines per square inch.
l'' = length of path in inches.

Consequently, in the given problem.

Ampere-turns required by airgap = 0.3133 × 40,000 × 0.125 = 1567

The total excitation for the entire circuit is 2060 + 1567 = 3627 ampere-turns.

It is important to observe that, although the length of path in the iron is 26.15/0.125 = 209 times that of the airgap, the latter takes 43 per cent, or nearly half, of the entire excitation. This result is due to the fact that the permeability of the iron* is 158.5 times that of the airgap, so that 1 in. of airgap requires as much excitation as 158.5 in. of cast iron when B'' = 40,000

2. Suppose now that it is required to find the total flux Φ which will be set up in the circuit of Fig. 41 if the total excitation is 4000 amp.-turns. This case exemplifies the condition discussed in connection with Eq. (41) of Art. 21, for H, B, and μ in the iron are not known. It is therefore necessary to *assume* a value for the flux density in the iron (therefore in the airgap also) and then to proceed as in case 1 to find the corresponding excitation; in general, this procedure will lead to a result that does not agree with the given excitation, and so another trial is made. Good judgment in assuming the values of B will lead to the correct result in three trials, thus:

In the problem under consideration, it is evident that the flux density will exceed 40,000 lines per sq. in., for that value corresponds to a total excitation of 3627 amp.-turns, which is less

* The flux density in the iron, in metric units, is $B = 40{,}000/(2.54)^2 = 6200$, and $H = 0.495 \times$ amp.-turns/in. $= 0.495 \times 79 = 39.1$; thus, we have $\mu = B/\mu_0 H = 158.5$.

than the given excitation. The calculations in case 1 are therefore equivalent to one trial solution of case 2.

Next assume that $B'' = 45{,}000$; the curve for cast iron in Fig. 37 shows that 110 amp.-turns per in. will be required, and so the total excitation will be

$$110 \times 26.15 + 0.3133 \times 45{,}000 \times 0.125 = 4635$$

This result is greater than the given excitation, and hence the flux density lies between 40,000 and 45,000.

Upon trying 42,000 lines per sq. in., corresponding to 91 amp.-turns per in. in the cast iron, the excitation is found to be

$$91 \times 26.15 + 0.3133 \times 42{,}000 \times 0.125 = 4025$$

or a little too large. If the assumed values of B'' and their corresponding values of ampere-turns are now plotted, as in Fig.

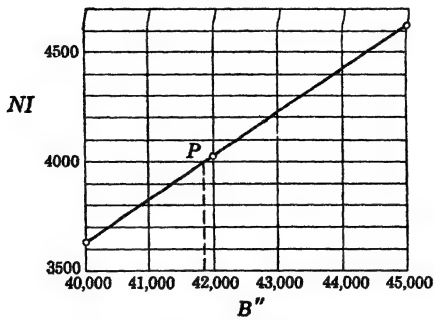


FIG. 42.—Graphical solution of problem.

42, and a smooth curve is drawn through the three points that have been determined, the curve will intersect the horizontal line through $NI = 4000$ at a point P of which the abscissa, in this case $B'' = 41{,}850$, is the required value. Accordingly, $\Phi = 4 \times 41{,}850 = 167{,}400$.

It should be noted that the particular values used in this last example relate to points on the magnetization curve of the cast iron that are so far beyond the knee of the curve as to account for the extreme flatness of the curve of Fig. 42. In such cases it is sufficiently accurate for practical purposes to compute the position of point P by a simple interpolation between the two points first located; in general, however, if the given conditions involve points on the knee of the magnetization curve, the curve

of Fig. 42 will bend appreciably, and at least three points are required to define its shape.

In numerous cases of this type, especially when the iron is well below the condition of saturation, the airgap will have by far the larger part of the total reluctance. As a first approximation to the final solution, it may be assumed that the airgap consumes all the given excitation; then

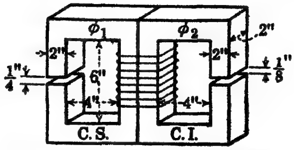


FIG. 43.—Circuits in parallel.

$$B'' = \frac{\text{given ampere-turns}}{0.3133 l''_{gap}}$$

as an upper limit for the required flux density.

3. In Fig. 43 the cast-iron and cast-steel circuits, with airgaps of ⅛ and ¼ in., are in parallel and are jointly subjected to the magnetizing action of a coil that *develops an excitation of 4,000 amp.-turns.* The cast-iron circuit, identical with that of Fig. 41, will develop a total flux of 167,400 lines, as in case 2. The flux in the cast-steel circuit must be found in the manner outlined in case 2. It is found to be approximately 190,000 lines, so that the total flux through the exciting winding is 357,400 lines.

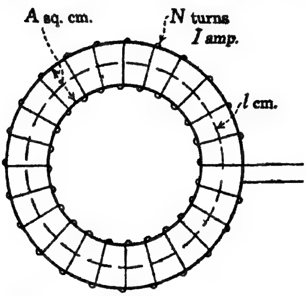


FIG. 44.—Ring-wound core.

25. Hysteresis Loss in Iron Cores.—Assume that a ring-shaped core of iron or steel, as shown in Fig. 44, has a mean length of l cm. and a cross-section of A sq. cm. and that it is wound with a uniformly distributed coil of N turns. Let the current in the coil be increased from I to $I + dI$, so that the magnetizing force

$$H = \frac{4\pi}{10}\frac{NI}{l}$$

increases by dH and the flux density by dB. The flux linked with the magnetizing coil will increase by the amount $d\varphi = A\,dB$, and the e.m.f. induced in the coil will be

$$E = -N\frac{d\varphi}{dt} \times 10^{-8} = -AN\frac{dB}{dt} \times 10^{-8} \quad \text{volts}$$

In accordance with Lenz's law this e.m.f. will be so directed as to oppose the original increase in the current from I to $I + dI$, and in order that the current may be held at the value I the source of the current must supply an equal and opposite e.m.f.; consequently in the time dt there will be consumed an amount of energy $dW = -EI\,dt$. Since

$$I = \frac{10}{4\pi}\frac{Hl}{N}$$

it follows that

$$dW = \frac{Al}{4\pi}H\,dB \quad \text{ergs}$$

But $H\,dB$ represents the area of the crosshatched element indicated in Fig. 45; hence, the total energy dissipated in the interval represented by the portion of the loop from a to b is

$$W = \frac{Al}{4\pi}\int_{B=0}^{B=B_{max}} H\,dB$$

where the integral represents the area $OabP$.

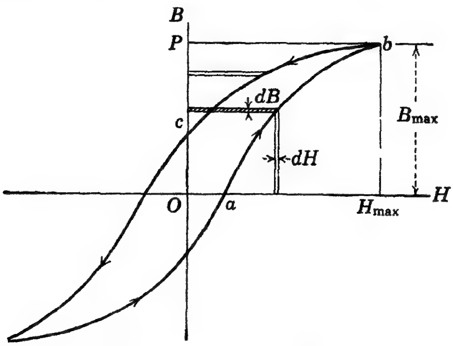


FIG. 45.—Energy loss due to hysteresis.

During the interval between points b and c the flux is decreasing instead of increasing, while the current is in the same direction as before, so that energy is now returned to the circuit instead of being absorbed. This returned energy is

$$W' = \frac{Al}{4\pi}\int_{B=B_{max}}^{B=Oc} H\,dB$$

and the integral now represents the area bPc. The net energy

loss in describing the part of the loop *abc* is therefore proportional to the area *Oabc*. If the remainder of the complete cycle is treated in a similar manner, the total dissipation of energy is

$$W = \frac{Al}{4\pi}\int_{-B_{max}}^{+B_{max}} H\,dB \quad \text{ergs} \tag{47}$$

the integral now representing the area included in the entire hysteresis loop.

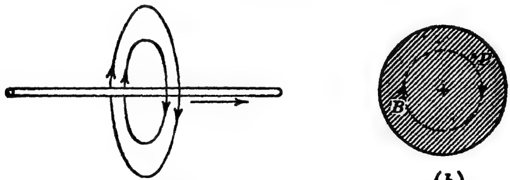
In Eq. (47) the product Al represents the volume of the iron core. It may therefore be concluded that the energy lost in describing one complete loop (that is, in one complete cycle) is proportional to the volume (and therefore also to the weight) of the core and also to the area of the loop expressed in appropriate units. Since the relation between B and H is wholly empirical, the integral cannot be evaluated mathematically, and it must therefore be determined experimentally. It has thus been found by Steinmetz that the loss is proportional to $(B_{max.})^n$ where n varies somewhat with different kinds of iron and steel but usually has a value in the neighborhood of 1.6. Therefore the hysteresis loss per cubic centimeter per cycle is approximately

$$W = \text{constant} \times (B_{max})^{1.6} \tag{48}$$

The experimental determination of points on the hysteresis loop can be made by means of the apparatus shown in Fig. 39; but instead of using the reversing switch S_1, the increments of current I and of H are produced by quick changes in R (as by using a plug box), the resulting swing of the galvanometer serving to measure the corresponding increment of B.

26. Self-induction. Inductance.—In the period following Oersted's discovery, several experimenters observed that the spark produced on breaking an electric circuit was noticeably increased in intensity if the wires, instead of being substantially straight, were wound in the form of coils. Joseph Henry, working with powerful electromagnets, studied this effect and in 1832 published its explanation, showing that it is due to the e.m.f. *self-induced* in the circuit by the collapse of its own magnetic field on breaking the circuit. Faraday, who worked on the same problem simultaneously with Henry, but independently, used the phrase "extra current" to describe the tendency of the current to persist (as shown by the spark) on rupturing the

circuit. When a straight wire (Fig. 46*a*) carries a current in the direction indicated, the wire will be surrounded by lines of magnetic force as shown. As the current increases from zero to any arbitrary value, the flux increases proportionally from



(*a*) (*b*)

FIG. 46.—*Lines of force surrounding a conductor.*

zero, provided that the wire is surrounded by air or other non-magnetic medium, and may be thought of as issuing from the center of the wire and expanding outward, like spreading ripples on a pond. The lines of force, thus expanding, cut across the wire in the manner indicated in Fig. 46*b*, which represents a cross-section of the wire in (*a*) when viewed from the left. The expanding line of force B is about to cut across the longitudinal filament of the wire shown at P, the motion of the line of force at this point being radially outward. Relatively, the effect is the same as though the filament were moving radially inward, so that if Fleming's (right-hand) rule is applied, it is found that the induced e.m.f. is directed outward from the plane of the paper, or in opposition to the direction of the current flow. The whole effect is in accord with Lenz's law; the original increase in the current strength that produced the change in the flux immediately calls into existence an opposing e.m.f. which tends to retard the change in current. Conversely, the same line of reasoning will show that an initial decrease of current induces an e.m.f. of reversed direction, which tends to maintain the current at its original strength. This e.m.f., being self-induced, is called the *e.m.f.* of *self-induction.*

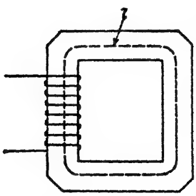


FIG. 47.—Inductive circuit.

Let Fig. 47 represent a coil of wire wound on a core having constant permeability $\mu_a = \mu\mu_0$, a cross-section of A sq. cm., and a mean length of magnetic path of l cm. The mean path is to be taken as passing through the center of gravity of the cross-section of the core. On passing a current of i amp. through

the coil there will be produced a flux

$$\phi = \frac{(4\pi/10)Ni}{l/\mu_a A} = \frac{4\pi}{10}\frac{Ni}{l}\mu_a A \tag{49}$$

and a change of current di will produce a change of flux

$$d\phi = \frac{4\pi}{10}\frac{N\,di}{l}\mu_a A$$

This change of flux will induce an e.m.f. (in volts)

$$e = -N\frac{d\phi}{dt} \times 10^{-8} = -\frac{4\pi}{10}\frac{N^2\mu_a A}{l}\frac{di}{dt} \times 10^{-8} = -L\frac{di}{dt} \tag{50}$$

where

$$L = \frac{4\pi}{10}\frac{N^2\mu_a A}{l} \times 10^{-8} \tag{51}$$

The quantity L is called the *coefficient of self-induction* or briefly the *inductance* of the circuit and in the practical system of units is measured in terms of a unit called the *henry*. It is evident from Eq. (51) that the inductance is proportional to the square of the number of turns linked with the flux and is dependent upon the shape, size, and material of the magnetic circuit. Its magnitude is of great importance in all electrical circuits in which the current is changing in strength, as, for instance, in those coils of a d-c generator or motor which are undergoing commutation (Chap. XII).

From Eq. (50) it is seen that the inductance L of a circuit is numerically equal to the e.m.f. induced in the circuit by a current which is changing at the rate of 1 amp. per sec. ($di/dt = 1$); that is, *a circuit has an inductance of* 1 *henry if a change of current of* 1 *amp. per sec. induces an e.m.f. of* 1 *volt.* The inductance may also be defined in another way. Thus, from Eq. (50),

$$L\frac{di}{dt} = N\frac{d\phi}{dt} \times 10^{-8}$$

or

$$L = N\frac{d\phi}{di} \times 10^{-8} \tag{52}$$

In this equation $d\phi/di$ is numerically equal to the rate of change of flux with current, or it is the number of lines of induction produced by 1 amp. Equation (52) states that the product of the number of lines produced by 1 amp., multiplied by the number of turns with which this flux links, and divided by 10^8, is equal to

the coefficient of self-induction. The product of flux per ampere by the number of turns with which this flux links is called the number of flux linkages per ampere, so that, briefly, *the inductance in henrys is equal to the number of flux linkages per ampere, divided by* 10^8.

If flux is expressed in webers, as in the m.k.s. system, the factor 10^{-8} disappears, in which case

$$L = N\frac{d\phi_1}{di}$$

27. Inductance of a Pair of Parallel Wires.—The inductance of parallel conductors carrying equal currents in opposite directions enters into all calculations concerning the performance

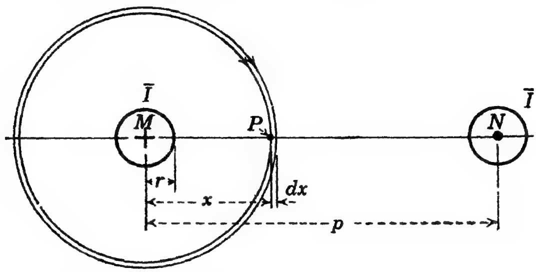


FIG. 48.—Flux linkages, pair of parallel wires.

characteristics of power-transmission lines and telephone circuits. It is therefore important to be able to compute its magnitude for given sizes and spacings of the conductors. The analysis given below is not entirely rigorous in theory, though the resultant formula is correct; an exact derivation is given in Art. 35.

In Fig. 48 are shown two wires M and N, each of radius r cm., spaced p cm. apart center to center, and carrying equal and opposite currents of $\bar{I}$ abamp. It is assumed that the wires are of nonmagnetic material and are surrounded by air. Considering wire M only, the flux density due to it at a point P on the line joining the centers of M and N is

$$B = \frac{2\mu_0\bar{I}}{x}$$

and the corresponding flux through a strip of width dx and length 1 cm. along the wires is

$$d\phi = \frac{2\mu_0\bar{I}}{x}dx$$

which may be regarded as linking with M alone. The flux thus linking with M continues past wire N to infinity, but the part with which we are concerned is that which lies between M and N; if the diameter of the wires $2r$ is small in comparison with p, it is a reasonable approximation to consider the flux ϕ between $x = r$ and $x = p$, and hence

$$\phi = 2\mu_0\bar{I}\int_r^p \frac{dx}{x} = 2\mu_0\bar{I}\ln\frac{p}{r} \qquad (53)$$

In addition to the flux ϕ thus encircling M, there is to be considered the magnetic field within the substance of M itself, as indicated in Fig. 49. It is shown in Art. 5 that at a distance ρ from the center the internal field is produced by that part of the total current which lies inside the circle of radius ρ; and if the total current ($\bar{I}$) is assumed to be uniformly distributed, this partial current is $\bar{I}\pi\rho^2/\pi r^2$. The field intensity at the element $d\rho$ is therefore

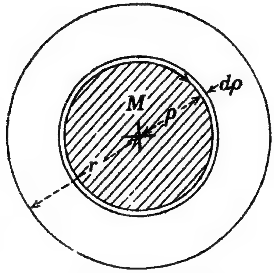


FIG. 49.—Internal flux linkages in conductor.

$$B = \frac{2\mu_0\bar{I}(\rho^2/r^2)}{\rho} = \frac{2\mu_0\bar{I}\rho}{r^2}$$

and the elementary internal flux, per centimeter length of wire, is

$$d\phi_i = \frac{2\mu_0\bar{I}\rho\,d\rho}{r^2}$$

Since this flux links only with the partial current $\bar{I}\rho^2/r^2$, $d\phi_i$ is equivalent to a proportionately reduced flux $d\phi'$, which links with the entire current $\bar{I}$; hence,

$$d\phi' = \frac{2\mu_0\bar{I}\rho^3\,d\rho}{}$$

and

$$\phi' = \frac{2\mu_0\bar{I}}{r^4}\int_0^r \rho^3\,d\rho = \frac{\mu_0\bar{I}}{2} \qquad (54)$$

The total linkage with conductor M is

$$\Phi = \phi + \phi' = \mu_0\bar{I}\left(2\ln\frac{p}{r} + 0.5\right)$$

and the inductance of this conductor (flux linkages per ampere $\times 10^{-8}$) is, per centimeter of its length,

$$L = \frac{\Phi}{I} \times 10^{-8} = \mu_0\left(2 \ln \frac{p}{r} + 0.5\right) \times 10^{-9} \quad \text{henry} \qquad (55)$$

The inductance of conductor N being equal to that of M, *the total inductance of both wires, per centimeter*, is

$$L = \mu_0\left(4 \ln \frac{p}{r} + 1\right) \times 10^{-9} \quad \text{henry} \qquad (56)$$

Formulas (55) and (56) are derived on the assumption that both the wires and the medium surrounding them are nonmagnetic, or that $\mu_a = \mu\mu_0 = 1$. If the relative permeability μ is greater than unity (but constant in magnitude), the values of L given in Eqs. (55) and (56) must be multiplied by μ.

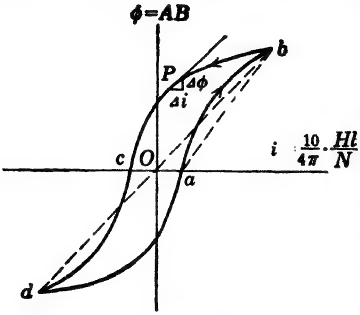


FIG. 50.—Variable inductance.

28. Inductance of Circuits Containing Iron.—The derivation of the equations in Arts. 26 and 27 is based on the explicit assumption that the medium associated with the electric circuit has constant permeability. This assumption is valid if the medium is nonmagnetic, such as air or vacuum, where $\mu = 1$; and, in circuits that include iron cores, if the ratio B/H remains constant, in which case the magnetization curve is a straight line through the origin. In general, however, the magnetization curve of iron or steel is not a straight line, except for a limited portion; hence, μ is variable and the inductance of circuits with such cores is also variable as the current changes.

In the general case of a circuit containing an iron core, where the current varies cyclically from a negative to an equal positive maximum, the relations between instantaneous values of flux ϕ and current i may be represented by Fig. 50. At the particular instant indicated by point P, a change of current Δi will be accompanied by a change of flux $\Delta\phi$ (both negative in the figure); therefore, at that instant, the inductance L will be proportional to the limit of the ratio $\Delta\phi/\Delta i$ as $\Delta\phi$ and Δi approach zero, in accordance with Eq. (52), which is independent of any assump-

tion concerning the constancy of μ. Thus, at any instant the inductance is proportional to the slope of the curve; it is positive unless the hysteresis loop has concave sections.

It follows that in the interval defined by the curve ab the *average* value of the inductance is proportional to the slope of the straight line ab, and for the entire cycle the average inductance is proportional to the slope of the line dOb.

On comparing Eqs. (49) and (51), it is seen that the flux ϕ may be expressed as

$$\phi = \frac{10^8}{N}Li \tag{57}$$

so that, when ϕ, L, and i are all variable, the e.m.f. induced by the variation of ϕ is

$$e = -N\frac{d\phi}{dt} \times 10^{-8} = -\left(L\frac{di}{dt} + i\frac{dL}{dt}\right) \tag{58}$$

which reduces to Eq. (50) if L is constant ($dL/dt = 0$).

29. Growth of Current in Inductive Circuit.—Let Fig. 51 represent a battery of constant e.m.f. E volts connected through a switch S to a circuit that has a total resistance of R ohms and an inductance of L henrys. The inductance L is provided by coiling the conductor, and it will be assumed that L is constant. In general, such a coil will have resistance as well as inductance; but this resistance, together with that of the battery, is lumped into the single equivalent resistance R, as though the resistance of the coil were wholly external to its own windings.

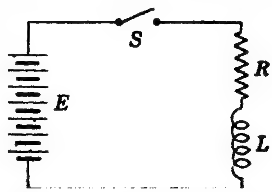


FIG. 51.—Inductive circuit.

Let the switch be closed at the time $t = 0$. At a general time t sec. later, the current, which was initially zero, has attained a value i amp. At that instant there is a drop of potential iR volts in the resistance; and if the current is increasing at the rate of di/dt amp. per sec., the e.m.f. of self-induction will be $e = -L(di/dt)$, acting in opposition to this increase of current. The battery must supply an equal and opposite e.m.f. if the current is to be maintained; hence,

$$E = iR + L\frac{di}{dt} \tag{59}$$

Equation (59), when rearranged to separate the variables i and t, becomes

$$\frac{di}{E - iR} = \frac{1}{L}dt \tag{60}$$

Since $i = 0$ when $t = 0$ and $i = i$ when $t = t$, it is possible to integrate both sides of this equation between these values as limits, or

$$\int_0^i \frac{di}{E - iR} = \frac{1}{L}\int_0^t dt$$

whence

$$\ln \frac{E - iR}{E} = -\frac{Rt}{L}$$

and

$$i = \frac{E}{R}(1 - e^{-Rt/L}) \tag{61}$$

Equation (61) shows that the current is an exponential function of the independent variable t, the time. At the first instant $t = 0$, the current is also zero; and after an infinite time ($t = \infty$), the current is $i = E/R$, which agrees with Ohm's law. Differentiating i with respect to t in Eq. (61),

$$\frac{\quad}{dt} \qquad\qquad E \tag{62}$$

which shows that at the first instant, $t = 0$, the current increases at its greatest rate E/L, and thereafter the rate gradually diminishes

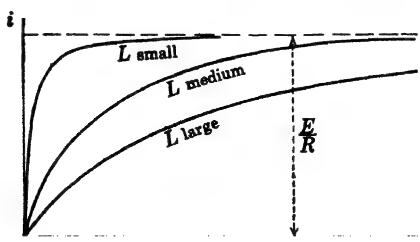


FIG. 52.—Growth of current in inductive circuit.

until the increase ceases after a theoretically infinite time, when $i = E/R$.

The graph of current as a function of time is indicated in Fig. 52. Though a theoretically infinite time is required to reach the limiting value E/R, this value is practically attained in a

comparatively short time; however, if L is very large as compared with R, as is the case in the field exciting windings of large generators and motors, the time required for the current to reach its final steady value may be very appreciable, a minute or two.

Inspection of the form of the exponential term $e^{-(Rt/L)} = e^{-\frac{t}{L/R}}$ shows that the ratio L/R must have the dimensions of time in order that $(t)/(L/R)$ may be a pure numeric. Accordingly, L/R is called the *time constant* of the circuit. If, in Eq. (61), $t = L/R$, the current becomes

$$i = \frac{E}{R}\left(1 - \frac{1}{e}\right) = \frac{E}{R}\left(1 - \frac{1}{2.718}\right) = 0.632\frac{E}{R}$$

and therefore the current reaches 63.2 per cent of its limiting value in the time $t = L/R$.

30. Decay of Current in Inductive Circuit.—Suppose that a circuit like that of Fig. 51, with the switch closed, has been left to itself for a sufficient time to allow the current to reach its limiting value of E/R amp. and that the circuit comprising R and L is then suddenly short-circuited through a switch S', Fig. 53, so that the resultant closed circuit has the same resistance as that of the original circuit in Fig. 51.

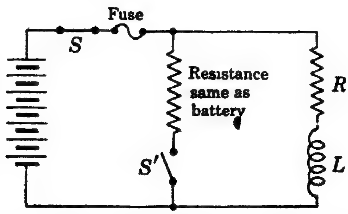


FIG. 53.—Discharge of inductive circuit.

Closure of the switch S' would short-circuit the battery, and damage must be prevented by means of the protective fuse indicated.

The time being counted from the moment when switch S' is closed and the current at this moment being $I = E/R$, the current i at any time t thereafter must satisfy the equation

$$0 = iR + L\frac{di}{dt} \tag{63}$$

which follows from Eq. (59) since the battery has been removed from the circuit. Therefore,

$$\frac{di}{i} = -\frac{R}{L}dt$$

and

$$\int_{E/R}^{i} \frac{di}{i} = -\frac{R}{L}\int_{0}^{t} dt$$

the limits being defined by the given physical conditions. It follows that

$$\ln \frac{i}{E/R} = -\frac{R}{L}t$$

or

$$i = \frac{E}{R}e^{-Rt/L} \tag{64}$$

The graph of this equation is shown in Fig. 54 for several values of L. It is readily seen, on comparing Eqs. (61) and (64), that the curve of decay of current is the same as the curve for growth, but turned upside down.

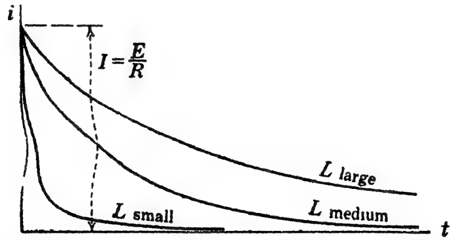


FIG. 54.—Decay of current, inductive circuit.

31. Energy Stored in Magnetic Field.—Equation (59) in Art. 29,

$$E = iR + L\frac{di}{dt}$$

states that, while the current is changing in magnitude, the battery e.m.f. is at all times equal to the sum of the resistance drop and the e.m.f. required to balance the e.m.f. of self-induction. At the first instant ($t = 0$) when $i = 0$, there is no drop in the resistance, and the battery e.m.f. is all consumed in overcoming the effect of the inductance; at the end of the process, when $di/dt = 0$, there is no e.m.f. of self-induction and the whole of the battery e.m.f. is consumed in overcoming the ohmic drop IR.

Multiplying both sides of Eq. (59) by $i\,dt$, we have

$$Ei\,dt = i^2R\,dt + Li\,di \tag{65}$$

Since $Ei\,dt$ is the energy expended by the battery in the time dt and $i^2R\,dt$ is by Joule's law the energy simultaneously converted into heat in the resistance R, the remaining term must represent energy absorbed in the inductive winding, this conclusion being a consequence of the law of conservation of energy. As the current increases from zero to its final value $I = E/R$, the total energy absorbed by the inductance becomes

$$\int_{i=0}^{i=I} Li\,di = \tfrac{1}{2}LI^2 \quad \text{joules} \tag{66}$$

The physical meaning of this result may be seen by considering the equation for the decay of current, namely, Eq. (63), which when multiplied by $i\,dt$ gives

$$-Li\,di = i^2R\,dt$$

To integrate this equation, it must be remembered that, when $t = 0$, $I = E/R$ and, when $t = \infty$, $i = 0$; therefore,

$$-\int_I^0 Li\,di = \tfrac{1}{2}LI^2 = \int_0^\infty i^2R\,dt$$

But $\int_0^\infty i^2R\,dt$ is the total energy absorbed by the resistor and converted into heat, all of which must have come from the energy $\tfrac{1}{2}LI^2$ previously absorbed by the inductance. It follows, therefore, that the energy absorbed by an inductance is *stored* in such a form that it can be recovered in its entirety when the current ceases to flow.

The energy thus stored in the inductive part of the circuit resides in the magnetic field linked with the circuit. It is equivalent to the stored *kinetic energy* of a mass M moving with velocity V, which is known to be equal to $\tfrac{1}{2}MV^2$; or it is equivalent to the kinetic energy $\tfrac{1}{2}J\omega^2$ of a flywheel of moment of inertia J, rotating at an angular velocity ω. In other words, the inductance L of an electric circuit corresponds to the mass, or inertia, of a physical body, and current corresponds to velocity. The latter analogy is by no means farfetched; for current is the time rate of displacement of electrical quantity (dq/dt) and velocity is the time rate of displacement of a physical body (dx/dt).

It is the energy stored in an inductive circuit that accounts for the spark observed on rupturing the current. This energy must go somewhere and appears as the heat, sound, and light in the

spark; and as energy can never be dissipated instantly, the spark endures for an appreciable time. For example, on opening a highly inductive circuit, like the field windings of a large generator, the stored energy is so considerable that even when the switch is very cautiously and slowly opened a vicious arc will appear at the switch opening and will "hang" there, flaring violently upward, for a considerable time. Opening such a circuit quickly is disastrous and dangerous; it is equivalent to stopping a rapidly moving vehicle in too short a time.

As an example, consider a large electromagnet having a resistance $R = 25$ ohms and an inductance $L = 5$ henrys, initially connected to a battery of which the e.m.f. E is 125 volts, as in Fig. 55. After a time a current of $125/25 = 5$ amp. will flow through the winding, and energy equal to $\frac{1}{2} \times 5 \times (5)^2 = 62.5$ joules will be stored in the magnetic field. At the same instant that switch S_1 is opened, let switch S_2 be closed, thereby discharging the winding through a lamp having a resistance of 250 ohms. The conditions are the same as those which led to Eq. (63) except that now the resistance in the circuit is $R + R_l$, and the current is given by the relation

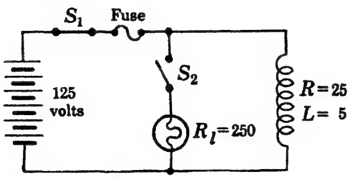


FIG. 55.—Voltage induced on opening inductive circuit.

$$0 = i(R + R_l) + L\frac{di}{dt}$$

or

$$\frac{di}{i} = -\frac{R + R_l}{L}dt$$

subject to the condition that, when $t = 0$, $i = E/R$.

$$\therefore \int_{E/R}^{i} \frac{di}{i} = -\frac{R + R_l}{L}\int_0^t dt$$

$$i = \frac{E}{R}e^{-\frac{R+R_l}{L}t} \tag{67}$$

and

$$\frac{di}{dt} = -\frac{E}{R}\frac{R + R_l}{L}e^{-\frac{R+R_l}{L}t} \tag{68}$$

At the first instant $t = 0$, the current continues to have the value

of 5 amp. that it had before the closure of S_2 and the opening of S_1; but the *rate* of decay, given by Eq. (68), means that the e.m.f. of self-induction is

$$e = -L\frac{di}{dt}\bigg|_{t=0} = \frac{E}{R}(R + R_l) = E\left(1 + \frac{R_l}{R}\right) = \frac{125}{25} \times 275 = 1375 \quad \text{volts}$$

and the potential difference across the lamp terminals is $IR_l = 5 \times 250 = 1250$ volts. It is obvious that the larger the value of R_l/R the greater will be the initial voltage; it is this large induced voltage that accounts for the vicious arc on suddenly introducing an airgap into an inductive circuit. For the same reason a

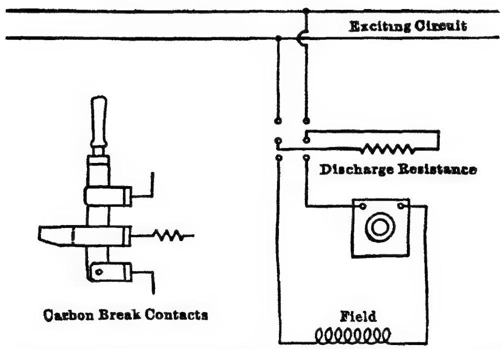


FIG. 56.—Diagram of connections of field-discharge resistance.

voltmeter (which always has high resistance) connected across an inductive circuit must invariably be removed or disconnected before opening the circuit; and for the same reason a special *field-discharge switch*, of the type illustrated in Fig. 56, must be used to disconnect the field windings of generators of large or even moderate rating. The energy stored in inductive circuits is utilized to advantage in the ignition circuits of internal-combustion engines that employ spark plugs.

The energy stored in a magnetic field, which is found to be

$$W = \tfrac{1}{2}LI^2 \quad \text{joules}$$

can be expressed in a somewhat different manner. Thus, when the current has a steady value of I amp. and the corresponding flux is Φ, it is seen from Eq. (52) that the inductance is

$$L = N\frac{\Phi}{I} \times 10^{-8}$$

so that

$$W = \tfrac{1}{2}N\Phi I \times 10^{-8} \quad \text{joules}$$
$$= \tfrac{1}{2}N\Phi I \times 10^{-1} \quad \text{ergs}$$

But since

$$\Phi = \frac{4\pi}{10}\frac{NI}{l}\mu_a A$$

and therefore

$$I = \frac{10}{4\pi}\frac{\Phi l}{N\mu_a A}$$

substitution of this value of I in the expression for W gives

$$W = \frac{\Phi^2}{8\pi}\frac{l}{\mu_a A} = \frac{B^2}{8\pi\mu_a}\cdot Al \quad \text{ergs}$$

As Al is the volume of the magnetic circuit,

$$w = \frac{W}{Al} = \frac{B^2}{8\pi\mu_a} \tag{69}$$

is the energy (in ergs) stored per cubic centimeter of the field.

Finally, since the inductance of a pair of parallel wires in air, per centimeter of their length, is

$$L = \mu_0\left(4 \ln \frac{p}{r} + 1\right) \times 10^{-9} \quad \text{henry}$$

the energy stored per centimeter length is

$$W = \frac{1}{2}LI^2 = \mu_0\left(2 \ln \frac{p}{r} + 0.5\right)I^2 \times 10^{-9} \quad \text{joules}$$
$$= \mu_0\left(2 \ln \frac{p}{r} + 0.5\right)I^2 \times 10^{-2} \quad \text{ergs}$$

This energy is the potential energy of one wire in the presence of the other, and since the derivative of potential with respect to any direction is (with a change of sign) equal to the force in that direction,

$$-\frac{\partial W}{\partial r} = f_r = \frac{2\mu_0 I^2}{r} \times 10^{-2} = \frac{2\mu_0 \bar{I}^2}{r} \quad \text{dynes per cm.}$$

which is in accord with the Biot-Savart law.

32. Mutual Induction.—If two circuits of N_1 and N_2 turns (Fig. 57) are so placed with respect to each other that the magnetic

field due to a current in one of them links in whole or in part with the other, a change in the current strength in the first circuit will induce an *e.m.f. of mutual induction* in the second circuit. In fact, it was this phenomenon which Faraday discovered in 1831, though the two coils in his original experiment were wound on the same magnetic core. It is clear that the magnitude of the induced e.m.f. will depend upon the geometrical shapes and relative positions of the two circuits, as well as upon the rate of change of current in the inducing circuit.

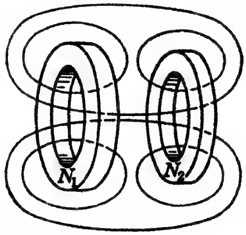


FIG. 57.—Mutual inductance of two coils.

Let a current of i_1 amp. in the first circuit produce a flux Φ_1 such that

$$\Phi_1 = \frac{(4\pi/10)N_1 i_1}{l_1/\mu_1 A_1} = C_1 N_1 i_1 \tag{70}$$

A part of this flux, or

$$\varphi_1 = K_1\Phi_1 = K_1 C_1 N_1 i_1 \tag{71}$$

(where $K_1 \leqq 1$) will link with the second circuit of N_2 turns, so that the total number of linkages with the second circuit is

$$\lambda_{21} = N_2\varphi_1 = K_1 C_1 N_1 N_2 i_1 \tag{72}$$

and if the second circuit is traversed by a current of i_2 amp. its potential energy in the presence of the first circuit is, by Eq. (35),

$$W_{21} = \lambda_{21}\frac{i_2}{10} = \frac{1}{10}K_1 C_1 N_1 N_2 i_1 i_2 \quad \text{ergs} \tag{73}$$

Similarly, the current i_2 in the second circuit will produce a total flux

$$\Phi_2 = \frac{(4\pi/10)N_2 i_2}{l_2/\mu_2 A_2} = C_2 N_2 i_2 \tag{74}$$

of which a part

$$\varphi_2 = K_2\Phi_2 = K_2 C_2 N_2 i_2 \tag{75}$$

(where $K_2 \leqq 1$) will link with the first circuit of N_1 turns, so that the total number of linkages with the first circuit is

$$\lambda_{12} = N_1\varphi_2 = K_2 C_2 N_1 N_2 i_2 \tag{76}$$

The potential energy of the first circuit in the presence of the second is

$$W_{12} = \lambda_{12}\frac{i_1}{10} = \frac{1}{10}K_2 C_2 N_1 N_2 i_1 i_2 \quad \text{ergs} \tag{77}$$

But W_{21} must be equal to W_{12}, since the potential energy of the system can have but one value;

$$\therefore K_1C_1N_1N_2 = K_2C_2N_1N_2 \tag{78}$$

From Eq. (72),

$$K_1C_1N_1N_2 = \frac{N_2\varphi_1}{i_1}$$

which is the number of flux linkages with the second circuit due to unit current in the first circuit; and, from Eq. (76),

$$K_2C_2N_1N_2 = \frac{N_1\varphi_2}{i_2}$$

which is the number of flux linkages with the first circuit due to unit current in the second circuit. Hence, from Eq. (78), it follows that *unit current in one circuit will produce the same number of linkages in the other as unit current in the latter will produce in the former.*

When the current in circuit N_1 changes, the e.m.f. induced in circuit N_2 is

$$e_2 = -N_2\frac{d\varphi_1}{dt} \times 10^{-8} = -K_1C_1N_1N_2\frac{di_1}{dt} \times 10^{-8} \quad \text{volts}$$

and when the current in circuit N_2 changes, the e.m.f. induced in circuit N_1 is

$$e_1 = -N_1\frac{d\varphi_2}{dt} \times 10^{-8} = -K_2C_2N_1N_2\frac{di_2}{dt} \times 10^{-8} \quad \text{volts}$$

From Eq. (78) the last two equations may be written

$$\left.\begin{aligned} e_2 &= -M\frac{di_1}{dt} \\ \text{and} \qquad e_1 &= -M\frac{di_2}{dt} \end{aligned}\right\} \tag{79}$$

where

$$M = K_1C_1N_1N_2 \times 10^{-8} = K_2C_2N_1N_2 \times 10^{-8} \tag{80}$$

is the *number of flux linkages with one circuit due to unit current* (the ampere) *in the other, divided by* 10^8. The quantity M is called the *coefficient of mutual induction*, or the *mutual inductance*, of the two circuits. It is of the same nature as self-inductance and is measured in henrys. From Eq. (79) it follows also that the *mutual inductance of two circuits is numerically equal to the e.m.f.*

induced in one of them when the current in the other changes at the rate of 1 amp. per sec.

It is clear from Eq. (70) that the self-inductance of circuit N_1 is

$$L_1 = \frac{N_1\Phi_1}{i_1} \times 10^{-8} = C_1N_1^2 \times 10^{-8} \tag{81}$$

and from Eq. (74) that the self-inductance of N_2 is

$$L_2 = \frac{N_2\Phi_2}{i_2} \times 10^{-8} = C_2N_2^2 \times 10^{-8} \tag{82}$$

Hence, from Eqs. (80), (81), (82),

$$M^2 = K_1K_2L_1L_2 \tag{83}$$

If the circuits are so related that there is no leakage of flux between them, that is, if all the flux produced by one circuit links with all the turns of the other,

$$K_1 = K_2 = 1$$

and

$$M = \sqrt{L_1L_2}$$

or the mutual inductance of two perfectly coupled circuits is a mean proportional between their self-inductances. The factor $\sqrt{K_1K_2}$ is called the *coefficient of coupling.*

The phenomenon of mutual induction is utilized in the induction coil and in the a-c transformer, each consisting of an iron core upon which are wound two coils, the primary and the secondary, insulated from the core and from each other. An interrupted or an alternating current in one winding sets up a periodically varying flux which in turn induces an alternating e.m.f. in the other winding. Mutual induction is also of importance as a factor in the commutation process.

It is clear from simple physical considerations that if two plane coils are mounted so that their planes are at right angles to each other their mutual inductance will be zero.

33. Energy Stored in Two Mutually Inductive Coils.—If two circuits have self-inductances L_1 and L_2 henrys and a mutual inductance M henrys, there is stored in the system an amount of energy

$$W = \tfrac{1}{2}L_1i_1^2 + \tfrac{1}{2}L_2i_2^2 + Mi_1i_2 \quad \text{joules} \tag{84}$$

if the two currents magnetize in the same direction. The deriva-

tion of the first two terms of Eq. (84) is obvious from Eq. (66); the last term Mi_1i_2 can be derived as follows:

The potential energy of one circuit in the presence of the other is, from Eqs. (73), (77),

$$W = \tfrac{1}{10}K_1C_1N_1N_2i_1i_2 = \tfrac{1}{10}K_2C_2N_1N_2i_1i_2 \quad \text{ergs}$$

and by Eq. (80) this becomes

$$W = \tfrac{1}{10}(M \times 10^8)i_1i_2 = Mi_1i_2 \times 10^7 \quad \text{ergs}$$
$$= Mi_1i_2 \quad \text{joules}$$

If the two currents magnetize in opposite directions, their *mutual* potential energy is evidently reversed in sign, so that the stored energy of the system is

$$W = \tfrac{1}{2}L_1i_1^2 + \tfrac{1}{2}L_2i_2^2 - Mi_1i_2 \qquad (85)$$

34. Tractive Effort of Electromagnets.—It is shown in Art. 5, Chap. II, that if a unit point pole is placed close to the face of a magnet of opposite polarity the force with which the pole is attracted is

$$F = \frac{2\pi\sigma}{\mu_0}$$

where $\sigma = m/A$ is the intensity of magnetization of the pole face.

If two bar magnets are placed end to end with a very small separation and if the intensities of magnetization of the adjacent surfaces are $+\sigma$ and $-\sigma$, the attraction of one of the magnets upon an elementary magnet pole of area dA on the other will be $dF = (2\pi\sigma/\mu_0) \times \sigma\, dA$; and the total attraction between the adjacent poles will be

$$F = \frac{2\pi\sigma^2 A}{\mu_0} \quad \text{dynes} \qquad (86)$$

Since $\sigma = m/A$, and since the flux issuing from a pole of m units is $\Phi = 4\pi m$, it follows that $\sigma = \Phi/4\pi A$; whence, from Eq. (86),

$$F = \frac{\Phi^2}{8\pi A\mu_0} = \frac{B^2A}{8\pi\mu_0} \quad \text{dynes} \qquad (87)$$

This is the fundamental equation underlying the design of tractive, or lifting, electromagnets.

As an example, consider the problem discussed in part 1 of Art. 24; the value of Φ in the circuit of Fig. 41 is 160,000, and the

area of the opposing pole faces is 4 sq. in., or $4 \times (2.54)^2$ sq. cm. The pull exerted between these two pole faces is therefore

$$\frac{(160{,}000)^2}{8\pi \times 4 \times (2.54)^2} = 39{,}600{,}000 \text{ dynes} = 89 \quad \text{lb.}$$

The formula for the pull of a magnet, Eq. (87), is expressed in c.g.s. units, B being in lines per square centimeter and A in square centimeters. If flux density (B'') is in lines per square inch and area (A'') is in square inches, the pull is

$$F = \frac{(B'')^2 A''}{72{,}130{,}000} \quad \text{lb.} \tag{88}$$

35. Rigorous Derivation of Formula for Inductance of Parallel Wires.*—The derivation given in Art. 27 is not rigorous for the reason that in integrating Eq. (53) the upper limit of the variable x (see Fig. 48) was arbitrarily taken equal to p. As a matter of fact, the flux due to conductor M continues indefinitely, penetrating the substance of conductor N in the manner shown in Fig. 58. It is seen that the flux links with a part of the total current in conductor N; and as this current must be regarded as negative if the current in M is positive, these internal linkages with N must be deducted from the positive linkages with M. It follows, therefore, that the net flux linkages with conductor M may be divided into three parts:

1. The linkages internal to M, given by Eq. (54), or $\phi' = \mu_0 \bar{I}/2$.
2. The linkages with M between the limits $x = r$ and $x = p + r$, Eq. (53) thus being changed to the form

$$\phi = 2\mu_0 \bar{I} \int_r^{p+r} \frac{dx}{x} = 2\mu_0 \bar{I} \ln \frac{p + r}{r} \tag{89}$$

3. The negative linkages ϕ'' of the flux from M with the conductor N between the limits $x = p - r$ and $x = p + r$. It is not necessary to go beyond $x = p + r$; for when x exceeds this value, the flux links with both $+\bar{I}$ and $-\bar{I}$, and the net linkage of such external flux is therefore zero.

It is clear from Fig. 58 that a tube of force of radius x and thickness dx links (negatively) with that part of conductor N represented by the crosshatched area. Its shape is so irregular that the area must be found by considering linkages with elementary

* A. RUSSELL, "Alternating Currents," Vol. I, p. 55.

subdivisions and then integrating the resulting differential expression. Let us select as element the area dA in the heavily shaded part of the annular ring of radius ρ and thickness $d\rho$; the entire annulus has an area $2\pi\rho\, d\rho$, and hence

$$dA = \frac{\theta}{\pi} \cdot 2\pi\rho\, d\rho = 2\theta\rho\, d\rho$$

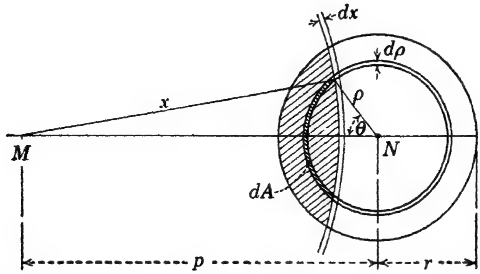


FIG. 58.—Partial linkage of flux with current in a conductor.

The entire area of conductor N being πr^2, the element dA represents a fractional part equal to $2\theta\rho\, d\rho/\pi r^2$, and linked with this part is the flux $(2\mu_0\bar{I}/x)dx$ in the elementary tube of which the length perpendicular to the plane of the drawing is 1 cm. The effect is the same as though the proportionally smaller flux $(2\theta\rho\, d\rho/\pi r^2) \cdot (2\mu_0\bar{I}\, dx/x)$ linked (negatively) with the whole current in N; accordingly, the elementary number of flux linkages, which must be considered as negative, is

$$d(\phi''') = -\int_{x=p-\rho}^{x=p+\rho} \frac{4\mu_0\bar{I}}{\pi r^2}\theta\rho\, d\rho \frac{dx}{x} \tag{90}$$

But it is seen from the diagram (Fig. 58) that

$$x^2 = p^2 + \rho^2 - 2p\rho \cos\theta$$

and when this expression is differentiated, ρ being held constant, there is obtained

$$x\, dx = p\rho \sin\theta\, d\theta$$

and

$$\frac{dx}{x} = \frac{x\, dx}{x^2} = \frac{p\rho \sin\theta\, d\theta}{p^2 + \rho^2 - 2p\rho\cos\theta}$$

Upon substituting this expression in Eq. (90), the variable x is replaced by θ, and the limits of the integral change accordingly

to $\theta = 0$ and $\theta = \pi$, and

$$d(\phi''') = -\int_0^\pi \frac{2\mu_0 \bar{I}}{\pi r^2} \theta\rho\, d\rho \frac{2p\rho \sin\theta\, d\theta}{p^2 + \rho^2 - 2p\rho\cos\theta} \tag{91}$$

It is to be observed that, if the last expression is integrated with respect to θ while ρ is held constant, the result will be the number of linkages with the *complete* annular element of radius ρ, produced by that part of the flux lying within the limit $x = p + \rho$. We can then write

$$d(\phi''') = -\frac{2\mu_0 \bar{I} \rho\, d\rho}{\pi r^2} \int_0^\pi \frac{\theta \cdot 2p\rho \sin\theta\, d\theta}{p^2 + \rho^2 - 2p\rho\cos\theta} \tag{92}$$

If the integral in Eq. (92) is designated as $\int_0^\pi (\quad)$, it can be integrated by parts in accordance with the relation

$$\int u\, dv = uv - \int v\, du$$

where

$$u = \theta \qquad dv = \frac{2p\rho\sin\theta\, d\theta}{p^2 + \rho^2 - 2p\rho\cos\theta}$$

$$du = d\theta \qquad v = \ln(p^2 + \rho^2 - 2p\rho\cos\theta)$$

Hence,

$$\int_0^\pi (\quad) = \theta \ln(p^2 + \rho^2 - 2p\rho\cos\theta)\Big]_0^\pi - \int_0^\pi \ln(p^2 + \rho^2 - 2p\rho\cos\theta)d\theta$$

$$= 2\pi \ln\frac{p+\rho}{p} - \int_0^\pi \ln\left(1 - \frac{2\rho}{p}\cos\theta + \frac{\rho^2}{p^2}\right)d\theta \tag{93}$$

It is proved in any good textbook on the differential calculus that

$$2\cos\theta = e^{j\theta} + e^{-j\theta}$$

where $j = \sqrt{-1}$, and on substituting this value for $2\cos\theta$ in the last expression in Eq. (93),

$$\int_0^\pi \ln\left(1 - \frac{2\rho}{p}\cos\theta + \frac{\rho^2}{p^2}\right)d\theta$$

$$= \int_0^\pi \ln\left[\left(1 - \frac{\rho}{p}e^{j\theta}\right)\left(1 - \frac{\rho}{p}e^{-j\theta}\right)\right]d\theta$$

$$= \int_0^\pi \left[\ln\left(1 - \frac{\rho}{p}e^{j\theta}\right) + \ln\left(1 - \frac{\rho}{p}e^{-j\theta}\right)\right]d\theta \tag{94}$$

By Maclaurin's theorem

$$\log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

and, on applying this expansion, Eq. (94) becomes

$$-2\int_0^{\pi}\left[\frac{\rho}{p}\cos\theta + \frac{1}{2}\frac{\rho^2}{p^2}\cos 2\theta + \frac{1}{3}\frac{\rho^3}{p^3}\cos 3\theta + \cdots\right]d\theta = 0$$

It thus appears that the integral in Eq. (92) is equal simply to $2\pi \ln (p + \rho)/p$, so that

$$d(\phi''') = -\frac{4\mu_0\bar{I}}{r^2}\rho\, d\rho \ln\frac{p + \rho}{p} \tag{95}$$

It must be remembered that $d(\phi''')$ is the number of linkages with the annular element due to the flux up to $x = p + \rho$; but the additional flux from $x = p + \rho$ up to $x = p + r$ also links with this entire annulus, which represents a part of the total area given by $2\pi\rho\, d\rho/\pi r^2 = 2\rho\, d\rho/r^2$, consequently contributing the additional (negative) linkages

$$d(\phi^{\text{iv}}) = -\int_{x=p+\rho}^{x=p+r}\frac{2\rho\, d\rho}{r^2}\cdot\frac{2\mu_0\bar{I}}{x}dx = -\frac{4\mu_0\bar{I}\rho\, d\rho}{r^2}\ln\frac{p + r}{p + \rho} \tag{96}$$

and the total number of linkages with the annular ring of current is

$$d(\phi'') = d(\phi''') + d(\phi^{\text{iv}}) = -\frac{4\mu_0\bar{I}}{r^2}\rho\, d\rho \ln\frac{p + r}{p} \tag{97}$$

The number of linkages with the entire area of conductor N is

$$\phi'' = -\frac{4\mu_0\bar{I}}{r^2}\ln\frac{p + r}{p}\int_0^r \rho\, d\rho = -2\mu_0\bar{I}\ln\frac{p + r}{p} \tag{98}$$

Finally, therefore, the total number of linkages with conductor M is

$$\begin{aligned}\Phi &= \phi + \phi' + \phi'' \\ &= 2\mu_0\bar{I}\ln\frac{p + r}{r} + \frac{\mu_0\bar{I}}{2} - 2\mu_0\bar{I}\ln\frac{p + r}{p} \\ &= \mu_0\bar{I}\left(2\ln\frac{p}{r} + 0.5\right)\end{aligned} \tag{99}$$

The inductance of conductor M, per centimeter, is

$$L = \frac{\Phi}{I}\times 10^{-8} = \mu_0\left(2\ln\frac{p}{r} + 0.5\right)\times 10^{-9}\quad\text{henry} \tag{100}$$

which is the same as Eq. (55)

CHAPTER IV

ELECTROSTATICS

1. Introductory.—The fundamental facts concerning electrostatic charges, which are discussed in the first five articles of Chap. I, are separated from the context of this chapter partly because of their historical priority, but chiefly because the concepts involved are believed to be essential to a complete understanding of the magnetic and electrodynamic phenomena thus far considered. The reader will find it advantageous to review these articles before proceeding with this chapter.

The domain of electrical engineering, beginning with the time it first developed as a recognized branch of the engineering profession in the early 1880's, was at first practically exclusively confined to applications of electromagnetic principles; for at that time the principal developments were concerned with the generation and distribution of electric current for general power and lighting purposes and for communication by telegraph and telephone. Electrostatic considerations were involved in submarine telegraphy but did not enter into power engineering except for the occasional use of condensers. Prior to 1890 the voltages used on power lines did not exceed a few hundred volts except in some series arc-lighting systems; but in that year 10,000-volt power systems employing alternating current were installed in Utah and between Lauffen and Frankfurt in Germany, and thereafter still higher voltages were gradually introduced. With increasing voltage, effects that at the original low potentials were too small to be observable came into prominence and were recognized as electrostatic in their nature, so that present-day electrical engineering is vitally concerned with this branch of the subject.

2. Electrification by Induction.—When an electrified body such as A (Fig. 1) is brought near an insulated metallic body B, which is originally in a neutral condition, B becomes *electrified by induction;* the charge on the side of B nearest to A has a sign

opposite that of A, whereas the charge on B remote from A is of the same sign as that of A. In terms of the electron theory, a positive charge on A is equivalent to a deficit of the normal number of electrons on its surface, with the result that the free electrons in B are drawn toward A. It is also possible to explain this effect on the ground that at b the potential due to A is higher than at b', thus tending to create a difference of potential between b and b'; but since B is a conductor, all parts of its surface, when in equilibrium, must have the same potential, and consequently the charges on B must be so distributed that the potential on B is everywhere the same.

With each element of charge on both A and B are associated lines of electric force, as indicated (in part) in the diagram. The lines of force are necessarily perpendicular to the surfaces of A and B at the points of entrance or exit, for otherwise there would be a component of electric force along the surface which would cause the charges to redistribute themselves. In other words, the surfaces of the conducting bodies A and B, when in equilibrium, are equipotential surfaces.* Perpendicularity between the surface and the line of force will not in general exist if the bodies are insulators.

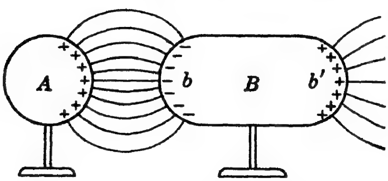


Fig. 1.—Induced charges.

The charge at b, Fig. 1, is a *bound charge*, being firmly held by the attraction due to A, and will remain so long as A retains its charge and does not move. On the other hand, the charge at b', equal to that at b but of opposite sign, is a *free charge*, which seeks to get as far away from A as possible because of their mutual repulsion. Consequently, if b' is momentarily connected to the earth by means of a conducting wire, or by a hand touching it (provided that the experimenter is standing on the ground), the free charge will escape to the ground and body B will be left with a charge equal and opposite to that of A; and on separating A and B to a considerable distance the charge on B will distribute itself over the entire surface. Thereafter, the charge on B can be conductively transferred to some third conductor C, and by repetition of this procedure the charge on C can be indefinitely accumulated.

* See Art. 12, Chap. II.

This process is utilized in the *electrophorus,* an early form of electrostatic generator. Later, the cumbersome procedure of moving body B back and forth by hand was improved by making B a part of a rotating system that accomplished the desired result continuously; thus there originated *electrostatic generators,* or *influence machines,* which played a large part in the early period of development of this subject and were extensively used for some time following Roentgen's discovery of X-rays in 1896.

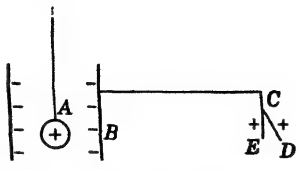


FIG. 2.—Faraday's ice-pail experiment.

In thus accumulating charges on conductor C, successively induced on B by A, the increasing potential energy of the accumulated charge is accounted for by the work that has to be done in overcoming the attraction between B and A and in bringing B up to C against their mutual repulsion.

3. Surface Charges on Conductors.—One of the many important discoveries made by Faraday was that the charge on a conductor resides wholly on its surface, or, to be more exact, that the effect of such a charge is wholly outside the surface. In the interior of a hollow conducting shell there is perfect screening from the effects of any external charges. These facts were demonstrated by what is called Faraday's ice-pail experiment, illustrated in Fig. 2: A is a positively electrified metal body, attached to an insulating thread or handle; it is inserted into an insulated metal cup B which in turn is connected to an electroscope C. A negative charge is induced on the inner surface of the cup B, and the free positive charge is repelled, the movable leaf D of the electroscope being thus likewise repelled from the fixed plate E. If A is now brought into contact with the inner wall of the cup and is then removed, the divergence of the leaf D remains unaltered, and no trace of electrification remains on A after its removal. The whole charge originally on A has thus been transferred to the exterior parts, and no charge remains inside the cup.

The screening effect of a metallic shell was checked by Faraday in other ways. Working with the most sensitive electrometer available, in the interior of a large cage charged to very high potentials, he was unable to detect the slightest evidence of any

internal electric field. The importance of this experiment is that it verifies the accuracy of the inverse square law discovered by Coulomb, as will now be shown.

Absence of Electric Field inside Conducting Shell.—Let P, Fig. 3, be any point within a thin, spherical, conducting shell of radius R which is uniformly charged. If the total charge thus uniformly distributed is Q_s e.s.u., the charge per unit area is $\sigma = Q_s/4\pi R^2$. With P as the vertex, construct a cone (of two nappes) enclosing the solid angle $d\omega$ and intersecting the sphere

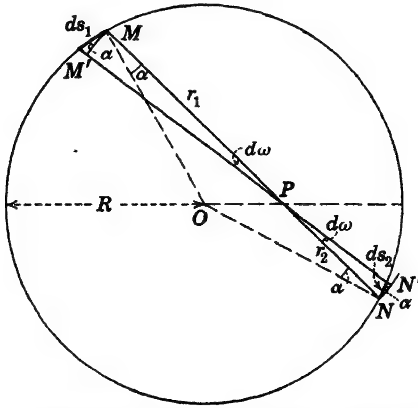


FIG. 3.—Electrical conditions inside a charged sphere.

in the two elementary areas ds_1 and ds_2. On joining the center O with the points M and N and passing through M and N spheres of radii $PM = r_1$ and $PN = r_2$, the spherical elements MM' and NN' intercepted by the cone will have areas $r_1^2\,d\omega$ and $r_2^2\,d\omega$. From the geometry of the figure it follows that

$$ds_1 = \frac{r_1^2\,d\omega}{\cos\alpha} \qquad \text{and} \qquad ds_2 = \frac{r_2^2\,d\omega}{\cos\alpha}$$

and the corresponding charges on these elementary areas are $\sigma r_1^2\,d\omega/\cos\alpha$ and $\sigma r_2^2\,d\omega/\cos\alpha$.

If the inverse square law holds, the force on a unit charge at P due to ds_1 will be $\sigma r_1^2\,d\omega/\epsilon_0 r_1^2\cos\alpha = \sigma\,d\omega/\epsilon_0\cos\alpha$ and the force due to ds_2 will be $\sigma r_2^2\,d\omega/\epsilon_0 r_2^2\cos\alpha = \sigma\,d\omega/\epsilon_0\cos\alpha$, and they are equal to each other and act in opposite directions. The same thing holds for all elementary cones like $d\omega$, and therefore for the entire sphere. It may be concluded, therefore, that the resultant

force in the interior will not be zero, as experiment proves it to be, unless the inverse square law holds.

There is one point in this chain of reasoning to which special attention should be directed, namely, the *assumption* that the sphere is *uniformly* charged. So long as the charged sphere is alone in space, or so far removed from other charges that there can be no disturbing effects due to them, the assumption of uniform distribution is clearly valid because of pure symmetry; but if the sphere is sufficiently close to other conductors to exert an inductive effect upon them, as in Fig. 1, the distribution of charge upon its surface ceases to be uniform, tending to be concentrated on the side facing the induced charge of opposite sign. Nevertheless, experiment shows that, even with such nonuniform surface distribution, the internal force remains zero throughout. The explanation must be, therefore, that, although a pair of elements such as are shown in Fig. 3 may have a resultant in one direction or the other, the *integrated* magnitude due to *all* such pairs must be zero, and the surface distribution must accordingly vary from point to point in such a way as to produce this result. This case will be considered in greater detail in Art. 15.

It is an interesting historical fact that Coulomb's discovery of the inverse square law was anticipated by several years by Cavendish, who derived the law by a process of reasoning based upon experiments he had made. Cavendish did not publish his work, however, and it was not until Maxwell obtained access to his notebooks nearly a century later that his work became known. Of course, the credit for the discovery has been accorded to Coulomb.

The effect of metallic shells in thus screening their interior from extraneous electrical disturbances is utilized in many ways. Delicate instruments, which might otherwise give false readings because of electrostatic attractions between their parts, are protected by enclosing them in copper-gauze cages which are connected to ground. The protection is practically complete even if the shield is made of wire mesh having fairly large openings. It is for this reason that the interior of steel-frame buildings is practically immune to damage from lightning; and overhead transmission lines acquire a high degree of protection from lightning if even a single wire, grounded at frequent intervals, is strung above them.

4. Dielectrics and Dielectric Flux. Permittivity.—In exactly the same way that the magnetic flux from a magnet pole of strength m units is shown to be $\Phi = 4\pi m$ (Art. 8, Chap. II), the *electric flux* issuing from a positive charge of Q_s e.s.u. is

$$\psi = 4\pi Q_s \tag{1}$$

Thus, in a medium characterized by the constant ϵ_a, the force on a unit point charge, r cm. from a point charge Q_s, is $E = Q_s/\epsilon_a r^2$ dynes; this fact is described by saying that the strength of the electric field at the point is E. Just as in the analogous magnetic case the quantity μ_a was called the absolute permeability, so here the quantity ϵ_a is called the *absolute permittivity* of the medium. The field intensity E is seen to be dependent upon ϵ_a, but the function Q_s/r^2 is independent of the medium and is called the *dielectric* flux density* at the point distant r cm. from Q_s. It is symbolized by the letter D, just as in the magnetic case B was used to represent m/r^2; and just as B can be thought of in terms of lines of magnetic flux per square centimeter, so can D be expressed in lines of dielectric flux per square centimeter.

Since $D = Q_s/r^2$, it follows that in general

$$E = \frac{Q_s}{\epsilon_a r^2} = \frac{D}{\epsilon_a}$$

or

$$D = \epsilon_a E \tag{2}$$

If the value of E (electric field intensity) that appears in Eq. (2) were to exist in free space (or, for all practical purposes, in air) the corresponding flux density would be

$$D_0 = \epsilon_0 E \tag{3}$$

so that

$$\frac{D}{D_0} = \frac{\epsilon_a}{\epsilon_0} = K \tag{4}$$

whence

$$D = K\epsilon_0 E \tag{5}$$

The quantity K is called the *relative permittivity*, or the *dielectric constant*, of the medium; it is the ratio of the dielectric flux den-

* The word dielectric was first used by Faraday to denote nonconducting substances, whether solid, liquid, or gaseous

sity produced by a given field intensity in a medium of absolute permittivity ϵ_a to the flux density that would be produced by the same field intensity in a vacuum (or in air).

Dielectric flux cannot exist in the substance of conductors, for it has been shown that the charge resides wholly on the surface of conducting materials and that there is no electric force inside their boundary surfaces; thus, both E and D are zero inside a conductor.

If a point charge q is placed at the center of a hollow spherical shell of dielectric material such as glass or paraffin (in the manner indicated in Fig. 4, Chap. II, for the case of a magnetic point pole), induced distributed charges $-q'$ and $+q'$ will appear on the inner and outer surfaces, the fact being thus accounted for that within the material of the dielectric the electric force is less than it would be in the same region if the dielectric were removed.

The distribution of the dielectric flux between two point charges Q and Q' in air is exactly the same as that of the magnetic flux between two point poles, as discussed in Art. 10, Chap. II; and if m and m' are there replaced by Q and Q', no other changes are necessary to make the analysis fit the electrical case. Accordingly, Figs. 9 and 11, Chap. II, correctly represent the dielectric lines of force between two point charges in air.

Just as with magnetic lines of force, there is a tension along lines of electric force, as though they were stretched elastic bands; there is a repulsion between lines that have the same direction, and an attraction between lines that are oppositely directed. In fact, the analogy between lines of magnetic force and lines of electric force is complete with one important exception, namely, that, whereas lines of magnetic force emanating from magnets are parts of continuous closed curves which, within the magnetic substance, are lines of magnetization, *lines of electric force associated with charged metallic bodies are not closed curves but originate on positive charges and terminate on negative charges.** The field of force in the vicinity of charges *always distributes itself in such a way that the total flux is a maximum.*

Dielectrics possess insulating qualities of two distinct kinds. One quality relates to the *insulation resistance,* expressible in

* Cases occur where lines of electric force are closed curves, as in electromagnetic waves radiated into space from the antenna of a radio transmitting station.

ohms (or megohms) per centimeter-cube (or any other appropriate unit for defining the physical dimensions of the material); this resistance determines the amount of the *leakage current* that will flow through the material on subjecting it to a difference of potential. Obviously, the higher the resistivity of an insulator, the better it will prevent the leakage of charge from the conductor it supports or surrounds; dry air, for example, is an almost perfect insulator in this sense.

The other insulation characteristic is of entirely different nature and is measured by the *number of volts per unit of thickness* (such as a mil, an inch, or a centimeter) required to produce a physical rupture or breakdown. This measure of the quality of the material is called its *dielectric strength,* and the number of volts

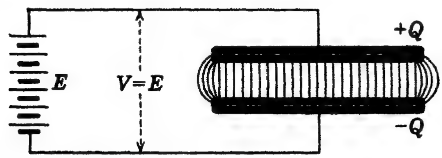


FIG. 4.—Parallel-plate condenser, air dielectric.

per unit thickness that expresses it is called the *critical voltage gradient.* Dry air, for instance, though an extremely good insulator from the standpoint of resistivity, has a much lower critical voltage gradient (about 75,000 volts per in.) than some other materials that have smaller resistivity than air.

5. Capacitance.—Consider a pair of insulated, parallel, metallic plates, separated by an air space and connected to a battery or other source of e.m.f., as in Fig. 4. This particular form of structure is not essential; in general a *condenser* (or *capacitor*) *consists of any pair of adjacent conducting bodies separated by a dielectric.* Electrons will be abstracted from one plate, which is then said to be positively charged, and transferred to the other negatively charged plate, until the difference of potential between the plates is exactly balanced by the e.m.f. of the battery. During the brief interval while this transfer is being effected, there will be a current in the battery and in the wires connecting it to the plates. Though this current is in general very small, it is detectable by a sufficiently sensitive galvanometer, but this flow of current ceases as soon as equilibrium is established.

It is found by experiment that the charge Q accumulated on the plates is directly proportional to the difference of potential $V(=E)$ impressed upon them, or

$$Q = CE = CV \tag{6}$$

where C is the proportionality constant that relates Q to V. If we agree to express the potential difference between the plates in statvolts V_s, and the charge in statcoulombs Q_s,

$$Q_s = C_sV_s \tag{7}$$

and the constant C_s, which is given by

$$C_s = \frac{Q_s}{V_s} \tag{8}$$

is called the capacitance of the condenser* consisting of the two conducting plates and the intervening dielectric; the name of the unit in which it is thus expressed in the e.s. system is the *statfarad*. It is seen from Eq. (8) that the capacitance, measured in this unit, is *numerically equal to the charge in statcoulombs that will be produced by a potential difference of* 1 *statvolt*.

Let the air between the plates of Fig. 4 be replaced by some other dielectric, glass, for example. Experiment shows that, if the potential difference is the same as before, the charge on the condenser plates is considerably increased, becoming, say Q'_s, where $Q'_s > Q_s$; but provided that the potential difference does not exceed a definite limiting value determined by the nature of the dielectric, Q'_s is again proportional to V_s, or

$$Q'_s = C'_sV_s \tag{9}$$

and

$$\frac{C'_s}{C_s} = \frac{Q'_s}{Q_s} = K(>1) \tag{10}$$

The effect of the dielectric is to increase the capacitance to K times its original magnitude, and K is accordingly called the *dielectric constant* of the insulating medium. In the early literature of the subject K was called *specific inductive capacity;* but

* Capacitance, comparable in form to the words resistance and inductance, is a property of the physical appliance that is the capacitor itself; the word condenser, often used interchangeably with capacitor, may be taken to denote the entire assembly required to constitute the capacitor, in the same way that a rheostat embodies a resistor.

more recently it has been called the *relative permittivity*, the word relative being introduced to indicate that the magnitude is relative to free space (or, practically, to air).

Inasmuch as the charge of the capacitor has thus been increased K times, from Q_s to Q_s', by changing the dielectric from air to, say glass, the dielectric flux between the plates has been increased from $\psi = 4\pi Q_s$ to $\psi' = 4\pi Q_s'$. More lines are therefore compacted in the same space than were originally there, as indicated by the closer spacing of the lines in Fig. 5 than in Fig. 4. The dielectric permits the passage of more lines of induction than does air, and hence the word permittivity; looked at in another way, the dielectric of Fig. 5 is a better conductor of dielectric flux than the air it replaces, and hence relative permittivity is a measure of dielectric conductance in the same way that (relative) permeability is a measure of magnetic conductance. Furthermore, just as the field intensity H in a magnetic substance is $H = B/\mu_a$, so the intensity of the electric force in a dielectric is

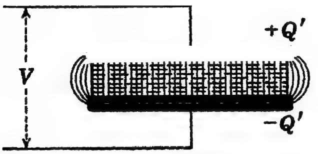


FIG. 5.—Parallel-plate condenser, solid or liquid dielectric.

$$E = \frac{D}{\epsilon_a} = \frac{\psi}{A\epsilon_0} = \frac{4\pi Q_s}{A\epsilon_a} \tag{11}$$

where $D = \psi/A$ is the dielectric flux density that corresponds to B in the magnetic circuit.

The relative permittivity, or dielectric constant, of various substances used in practice is given in Appendix D. It will be observed that in most of the commonly used materials $K = \epsilon_a/\epsilon_0$ varies between 1 and 10.

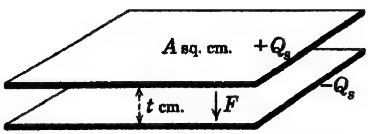


FIG. 6.—Parallel-plate condenser.

6. Capacitance of Parallel-plate Condenser.—Let Fig. 6 represent a parallel-plate condenser of which the plates, each A sq. cm. in area, have linear dimensions that are large in comparison with their distance apart, t cm. The charge on each plate being Q_s statcoulombs, the charge per unit area is $\sigma = Q_s/A$. A unit point charge placed anywhere between the plates (except near the edges, the effect of which will be neglected) will experience a force of $2\pi\sigma/\epsilon_0$ dynes* due to each plate, or $4\pi\sigma/\epsilon_0$ dynes due to

* The proof of this statement is exactly similar to the derivation for the

both plates, provided that the dielectric between the plates is air (more exactly, vacuum). For any other dielectric of permittivity ϵ_a, the force is $4\pi\sigma/\epsilon_a$.

The work required to carry a unit charge from the negative to the positive plate is $Ft = 4\pi\sigma t/\epsilon_a$; but this amount of work (in ergs) is by definition equal to the difference of potential between the plates, or

$$V_s = \frac{4\pi\sigma t}{\epsilon_a} = Ft = \frac{Q_s}{C_s}$$

and on substituting $\sigma = Q_s/A$ there results

$$C_s = \frac{\epsilon_a A}{4\pi t} = \frac{K\epsilon_0 A}{4\pi t} \tag{12}$$

Capacitance is thus seen to be a function of the dimensions and nature of the dielectric material, as was found to be true for conductance and inductance.

Since capacitance is inversely proportional to the separation of the plates, it follows that, as t becomes smaller, C_s becomes larger, approaching infinity as t approaches zero. But when the plates are actually in contact ($t = 0$), they constitute merely a local enlargement of the wires of the circuit to which they are connected, so that we have the seeming paradox that a circuit in which there is no capacitor has infinite capacitance.

7. Units of Capacitance.—It is plain that, without regard to the units in which charge and potential difference may be measured, the proportionality between them remains a physical fact. In the e.s. system of units,

$$C_s \text{ (in statfarads)} = \frac{Q_s}{V_s} = \frac{\text{number of statcoulombs}}{\text{number of statvolts}}$$

whereas in the e.m. and the practical system

$$\bar{C} \text{ (in abfarads)} = \frac{\bar{Q}}{\bar{V}} = \frac{\text{number of abcoulombs}}{\text{number of abvolts}}$$

and

$$C \text{ (in farads)} = \frac{Q}{V} = \frac{\text{number of coulombs}}{\text{number of volts}}$$

---

force on a unit point pole due to a distributed magnet pole (see Art. 5, Chap. II).

It has already been stated that

$$\begin{aligned} 1 \text{ coulomb} &\equiv \tfrac{1}{10} \text{ abcoulomb} \equiv 3 \times 10^9 \quad \text{statcoulombs} \\ 1 \text{ volt} &\equiv 10^8 \text{ abvolts} \equiv \tfrac{1}{300} \quad \text{statvolt} \end{aligned}$$

and from these relations it is readily found that

$$1 \text{ farad} \equiv 10^{-9} \text{ abfarad} \equiv 9 \times 10^{11} \quad \text{statfarads}$$

or

$$1 \text{ abfarad} \equiv 10^9 \text{ farads} \equiv 9 \times 10^{20} \quad \text{statfarads}$$

As an example, consider two square metal plates each measuring 10 by 10 cm., spaced 0.1 cm. apart, with air as the dielectric. Their capacitance is

$$C_s = \frac{10 \times 10}{4\pi \times 0.1} = 79.6 \quad \text{statfarads}$$

and this is equivalent to

$$C = \frac{79.6}{9 \times 10^{11}} = 8.84 \times 10^{-11} \quad \text{farad}$$

If the capacitance were to be made 1 farad by increasing the area A but leaving the separation t at 0.1 cm.,

$$C = 1 = \frac{A}{4\pi \times 0.1 \times 9 \times 10^{11}}$$

whence $A = 36\pi \cdot 10^{10}$, and the side of each square plate would be $6\sqrt{\pi} \times 10^5 = 1{,}062{,}000$ cm., which is equivalent to 6.6 miles. The farad is much too large a unit for practical sizes of condensers, so that the unit ordinarily used is the *microfarad*, which is one-millionth farad. In the example here given, the 10- by 10-cm. plates would have a capacitance of 8.84×10^{-5} microfarad.

8. Capacitance of Cylindrical Condenser.—A form of condenser that occurs frequently in practice consists of concentric conducting cylinders, as in Fig. 7. Such an arrangement appears, for example, in single-conductor cables that have a metallic sheath, and it is used also in making accurate measurements in high-voltage circuits. The capacitance of such a condenser may be computed with a high degree of precision because of the simple geometrical forms that are involved.

Assume that the concentric cylinders of Fig. 7 are indefinitely long, so that the disturbing effects of the ends are negligible in the greater part of the interior. Because of symmetry, the lines of electric force will pass from cylinder to cylinder as uniformly distributed radial lines. Let the charge per centimeter length of the cylinders be Q_s statcoulombs; that is, the charge on one cylinder is $+Q_s$ units per cm., whereas that on the other is $-Q_s$ units per cm. The total flux crossing the dielectric in each

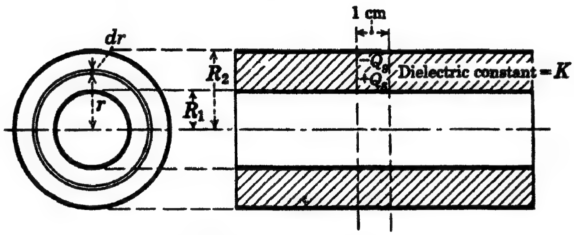


FIG. 7.—Cylindrical condenser.

centimeter length along the axis will be $\psi = 4\pi Q_s$, the flux density on the cylindrical surface of radius r will be

$$D = \frac{4\pi Q_s}{2\pi r} = \frac{2Q_s}{r} \tag{13}$$

and the force acting on a unit point charge at a distance r from the axis will be

$$F = E = \frac{D}{\epsilon_a} = \frac{2Q_s}{\epsilon_a r} \quad \text{dynes} \tag{14}$$

Incidentally, it may be observed that Eq. (14) is similar in form to the Biot-Savart equation $H = 2\bar{I}/r$ for the strength of magnetic field at a distance r from a long, straight wire carrying a current $\bar{I}$. It is also seen that $2Q_s/\epsilon_a r$ is the strength of electric field at a distance r from an infinitely long, statically charged filament; otherwise stated, the charge on the inner cylinder of Fig. 7 acts, so far as external points are concerned, as if it were concentrated along the central axis.

The work done in moving a unit charge through a distance dr is $(2Q_s/\epsilon_a r)dr$ ergs, and the total work, from cylinder to cylinder, is

$$V_s = \int_{R_1}^{R_2} \frac{2Q_s}{\epsilon_a r} dr = \frac{2Q_s}{\epsilon_a} \ln \frac{R_2}{R_1} \quad \text{statvolts} \tag{15}$$

so that the capacitance is

$$C_s = \frac{Q_s}{V_s} = \frac{\epsilon_a}{2 \ln (R_2/R_1)} = \frac{K\epsilon_0}{2 \ln (R_2/R_1)}, \quad \text{statfarads per cm.} \qquad (16)$$

9. Capacitance of Spherical Condenser.—Let the concentric spheres of Fig. 8 be charged with Q_s statfarads of opposite sign; from the inner sphere the flux $\psi = 4\pi Q_s$ radiates uniformly in all directions, and the flux density at the intermediate spherical surface of radius r is

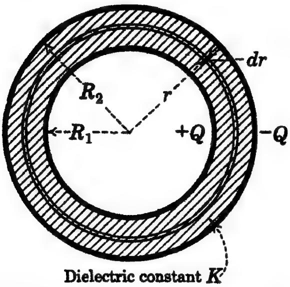


FIG. 8.—Spherical condenser.

$$D = \frac{4\pi Q_s}{4\pi r^2} = \frac{Q_s}{r^2}$$

That is, so far as points external to the inner sphere are concerned, the surface charge acts as though it were concentrated at the center. The work done in carrying a unit charge from the inner to the outer sphere is

$$V_s = \int_{R_1}^{R_2} F\,dr = \int_{R_1}^{R_2} \frac{D}{\epsilon_a} dr = \frac{Q_s}{\epsilon_a} \int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{Q_s}{\epsilon_a}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

and

$$C_s = \frac{Q_s}{V_s} = \frac{K\epsilon_0}{(1/R_1) - (1/R_2)} = \frac{K\epsilon_0 R_1 R_2}{R_2 - R_1} \quad \text{statfarads} \qquad (17)$$

If $R_2 = \infty$, which means that the inner sphere is alone in space of permittivity ϵ_a,

$$C_s = \epsilon_a R_1$$

and if $\epsilon_a = \epsilon_0 = 1$, as in air or free space.

$$C_s = \epsilon_0 R_1 = R_1 \qquad (18)$$

or the capacitance of an isolated sphere, expressed in statfarads, is *numerically* equal to its radius in centimeters.

10. Energy Stored in Condenser.—If a condenser of capacitance C_s is charged to a difference of potential v_s, the charge is $q_s = C_s v_s$, or $v_s = q_s/C_s$. If a charge dq_s is now added to the charge already there, the work done is $v_s\,dq_s = q_s\,dq_s/C_s$, and

the total work done in building up the charge from zero to Q_s is

$$W = \int_0^{Q_s} \frac{q_s\, dq_s}{C_s} = \frac{1}{2}\frac{Q_s^2}{C_s} \quad \text{ergs} \tag{19}$$

Substituting the relation $Q_s = C_s V_s$, where V_s is the final difference of potential,

$$W = \tfrac{1}{2} C_s V_s^2 \quad \text{ergs} \tag{20}$$

which is an alternative expression for the stored energy.

If capacitance, charge, and potential difference are expressed in practical units (farads, coulombs, and volts) the energy stored is

$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2 \quad \text{joules} \tag{21}$$

The same result may be derived in another way that is illuminating as to the mechanism involved in the energy storage. Thus, in the case of the parallel-plate condenser (Fig. 6) it is known that if the distance between the plates is small they attract each other with a force $\int \frac{(2\pi\sigma)(\sigma\, dA)}{\epsilon_a} = \frac{2\pi\sigma^2 A}{\epsilon_a}$ dynes, where $\sigma = Q_s/A$, and this force will be constant for all values of the separation that are small compared with the linear dimensions of the plates. If the plates are originally an infinitesimal distance apart and are then pulled apart against their mutual attraction, the work done in separating them through a distance t is

$$W = \frac{2\pi\sigma^2 A t}{\epsilon_a} = \frac{2\pi Q_s^2 t}{\epsilon_a A} \tag{22}$$

But since $C_s = \epsilon_a A/4\pi t$,

$$W = \frac{1}{2}\frac{Q_s^2}{C_s} = \frac{1}{2} C_s V_s^2$$

or the same as Eqs. (19) and (20).

It is thus apparent that the energy $\tfrac{1}{2}CV^2$ stored in the condenser is of exactly the same sort as that involved in a stretched spring, and it is accordingly *potential energy,* as distinguished from the kinetic energy implied in the corresponding expression $\tfrac{1}{2}LI^2$, which arises in connection with the energy of an electromagnetic field.

If in Eq. (22) Q_s is eliminated by means of the relation $\psi = 4\pi Q_s$,

$$W = \frac{\psi^2 t}{8\pi\epsilon_a A}$$

and since $\psi = DA$,

$$W = \frac{D^2}{8\pi\epsilon_a} At$$

As At is the volume of the dielectric,

$$w = \frac{W}{At} = \frac{D^2}{8\pi\epsilon_a} \tag{23}$$

is the energy stored per cubic centimeter of the dielectric. This expression compares term by term with the corresponding expression for the energy stored per cubic centimeter in a magnetic field, namely, $w = B^2/8\pi\mu_a$.

11. Elastance and Elastivity.—The analogy between the energy storage in a condenser and that in a coiled spring (or its equivalent) is strengthened when one realizes that in separating the plates, say in Fig. 6, it is necessary to overcome the tension which acts along the lines of force as though they were elastic bands. Moreover, when a spring having a constant s (where s is the force required to elongate or compress it through a unit distance) is subjected to a pull P, the resultant stretch δ is given by the relation $P = s\delta$, and a further stretch through an elementary distance $d\delta$ requires an amount of work equal to

$$dW = P\,d\delta = s\delta \cdot d\delta$$

The total amount of work done in stretching the spring from its equilibrium position, $\delta = 0$, to a final displacement $\delta = \Delta$ is

$$W = \int_0^\Delta s\delta\,d\delta = \tfrac{1}{2}\,s\Delta^2 \tag{24}$$

which is analogous to Eq. (19), or

$$W = \frac{1}{2}\frac{Q_s^2}{C_s}$$

The reason underlying the analogy is that the displacement of the spring, Δ, is the mechanical analogue of the displacement of electric charge, Q_s, which takes place in such a system as is shown in Fig. 1 or 2. This being admitted, the reciprocal of the capaci-

tance C_s is the analogue of the spring constant s, which is a measure of the elasticity or stiffness of the spring; for this reason,

$$S_s = \frac{1}{C_s} \tag{25}$$

is called the *elastance* of the condenser. Accordingly, a small value of C_s means a large value of S_s, equivalent in the mechanical analogy to a stiff spring; vice versa, a large value of C_s means a small value of S_s, corresponding to a weak or easily deformable spring. The unit in which elastance is expressed has been called the *daraf* in the practical system, and the prefixes *stat* and *ab* are used when this quantity is to be expressed in e.s. and e.m. units, respectively.

Since in every case the formula for the capacitance of a condenser involves the relative permittivity K in the numerator, the elastance is always proportional to $1/K$, which is called the *elastivity*.

12. Law of the Dielectric Circuit.—In general, whatever the size or geometrical shape of a condenser, the basic relation

$$Q = CV = \frac{V}{S}$$

is rigorously true, provided that the "elastic limit" of the dielectric is not exceeded. Since the dielectric flux ψ is given by $4\pi Q$, it is possible to write

$$\psi = \frac{4\pi V}{S} \tag{26}$$

or, in words,

$$\text{Dielectric flux} \propto \frac{\text{potential difference}}{\text{elastance (or dielectric resistance)}}$$

Thus the "dielectric circuit" obeys the same general type of law that holds for the electric and magnetic circuits, and Eq. (26) is for this reason sometimes referred to as Ohm's law for the dielectric circuit. The word circuit must be used with caution, for dielectric lines of force are not always closed curves.

It is obvious that Kirchhoff's laws will apply, with suitable changes in nomenclature.

13. Series and Parallel Groupings of Condensers. 1. *Condensers in Series.*—Let any number of condensers be connected in

series, Fig. 9, with a potential difference V volts across the outer terminals. The electric displacement that occurs throughout the circuit, in the manner illustrated in Figs. 4 and 5, implies that the charge Q in each of the condensers is the same. In the wire connecting condenser C_1 to C_2, for instance, there is no conductive connection with any other part of the circuit, and so the charges at its ends must be equal and opposite, just as in body B, Fig. 1. Accordingly, the potential differences V_1, V_2, . . . must satisfy the relations $V_1 = Q/C_1$, $V_2 = Q/C_2$, . . . , and by Kirchhoff's second law

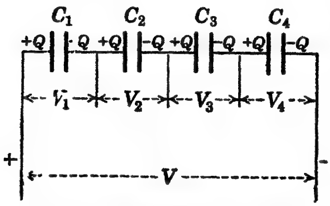


FIG. 9.—Condensers in series.

$$V = V_1 + V_2 + V_3 + \cdots = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots\right) \quad (27)$$

The single capacitance that will replace the series grouping and yield the same charge Q as before is $C = Q/V$, and it follows, therefore, that

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

or

$$S = S_1 + S_2 + S_3 + \cdots \quad (28)$$

or capacitances in series conform to the same law that holds for resistances in parallel, whereas the elastances are additive like resistances in series.

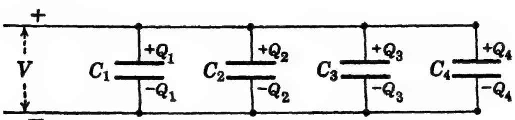


FIG. 10.—Condensers in parallel.

2. *Condensers in Parallel.*—When a number of condensers are in parallel (Fig. 10), all have the same potential difference across their terminals, so that $Q_1 = C_1V$, $Q_2 = C_2V$, ... The sum Q of all the individual charges is the total charge supplied from the source of e.m.f., or

$$Q = Q_1 + Q_2 + Q_3 + \cdots = V(C_1 + C_2 + C_3 + \cdots)$$

and the single capacitance that is equivalent to the parallel-connected group is

$$C = C_1 + C_2 + C_3 + \cdots \quad (29)$$

So far as elastances are concerned,

$$\frac{1}{S} = \frac{1}{S_1} + \frac{1}{S_2} + \frac{1}{S_3} + \cdots \quad (30)$$

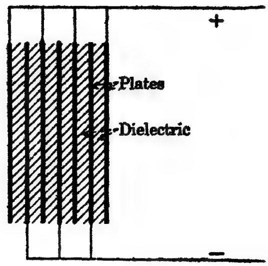


FIG. 11.—Condenser with interleaved plates.

The simple additive relation that makes a number of parallel-connected capacitors equivalent to a single capacitor of greater capacitance than any of the parts is utilized in the construction of large capacitors. This is accomplished in the manner shown in Fig. 11.

Condensers for telephone circuits and other applications, where the capacitor must have small or moderate over-all dimensions, are made of long strips of tin foil separated by a paper strip, the combination being rolled up spirally for insertion in a sealed cartridge or container.

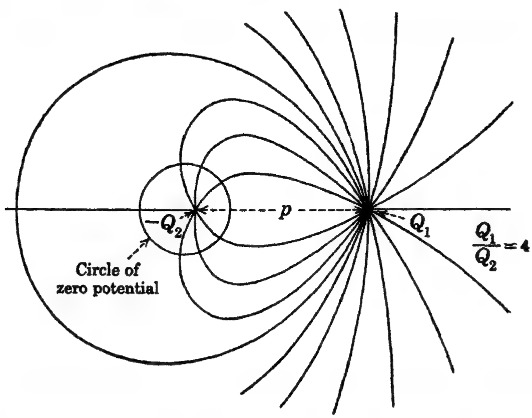


FIG. 12.—Lines of electric force between point charges of opposite sign.

14. Electrical Images.—Attention has already been called to the fact that the diagram representing the lines of magnetic force between two point poles m and m', Figs. 9 and 11, Chap. II, is the same as the diagram for two point charges Q_1 and Q_2, provided that $Q_1/Q_2 = m/m'$. The first of these diagrams, representing the case of opposite polarities, $+Q_1$ and $-Q_2$, is reproduced as Fig. 12, for the condition $Q_1/Q_2 = 4$.

At any general point P in this field of electric force (assumed to be in free space) the potential due to Q_1 is $Q_1/\epsilon_0 r_1$ and that due to Q_2 is $-Q_2/\epsilon_0 r_2$, where r_1 and r_2 are the distances from P to Q_1 and Q_2, as in Fig. 13; the resultant potential at P is

$$V_P = \frac{1}{\epsilon_0}\left(\frac{Q_1}{r_1} - \frac{Q_2}{r_2}\right) \tag{31}$$

If the condition is imposed that V_P is constant, Eq. (31) establishes a relation between r_1 and r_2 that will define the positions of all points that lie on the equipotential line corresponding to the chosen value of V_P. The whole family of equipotential lines can thus be drawn by making point by point computations.

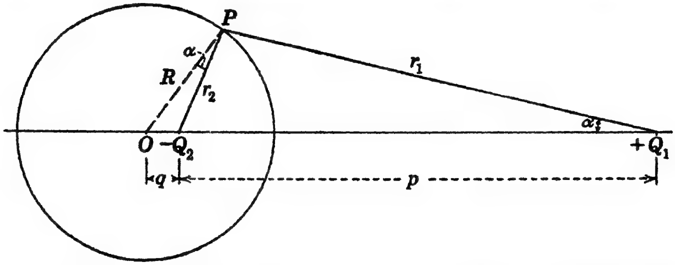


FIG. 13.—Circle of zero potential—two point charges of opposite sign.

In particular, if $V_P = 0$, the curve of zero potential will be determined; its equation is

$$\frac{Q_1}{r_1} - \frac{Q_2}{r_2} = 0$$

or

$$\frac{r_1}{r_2} = \frac{Q_1}{Q_2} = \text{constant} = c \tag{32}$$

Under the conditions assumed in making Fig. 12, the constant is $c = 4$, and the fixed distance between Q_1 and Q_2 is p. In Fig. 13, let P be any point that satisfies the condition $r_1/r_2 = c$, and from P draw the line PO so that the angle α between it and r_2 is the same as the angle between r_1 and p; PO intersects the axis at a point O which is distant q from Q_2, and let $OP = R$. By construction, the triangles OPQ_2 and OPQ_1 are equiangular and therefore similar, so that

$$\frac{r_1}{r_2} = \frac{R}{q} = c \tag{33}$$

and

$$\frac{r_1}{r_2} = \frac{p + q}{R} = c \tag{34}$$

Equations (33) and (34) involve the two quantities R and q in terms of the given constants p and c and when solved for q and R give the relations

$$q = \frac{p}{c^2 - 1} \tag{35}$$

$$R = \frac{cp}{c^2 - 1} \tag{36}$$

which prove that both q and R are constant for all positions of the point P. The locus of point P is therefore a circle of radius R, which in this case ($c = 4$) is equal to $\frac{4}{15}p$; and its center lies to the left of Q_2 at a distance $\frac{1}{15}p$. This circle is shown to scale in Fig. 12, and it is seen that it satisfies the condition that the lines of force intersect it at right angles.

The diagram of Fig. 12 is a section of the field of force in the plane of the paper, but the distribution is the same in all other planes through the point charges Q_1 and Q_2; hence, if the diagram is rotated around the line Q_1Q_2, tubes of force will be generated, and the circle of zero potential becomes a *sphere of zero potential*. If now a thin metallic coating is applied to the sphere and the spherical shell thus formed is connected to the earth (that is, grounded) so as to maintain it at zero potential, the conditions external to the sphere remain absolutely unchanged. Within the hollow conducting shell there can be, however, no electric field, as proved by Faraday's ice-pail experiment. There could be a charge $-Q_2$ located as at the beginning, but a positive charge $+Q_2$ will appear on the inner surface of the sphere, and an equal and opposite charge $-Q_2$ will appear on the outer surface. Inside the sphere, the original charge $-Q_2$ and the induced surface charge $+Q_2$ exactly neutralize each other at all points, the effect being the same as though there were no internal charge at all. But outside the sphere there is now a negative charge of magnitude $-Q_2$ distributed over the entire surface. What has occurred is that charge $-Q_2$, originally concentrated at a point, has been expanded to cover the surface of zero potential which at first existed only as a mathematical concept but which has become a grounded metallic surface.

The physical meaning of this analysis is that if a point charge $+Q_1$ is brought into the vicinity of a grounded metallic sphere (Fig. 14), a negative bound charge $-Q_2$ will be induced on the outer surface of the sphere, and the resultant field of electric force *external* to the sphere will be exactly the same as if the spherical distribution were replaced by a negative point charge

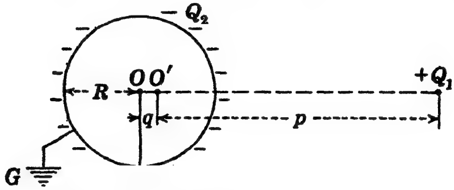


FIG. 14.—Image of point charge in grounded conducting sphere.

$-Q_2$, at the *inverse* point O' of the circle. The charge $-Q_2$ at the inverse point is in effect the *electrical image* of the entire surface distribution on the sphere. Conversely, a negative charge $-Q_2$ placed inside a grounded conducting sphere will induce a positive charge distributed over the inner surface, and *the effect of this distributed charge will be the same, at points within* the sphere, as though it were imaged by a positive charge $+Q_1$ at the outer inverse point.

In the case shown in Fig. 14, the given magnitudes are Q_1, R, and the distance $(p + q)$ between Q_1 and the center of the sphere; in accordance with Eqs. (33) and (34),

$$p + q = c^2q = \frac{R^2}{q}$$

or

$$(p + q)q = R^2 \tag{37}$$

and R is a mean proportional between q and $(p + q)$. The distance q is readily found from the relation

$$q = \frac{R^2}{p + q} \tag{38}$$

and the charge Q_2 is

$$Q_2 = \frac{Q_1}{c} = \frac{Q_1q}{R} = \frac{Q_1R}{p + q} \tag{39}$$

It has now been shown that the point charge $-Q_2$ can be redistributed on the surface of zero potential that originally surrounded

it. From this fact it is a simple step to the conclusion that $-Q_2$ can likewise be transferred to any one of the other equipotential surfaces (having positive or negative potentials) around it, provided that the surface is conducting and is maintained at the appropriate potential; in general, however, if Q_1 and Q_2 are unequal, these other equipotential surfaces are not spherical.

The theory of electrical images, of which this case is an example, was first developed by William Thomson, later Lord Kelvin. It is of great interest and importance, and further examples of its use will be given.

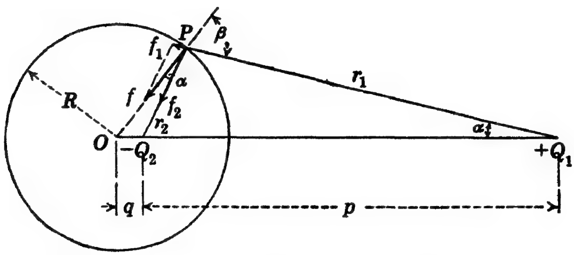


FIG. 15.—Field intensity at zero potential surface.

15. Distribution of Charge and Dielectric Flux on Sphere of Zero Potential.—Assume first that two point charges $+Q_1$ and $-Q_2$, p cm. apart in air, produce the field distribution shown in Fig. 12. At the zero-potential surface (considered merely as a geometrical surface) the force on a unit positive charge at any point P, Fig. 15, is $f_1 = Q_1/\epsilon_0 r_1^2$ due to Q_1, and $f_2 = Q_2/\epsilon_0 r_2^2$ due to Q_2, the former being a repulsion and the latter an attraction. The resultant of these two forces, indicated by f in Fig. 15, must be directed along the radius of the circle of zero potential. Consequently, f is the sum of the radial components of f_1 and f_2, or

$$f = E = \frac{Q_2}{\epsilon_0 r_2^2} \cos \alpha + \frac{Q_1}{\epsilon_0 r_1^2} \cos \beta \tag{40}$$

where the angles α and β are determined by the relations

$$q^2 = R^2 + r_2^2 - 2Rr_2 \cos \alpha \tag{41}$$

$$(p + q)^2 = R^2 + r_1^2 - 2Rr_1 \cos \beta \tag{42}$$

In Eq. (40), $\cos \alpha$ and $\cos \beta$ can be replaced by means of Eqs. (41), (42); Q_1 can be replaced by its equivalent $Q_2(p + q)/R$, from Eq. (39); r_1 can be replaced by using the relation

$$\frac{r_1}{r_2} = \frac{Q_1}{Q_2} = \frac{p + q}{R}$$

and finally, by Eq. (37), $(p + q)$ can be replaced by R^2/q, so that f reduces to

$$f = E = \frac{R^2 - q^2}{R} \frac{Q_2}{\epsilon_0 r_2^3} \tag{43}$$

Now let the purely geometrical sphere be replaced by a grounded, conducting, spherical shell of equal radius, as in Fig. 14, the charge $+Q_1$ remaining as before. A bound charge $-Q_2$ will distribute itself over the surface of the sphere, and inside the sphere there will be no electric force; but outside the sphere the field will be undisturbed, and immediately adjacent to its surface the field intensity at a point P, Fig. 15, will be such that the force on a unit charge will be given by Eq. (43). But it is known from Eq. (5) that $f = E = D/\epsilon_0$, where D is the dielectric flux density, so that

$$D = \frac{R^2 - q^2}{R} \frac{Q_2}{r_2^3}$$

Moreover, if the charge per unit area at the point P is represented by σ, the electrostatic flux emitted from an elementary area dA will be

$$d\psi = 4\pi\sigma \, dA$$

and all this flux will be external to the sphere since there is no force inside it. Consequently,

$$\frac{d\psi}{dA} = D = 4\pi\sigma$$

and

$$\sigma = \frac{R^2 - q^2}{4\pi R} \frac{Q_2}{r_2^3} \tag{44}$$

The last equation shows that the density of the charge on the sphere varies inversely as the cube of r_2; it is therefore greatest at the point immediately facing the inducing charge Q_1 and falls off rapidly to a minimum value at the diametrically opposite point. The dielectric flux density surrounding the sphere is similarly distributed, since D is proportional to σ.

In general, the density of charge and the resultant concentration of dielectric flux tend to be greatest where the radius of curvature of a charged body is least, as at sharp projections, and thus, in designing the parts of high-tension apparatus, sharp edges

and points must be avoided. But it is not correct to assume that the dielectric flux density will be uniform merely because of uniform surface curvature, as this case of the charged sphere plainly shows. The density of charge depends not only upon the curvature but also on the shape and position of neighboring charged bodies.

16. Capacitance of Parallel Wires.—A pair of parallel wires separated from each other by air or other dielectric conforms to the general definition of a condenser; the calculation of the capacitance of the wires is of importance in connection with power-transmission lines and telephone circuits.

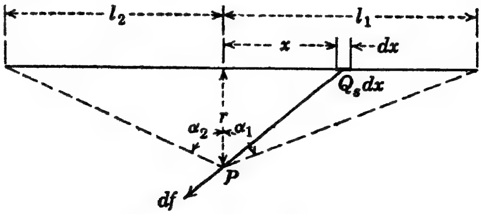


FIG. 16.—Force due to a charged wire.

Assume for the time being that the wires have such small cross-section that they may be considered to be geometrical lines. Let the charges per centimeter length be $+Q_s$ and $-Q_s$ e.s.u., and assume that the dielectric is air. At a perpendicular distance r from the positively charged wire Fig. 16, the force on a unit positive charge at point P due to the elementary charge $Q_s\,dx$ is

$$df = \frac{Q_s\,dx}{\epsilon_0(r^2 + x^2)}$$

and the component in the direction of r is

$$df_r = \frac{Q_s\,dx}{\epsilon_0(r^2 + x^2)} \cdot \frac{r}{\sqrt{r^2 + x^2}}$$

The entire force in the direction r is

$$f_r = \int_{-l_2}^{+l_1} \frac{Q_s r\,dx}{\epsilon_0(r^2 + x^2)^{3/2}} = \frac{Q_s}{\epsilon_0 r}(\sin\alpha_1 + \sin\alpha_2) \qquad (45)$$

and if the wire is indefinitely long in both directions,

$$\epsilon_0 f_r = D = \frac{2Q_s}{r} \qquad (46)$$

The components of force parallel to the wire cancel out completely under this last condition, and the lines of dielectric flux therefore radiate perpendicularly from the wire in all directions.

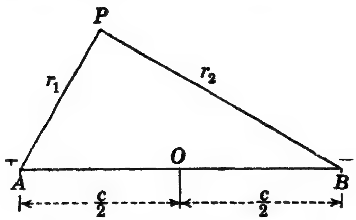


FIG. 17.—Potential due to pair of parallel charged wires.

Now let A and B, Fig. 17, represent a cross-sectional view of the two charged wires which are c cm. apart. At a point distant r cm. from the positively charged wire A the force on a positive unit charge is $2Q_s/\epsilon_0 r$; and in moving the unit charge from point P to point O (midway between A and B), the work done, which is the difference of potential between P and O, is

$$\int_{V_{P_A}}^{V_0} dV = -\int_{r_1}^{c/2} f_r\, dr = -\frac{2Q_s}{\epsilon_0}\int_{r_1}^{c/2}\frac{dr}{r}$$

whence

$$V_{P_A} - V_0 = \frac{2Q_s}{\epsilon_0}\ln\frac{c}{2r_1}$$

Similarly, the potential at P due to the charge on wire B is determined by

$$V_{P_B} - V_0 = -\frac{2Q_s}{\epsilon_0}\ln\frac{c}{2r_2}$$

Because of the symmetry of arrangement of the wires, V_0 is zero; hence the resulting potential at P is

$$V_P = V_{P_A} + V_{P_B} = \frac{2Q_s}{\epsilon_0}\ln\frac{r_2}{r_1} \tag{47}$$

If, in Eq. (47), the condition is now imposed that V_P shall have a constant value, say V,

$$\frac{r_2}{r_1} = e^{V\epsilon_0/2Q_s} = \text{constant} \tag{48}$$

which fixes the equation of the equipotential line corresponding to V. The equipotential curves are circles that have A and B as their inverse points, since their equation is the same as Eq. (32) in Art. 14, where the proof was given. Several of these circles are shown in Fig. 18.

In exactly the same manner as in Art. 14, the charge Q_s on wire A can be transferred to the outer surface of a conducting

cylinder bounded by any of the encircling equipotential cylinders without in any way disturbing the external field; and similarly for the charge $-Q_s$ on wire B. If the circles surrounding A and B are made equal to each other, with radii r cm. and a distance center to center of d cm., we have a pair of parallel conductors of finite size, as in actual practice.

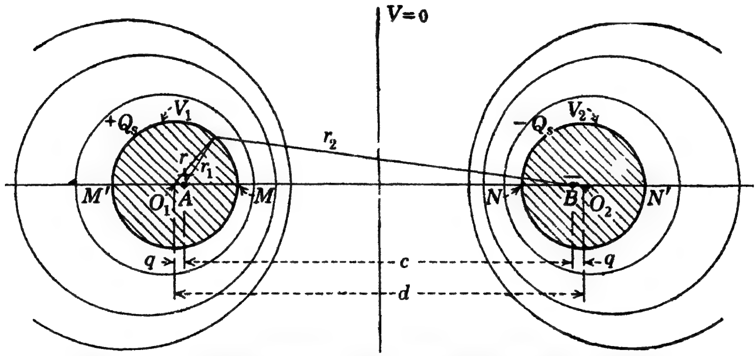


FIG. 18.—Equipotential curves, parallel charged conductors

At every point on the shaded circle surrounding A, including points M and M', the potential is, in accordance with Eq. (47),

$$V_1 = \frac{2Q_s}{\epsilon_0} \ln \frac{r_2}{r_1} = \frac{2Q_s}{\epsilon_0} \ln \frac{BM}{AM}$$

and because of symmetry the potential of the corresponding circle surrounding B is equal and opposite in sign. The difference of potential between the two conductors is, therefore,

$$V_s = 2V_1 = \frac{4Q_s}{\epsilon_0} \ln \frac{BM}{AM}$$

and the capacitance of the conductors, per centimeter of their length, is

$$C_s = \frac{Q_s}{V_s} = \frac{\epsilon_0}{4 \ln (BM/AM)} \quad \text{statfarads}$$

From the geometry of Fig. 18,

$$BM = c - (r - q)$$
$$AM = r - q$$

Also, it is known from Art. 14 that

$$(c + q)q = r^2$$

and the diagram shows that

$$c + 2q = d$$

Solving the last two equations for c and q in terms of r and d,

$$c = 2\sqrt{\frac{d^2}{4} - r^2}$$

$$q = \frac{d}{2} - \sqrt{\frac{d^2}{4} - r^2}$$

whence

$$\frac{BM}{AM} = \frac{\sqrt{\frac{d^2}{4} - r^2} + \left(\frac{d}{2} - r\right)}{\sqrt{\frac{d^2}{4} - r^2} - \left(\frac{d}{2} - r\right)} = \frac{d + \sqrt{d^2 - 4r^2}}{2r}$$

so that the capacitance of two parallel wires per centimeter of length, with air as dielectric, is

$$C_s = \frac{\epsilon_0}{4 \ln \frac{d + \sqrt{d^2 - 4r^2}}{2r}} \text{ statfarads per cm.} \qquad (49)$$

If d is large in comparison with r, Eq. (49) reduces to the approximate value

$$C_s = \frac{\epsilon_0}{4 \ln (d/r)} \qquad (50)$$

and if the wires are surrounded by a medium of which the dielectric constant is K,

$$C_s = \frac{K\epsilon_0}{4 \ln (d/r)} \qquad (51)$$

The approximate formulas (50) and (51) give very misleading results when d is relatively small, and in that case Eq. (49) must be used.

17. Capacitance of Wire Parallel to Earth.—Figure 19 represents a cross-section of a conductor of radius r cm. of indefinitely great length, mounted parallel to the earth's surface and at a distance h cm. above the ground, measured to its center. On charging the conductor to a potential V_s statvolts above ground (zero) potential, the charge per unit length of conductor will be Q_s statcoulombs; and an equal and opposite charge, $-Q_s$ units,

will be induced on the earth's surface, but spread out on both sides of the foot of the perpendicular h.

On comparing Fig. 19 with Fig. 18, it is seen that the given conductor in Fig. 19 corresponds to conductor O_1 in Fig. 18, and that the earth's surface in Fig. 19 is the equivalent of the plane of zero potential* ($V = 0$) in Fig. 18. For if in Fig. 18

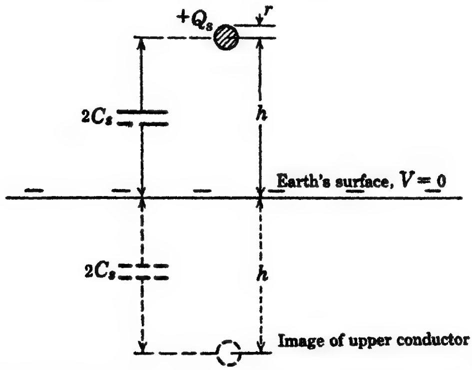


FIG. 19.—Capacitance of wire parallel to earth's surface.

a grounded conducting plate of infinite extent were inserted midway between conductors O_1 and O_2, the charge $-Q_s$ could be transferred to this surface of zero potential in accordance with the principle developed in Art. 14. That is, conductor O_2 of Fig. 18 can be regarded as the electrical image of conductor O_1 provided that the surface of zero potential is made of conducting material.

The capacitance C_s of the two conductors in Fig. 18 is equivalent to two capacitances, each equal to $2C_s$, connected in series, as indicated in Fig. 19. It follows, therefore, that the capacitance of the single conductor of Fig. 19 is double the capacitance given by Eq. (49), and the only other change that need be made is to substitute h for $\frac{1}{2}d$. Hence, the capacitance of the single conductor, in air, is

$$C_s = \frac{\epsilon_0}{2 \ln \dfrac{h + \sqrt{h^2 - r^2}}{r}} \text{ statfarads per cm.} \qquad (52)$$

18. Lines of Force from Parallel Charged Conductors.—In Art. 16 it is shown that a pair of parallel geometrical lines is

* The plane of zero potential is merely the limiting case of the family of circles and corresponds to $r_1 = r_2$.

exactly equivalent to a pair of conductors of finite circular cross-section, provided that the geometrical lines pass through the inverse points of the two circular sections. The converse proposition also holds, and a pair of actual conductors will now be replaced by an equivalent pair of geometrical lines in order to determine the distribution of dielectric flux around the actual conductors.

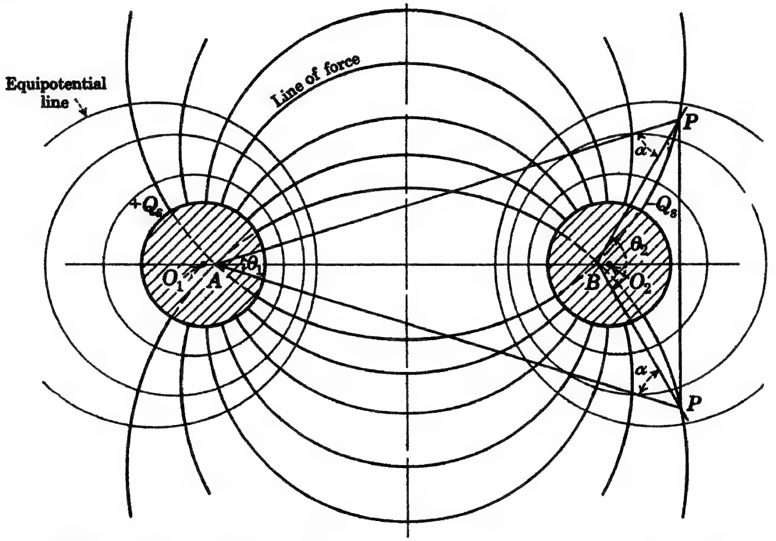


FIG. 20.—Lines of electric force surrounding parallel charged wires.

In Fig. 20, O_1 and O_2 are the centers of the two equal circular cross-sections, and A and B are the inverse points, exactly as in Fig. 18. Each linear wire, having a charge of Q_s statcoulombs per cm. (one charge being positive, the other negative), has associated with it a flux of $4\pi Q_s$ lines per cm. of its length; and if each wire were alone in space, these lines would radiate outward along straight lines in the case of $+Q_s$, and inward, in the case of $-Q_s$.

Let the planes of which the traces are AP and AP' be passed through filament A, enclosing the dihedral angle θ_1; and let similar planes, enclosing angle θ_2, be drawn through B. The outward flux enclosed by θ_1 will be, per centimeter length of wire,

$$\psi_1 = \frac{\theta_1}{2\pi} \cdot 4\pi Q_s = 2Q_s\theta_1$$

and the inward flux enclosed by θ_2, per centimeter length of wire, will be

$$\psi_2 = 2Q_s\theta_2$$

The resultant inward flux through the rectangle of which the vertical dimension is PP' and which is 1 cm. wide perpendicular to the plane of the paper is

$$\psi = \psi_2 - \psi_1 = 2Q_s(\theta_2 - \theta_1) \tag{53}$$

If the condition is imposed that ψ is to remain constant, the locus of P and of P', as θ_1 and θ_2 change, will be a line of force; that is,

$$\theta_2 - \theta_1 = \text{constant} = 2\alpha$$

is the condition that determines a line of force, the angle α being indicated in Fig. 20. But if α is constant the locus of P will be a circle which passes through A and B and of which the center lies on the vertical line midway between A and B. Several of these lines of force have been drawn in Fig. 20, and it will be noted that in all cases they intersect the circular lines of equipotential (constructed as in Fig. 18) at right angles.

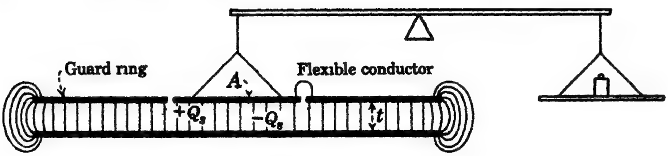


FIG. 21.—Absolute electrometer.

It is an interesting fact that the lines of electric force around two statically charged parallel wires (Fig. 20) are identical with the equipotential lines of the magnetic field which surrounds the same wires when they carry current. It is also true that the equipotential lines of the electric field in Fig. 20 are the same as the lines of magnetic force which surround the same wires when a current flows in the wires.

19. Absolute Electrometer.—In Art. 6 it is shown that the capacitance of a parallel-plate condenser is

$$C_s = \frac{A\epsilon_0}{4\pi t} \quad \text{statfarads}$$

and that a difference of potential of V_s statvolts between the plates will charge the condenser with $Q_s = C_sV_s$ statcoulombs, provided that the disturbing effect of the edges is neglected. If the upper plate in Fig. 21 is constructed so that a circular

hole in it is almost filled by a circular metal disk of area A, connected to the main part of the plate by a fine, flexible wire so that both parts are at the same potential, the lines of electric force at the surface of the small disk will be normal to the surface. The formula of Art. 6 for the capacitance will then apply to the small condenser consisting of the circular disk and an equal area A on the bottom plate.

If the density of the electric charge is $\sigma = Q_s/A$, the force of attraction between the circular disk and the bottom plate is

$$f = \frac{2\pi\sigma^2 A}{\epsilon_0} = \frac{2\pi Q_s^2}{A\epsilon_0} = 2\pi\frac{C_s^2 V_s^2}{A\epsilon_0}$$

and, on substituting the value of C_s,

$$f = \frac{V_s^2 A\epsilon_0}{8\pi t^2}$$

or

$$V_s = t\sqrt{\frac{8\pi f}{A\epsilon_0}} \quad \text{statvolts} \tag{54}$$

Since the force f is easily measured by some form of balance, as indicated in Fig. 21, and since A and t can be readily measured, there is provided a simple device for the direct measurement of the difference of potential applied to the plates. The instrument as a whole is called the *absolute electrometer* and was invented by William Thomson. The condition of exact balance requires that the disk A must be in the same plane as the stationary part of the upper plate.

The stationary upper plate, which ensures the existence of a uniform field over the area A of the movable disk, is called a *guard ring*. It nullifies the effect of the concentration of dielectric flux, which would otherwise occur at the edges, and in this case transfers the concentration to the edges of the stationary plates where it does no harm. It will be seen later that the principle of the guard ring has numerous important applications.

20. Use of Condensers for High-voltage Measurements; Subdivision of Voltage.—In making measurements of very high voltages it is in general necessary to use instruments of which the range is much less than the voltage to be measured and to multiply the observed reading by an appropriate factor. One way to

accomplish the desired result is to use two condensers in series, in the manner indicated in Fig. 22a, where the high-tension conductor is connected to the upper insulated plate. The low-range voltmeter V_2 is connected between the intermediate condenser plate and the grounded bottom plate, so that the maximum

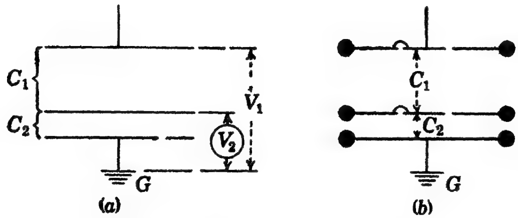


FIG. 22.—Voltage subdivider.

voltage to which the instrument (and the observer) are subjected is the potential difference V_2.

As the potential difference across the upper capacitance C_1 is $V_1 - V_2$,

$$Q = C_1(V_1 - V_2) = C_2V_2$$

whence

$$V_1 = \frac{C_1 + C_2}{C_1}V_2 \tag{55}$$

The potential difference V_2 may be made equal to any desired fraction of V_1 by adjusting the position of the intermediate plate; when this plate is close to the bottom plate, C_2 will be large in comparison with C_1. The upper plate must be sufficiently elevated to prevent the possibility of a flash-over to points of lower potential. The accuracy of the measurement of V_1 depends upon the precision with which C_1 and C_2 can be computed from the known dimensions and separations of the plates.

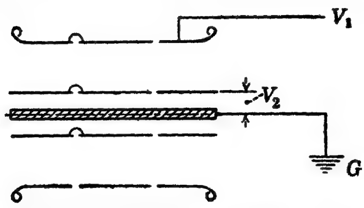


FIG. 23.—Cylindrical condensers for high-voltage measurement.

In order that C_1 and C_2 may be computed with precision, the guard-ring principle is utilized in the manner indicated in Fig. 22b; this arrangement ensures that the electrostatic fields in the parts actually used in the computations will be uniform, without end-effect disturbances. To limit the dielectric flux concentration at the edges of the plates, either the edges are rolled to a

large radius, or else there are attached to the edges wooden rings coated with tin foil, electrically connected to the plates.

Instead of using parallel plates, concentric cylindrical condensers may be used, as shown in Fig. 23.

21. Voltage Multiplication by Means of Condenser.—Suppose that a parallel-plate condenser C_{s_1} (Fig. 24) is charged to a difference of potential V_{s_1}, so that the charge is

$$Q_s = C_{s_1} V_{s_1}$$

Assume that both plates are thoroughly insulated, so that the charge Q_s, once imparted, remains without change. On increas-

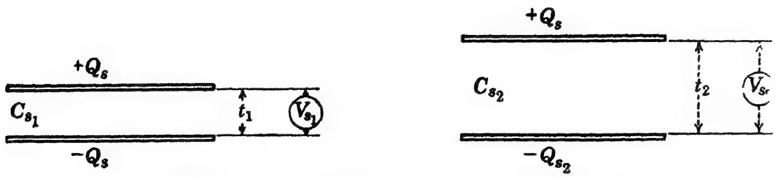


FIG. 24.—Effect of separating plates of a condenser.

ing the separation of the plates from the original amount t_1 cm. to t_2 cm., the capacitance will change from

$$C_{s_1} = \frac{\epsilon_0 A}{4\pi t_1}$$

to

$$C_{s_2} = \frac{\epsilon_0 A}{4\pi t_2}$$

and the potential difference V_{s_2} will be

$$V_{s_2} = \frac{Q_s}{C_{s_2}} = \frac{C_{s_1} V_{s_1}}{C_{s_2}} = \frac{t_2}{t_1} V_{s_1} \tag{56}$$

or the difference of potential will increase in direct proportion to the distance between the plates, provided that the formulas for the capacitances remain valid. Because of the increased fringing, or bulging, of the dielectric flux at the edges of the plates as they are separated more and more, this direct proportionality will be departed from, but the potential difference will increase. The energy stored in the condenser, originally equal to

$$W_1 = \tfrac{1}{2} C_{s_1} V_{s_1}^2$$

becomes

$$W_2 = \frac{1}{2}C_{s_2}V_{s_2}^2 = W_1\frac{t_2}{t_1}$$

and the increase of stored energy is equal to the work done in separating the plates against their mutual attraction.

22. Series and Parallel Groupings of Dielectrics. 1. *Series Grouping.*—Figure 25 represents a pair of conducting plates separated by three layers of dielectric materials of which the dielectric constants are K_1, K_2, K_3. If the charge on the conducting plates is Q_s units, a total flux $\psi = 4\pi Q_s$ will pass through the successive layers, if it is assumed that the lines of force pass straight across without fringing (bulging) at the sides. The flux density will therefore be the same in all the successive layers.

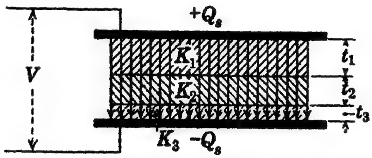


FIG. 25.—Series grouping of dielectrics.

If we imagine that there is an infinitesimally thin film of conducting material between each pair of dielectric layers, positive and negative charges equal to Q_s will be induced on the faces of each film; but the charges are so close together that they neutralize, just as if the conducting film were absent, as in reality it is. This idea enables us to visualize the assembly as a number of capacitors in series, the capacitances being $C_1 = K_1\epsilon_0 A/4\pi t_1$, $C_2 = K_2\epsilon_0 A/4\pi t_2$, $C_3 = K_3\epsilon_0 A/4\pi t_3$. The charge of each being Q_s, the differences of potential across the successive layers are $V_1 = 4\pi Q_s t_1/\epsilon_0 A K_1$, $V_2 = 4\pi Q_s t_2/\epsilon_0 A K_2$, $V_3 = 4\pi Q_s t_3/\epsilon_0 A K_3$. The voltage across the outer plates is

$$V = V_1 + V_2 + V_3 = \frac{4\pi Q_s}{\epsilon_0 A}\left(\frac{t_1}{K_1} + \frac{t_2}{K_2} + \frac{t_3}{K_3}\right)$$

so that the capacitance of the assembly as a whole is

$$C = \frac{Q_s}{V} = \frac{A\epsilon_0}{4\pi\left(\frac{t_1}{K_1} + \frac{t_2}{K_3} + \frac{t_3}{K_3}\right)} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

just as might be anticipated from Eq. (28). But the point of importance is that

$$V_1 : V_2 : V_3 = \frac{t_1}{K_1} : \frac{t_2}{K_3} : \frac{t_3}{K_3} \qquad (57)$$

and

$$\frac{V_1}{t_1}:\frac{V_2}{t_2}:\frac{V_3}{t_3} = \frac{1}{K_1}:\frac{1}{K_2}:\frac{1}{K_3} \tag{58}$$

which means that the *voltage gradients* (statvolts per centimeter, or volts per inch) *are inversely proportional to the relative permittivities* K_1, K_2, K_3.

Thus, if $K_1 = 8$, $K_2 = 5$, and $K_3 = 1$, and $t_1 = t_2 = t_3 = 1$ cm., a total difference of potential V of 60,000 volts will divide so that $V_1 = 5661$, $V_2 = 9056$, and $V_3 = 45{,}283$ volts. The value $K_3 = 1$ corresponds to an air space, so that under the assumed conditions the 1 cm. layer of air is subjected to a potential difference of 45,283 volts, which is more than its critical or breakdown gradient of 30,000 volts per cm. Had the two upper layers of relative permittivities K_1 and K_2 been removed, the 3 cm. air space would have been subjected to only 20,000 volts per cm., which is safely within its breakdown limit. *We therefore have the apparent paradox that the insertion of insulation between charged metallic surfaces may actually promote the breakdown of the insulation.*

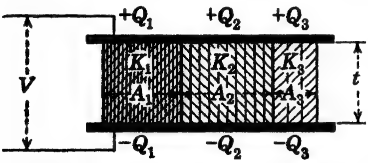


FIG. 26.—Parallel grouping of dielectrics.

2. *Parallel Grouping.*—If several dielectric materials are placed side by side, filling the space between two conducting plates (Fig. 26), and if it is assumed that the lines of dielectric flux pass perpendicularly from plate to plate, the flux densities will be proportional to the corresponding permittivities, and the plate charges will be greatest adjacent to the material of greatest permittivity. Each material will be subjected to the same voltage gradient V/t. The total flux from plate to plate divides in proportion to the permittivities, as one would expect from the fact that a high permittivity implies a high conducting capacity for dielectric flux. Because of the fact that, in the diagram, $K_1 > K_2 > K_3$, the material on the left will tend to rob the middle column of some of its flux, and the middle column will in turn tend to take flux away from the one on the right; the result will be that the lines of flux in the regions A_2 and A_3, instead of passing perpendicularly from plate to plate, will be distorted to the left, since region A_1 offers better dielectric conductance.

23. Refraction of Dielectric Flux.—Dielectric flux that crosses the boundary between media of different permittivities obeys the same laws as those discussed in Art. 20, Chap. II, in connection with the refraction of lines of magnetic induction. That is, the normal components of the dielectric flux on the two sides of the surface of separation are necessarily equal because the same number of lines continues through both media; and the tangential components of the electric force are equal for the same reasons that apply in the case of magnetic substances. Consequently,

$$E_1 \sin \theta_1 = E_2 \sin \theta_2$$

and

$$D_1 \cos \theta_1 = D_2 \cos \theta_2$$

where θ_1 and θ_2 are the angles between the direction of the flux and the normal to the surface, and therefore

$$\frac{(D_1/E_1) = \epsilon_1}{\tan \theta_1} = \frac{(D_2/E_2) = \epsilon_2}{\tan \theta_2} \tag{59}$$

The range of values of permittivity being much less than for magnetic permeability, the angles θ_1 and θ_2 in dielectric refraction do not exhibit the same marked differences of direction that are found in magnetic refraction.

24. Graded Insulation in Cable Construction.—Figure 27*a* represents a cross-section of a single-conductor, high-tension cable of radius R_1, having a grounded metal sheath S of which the internal radius is R_2; the intermediate insulation is assumed to be homogeneous and to have dielectric constant K. The cable with its sheath constitutes a cylindrical condenser; and in accordance with Eqs. (14), (16) of Art. 8 the field intensity at a distance r from the center is

$$E = \frac{2Q_s}{K\epsilon_0 r}$$

and the capacitance (per centimeter length of cable) is

$$C_s = \frac{Q_s}{V_s} = \frac{K\epsilon_0}{2 \ln (R_2/R_1)}$$

where V_s is the difference of potential between the wire O and the grounded sheath S, and therefore also the potential of the conductor.

The quantity $F(-dr)$ is the work done in moving a unit positive charge radially inward through the distance $(-dr)$, and by definition this work equals the difference of potential dV between layers that are dr cm. apart; consequently,

$$-\frac{dV}{dr} = E = \frac{2Q_s}{K\epsilon_0 r} = \frac{V_s}{\ln (R_2/R_1)} \cdot \frac{1}{r} \tag{60}$$

But $-dV/dr$ is the potential gradient at the distance r from the center and is seen to vary inversely as r; hence, it is greatest when $r = R_1$ and least when $r = R_2$, the variation being shown in the lower part of Fig. 27*a*. Both Eq. (60) and the gradient diagram

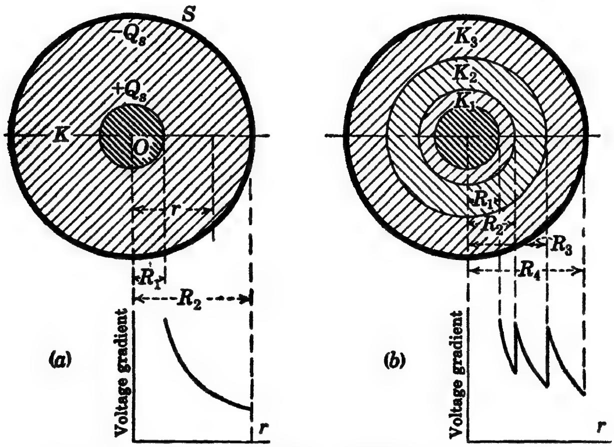


FIG. 27.—Voltage gradient in single-conductor cable.

show that the dielectric immediately adjacent to the high-tension core of the cable is subjected to much greater stress than are the layers nearer the sheath, and these inner layers may be expected to fail first when the gradient reaches the critical value of the insulating material. If breakdown occurs, which means that the material becomes conducting, the entire voltage V_s is then applied to a reduced thickness of insulation and the breakdown of the material may progress rapidly outward until the inner core is short-circuited.

Under ideal conditions, all parts of the insulation should be subjected to the same stress; that is, $E = 2Q_s/K\epsilon_0 r$ should be uniform. But this condition could occur only if K varied continuously in inverse proportion to r, which is impossible of attainment. However, by *grading* the insulation, using material of high

permittivity near the center and then another two or three layers of successively lower permittivity, an approximation to the ideal can be attained.

For example, Fig. 27b shows three layers of different permittivities, where $K_1 > K_2 > K_3$. In the inner layer, at a general distance r from the center, the force on a unit charge is

$$E_1 = \frac{2Q_s}{K_1\epsilon_0 r}$$

and the drop of potential between $r = R_1$ and $r = R_2$ is

$$V_1 = \int_{R_1}^{R_2} \frac{2Q_s}{K_1\epsilon_0 r} dr = \frac{2Q_s}{K_1\epsilon_0} \ln \frac{R_2}{R_1}$$

Similarly, the drops across the middle and outer layers are

$$V_2 = \frac{2Q_s}{K_2\epsilon_0} \ln \frac{R_3}{R_2}$$

and

$$V_3 = \frac{2Q_s}{K_3\epsilon_0} \ln \frac{R_4}{R_3}$$

so that

$$V_s = V_1 + V_2 + V_3 = \frac{2Q_s}{\epsilon_0}\left(\frac{1}{K_1} \ln \frac{R_2}{R_1} + \frac{1}{K_2} \ln \frac{R_3}{R_2} + \frac{1}{K_3} \ln \frac{R_4}{R_3}\right)$$

and

$$C_s = \frac{Q_s}{V_s} = \frac{\epsilon_0}{2\left(\frac{1}{K_1} \ln \frac{R_2}{R_1} + \frac{1}{K_2} \ln \frac{R_3}{R_2} + \frac{1}{K_3} \ln \frac{R_4}{R_3}\right)} \tag{61}$$

the last expression giving the capacitance of the composite cable in statcoulombs per centimeter of its length; the form of the expression shows that the component capacitors are in series. The maximum voltage gradients in the three layers are

$$(E_1)_{max.} = \frac{2Q_s}{K_1\epsilon_0 R_1} = \frac{2C_sV_s}{K_1\epsilon_0 R_1} \tag{62}$$

$$(E_2)_{max} = \frac{2C_sV_s}{K_2\epsilon_0 R_2} \tag{63}$$

$$(E_3)_{max} = \frac{2C_sV_s}{K_3\epsilon_0 R_3} \tag{64}$$

Consequently, if, in addition to the values of the permittivities, the maximum permissible gradient for each material is known

from actual breakdown tests, the radii can be computed so that all the layers will simultaneously be stressed to their allowable limits.

25. Condenser Bushings.—High-tension equipment, such as transformers and circuit breakers, in which the working parts are enclosed in steel casings, presents the important problem of properly insulating the terminals where they pass through the walls of the grounded enclosure. To a certain extent the conditions are similar to those just discussed in the preceding article, but with the important difference that the cylindrical surface of the hole in the present case is of short length, a condition that

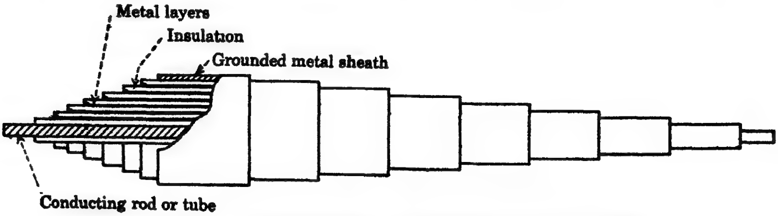


FIG. 28.—Construction of condenser-type terminal.

tends to create an intense concentration of dielectric flux in the space between it and the high-tension conductor.

In the condenser bushing, originated by the Westinghouse Electric and Manufacturing Company and illustrated diagrammatically in Fig. 28, the problem is met by subdividing the insulating material into concentric layers of approximately equal thickness, separated by tin foil, thus constituting a number of condensers in series. If all the condensers were of the same axial length, the capacitance would increase in a logarithmic manner from a minimum value for the inner layer to a maximum value for the outer layer, the inner layer being thus subjected to a disproportionately high electric stress. To equalize the voltage drops across the successive condensers, these are accordingly made of successively shorter axial lengths.

Concentration of dielectric flux at the edges of the metal separators is prevented by the use of rings of circular section.

In practice, bushings of this type are enclosed in an insulating cylinder filled with an insulating compound. The inner conductor is made of a brass tube the diameter of which is sufficiently large to limit the flux concentration at its surface to a value safely within the limit of the insulation.

26. Displacement Current in Dielectrics. Dielectric Hysteresis.—The charge in a condenser of capacitance C being equal to

$$q = Cv$$

where q and v are corresponding instantaneous values of charge and potential difference, any variation of v must be accompanied by a proportional change in q, provided that v does not exceed the limit beyond which the capacitance C does not remain constant. Therefore, when v changes with the time t,

$$\frac{dq}{dt} = i = C\frac{dv}{dt} \tag{65}$$

so that, when $dv/dt = 1$, $i = C$ This statement is equivalent to saying that *the capacitance of a condenser (in farads) is numerically equal to the current (in amperes) that will flow in the supply circuit when the impressed voltage varies at the rate of 1 volt per sec.*

In the ordinary types of circuits discussed in Chap. I, there is a continuous, closed, conducting path for the current. But it is clear from Eq. (65) that when the circuit includes a condenser, that is, when an insulated gap exists in the otherwise continuous, conducting circuit, it is still possible to have a finite current, either continuous in one direction or alternating in both directions, depending upon the nature of the variation of the potential applied to the condenser. This current does not "get through" the condenser in the ordinary sense of a flow of electrons, such as exists in metallic conductors. In the dielectric that constitutes the insulating gap in the circuit, the current is to be thought of as a *displacement current.* The molecules of the dielectric are subjected to a physical stress and strain because of the electric force established by the potential gradient; they may be thought of as elongating and contracting as the electric force is varied, becoming, as it were, minute electrostatic needles of variable length called *dipoles*, with positive and negative charges at their opposite ends, which tend to orient themselves in line with the electric force. The movement of the charges on the ends of the dipoles, while the voltage gradient is changing, constitutes the displacement current.

In an a-c circuit the instantaneous potential difference across the condenser terminals will vary, say in accordance with the equation

$$v = V_{max} \sin \omega t \tag{66}$$

where V_{max} is the maximum value of the sinusoidally varying voltage, and ω is a constant. Hence, from Eq. (65),

$$i = \omega C V_{max} \cos \omega t = \omega C V_{max} \sin\left(\omega t + \frac{\pi}{2}\right) \tag{67}$$

and the maximum value of the alternating current, which occurs when $\omega t + \pi/2 = 90$ deg., and periodically thereafter, is

$$I_{max} = \omega C V_{max} \quad \text{amp.} \tag{68}$$

where C is expressed in farads and V_{max} in volts.

Under these conditions, namely, when the impressed voltage varies harmonically from a positive to a negative maximum and then back again, the dipoles in the dielectric elongate first one way, then the other, with a change of polarity at their ends. If they were perfectly elastic and devoid of anything resembling friction between adjacent molecules, these internal distortions would not consume any energy; but experiment shows that, though most dielectrics show no appreciable departure from the kind of perfect elasticity implied by Hooke's law in mechanics, there is a loss of energy to which the name *dielectric hysteresis* has been given.

It is not certain that dielectric hysteresis implies the same type of lag of dielectric flux behind the electric force that exists between magnetic induction and magnetizing force; but there must be some lag to account for the known loss of energy, which appears as sensible heat in the dielectric and raises its temperature. Moreover, when a condenser has a solid dielectric and is discharged after being charged, it is found, upon waiting after the first discharge has taken place, that a second, but feebler discharge, can be obtained; and in some cases additional discharges may be obtained if sufficient time is allowed to elapse. The entire energy of the initial charge is not drained off by the first discharge, the effect being as if there exists a kind of dielectric viscosity quite similar in its effects to magnetic remanence. These secondary discharges are a source of danger in handling condensers and must be carefully guarded against. The electricity that appears in the second and subsequent discharges is referred to as the *residual charge* and must be due to the slow release of the dipoles from their initial state of stress.

27. Charge and Discharge of Condensers, Noninductive Circuit. 1. *Charge.*—Assume that a condenser of capacitance C farads, originally uncharged, is connected to a source of e.m.f. E volts through a resistance R ohms (Fig. 29); let the switch S be closed at the moment when the time $t = 0$. At any instant thereafter, when $t = t$, the current in the circuit will have a general value i and the charge in the condenser will be q. The entire e.m.f. E will be used to overcome the drop iR in the resistor and to balance the potential difference q/C that appears across the terminals of the condenser; in accordance with Kirchhoff's second law,

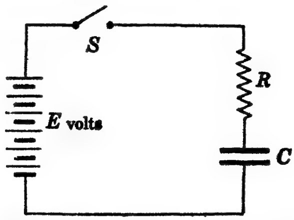


FIG. 29—Charging a condenser.

$$E = iR + \frac{q}{C} \tag{69}$$

But we also have the relation

$$i = \frac{dq}{dt} \tag{70}$$

whence

$$E = R\frac{dq}{dt} + \frac{q}{C}$$

and

$$\int_0^t dt = \int_0^q \frac{RC\,dq}{EC - q} \tag{71}$$

Integration of Eq. (71) gives

$$t = -RC \ln \frac{EC - q}{EC}$$

which may be written

$$q = CE(1 - e^{-t/RC}) \tag{72}$$

and, on differentiating Eq. (72),

$$\frac{dq}{dt} = i = \frac{E}{R}e^{-t/RC} \tag{73}$$

The graphs of these equations for q and i are shown in Fig. 30. It is seen that, at the first instant ($t = 0$), the charge q builds up rapidly, then more and more slowly, approaching a limiting

value $Q = CE$ after a theoretically infinite time, but practically in a finite time. These relations are analogous to those that relate the quantity of air in a tire to the time during the period of inflation.

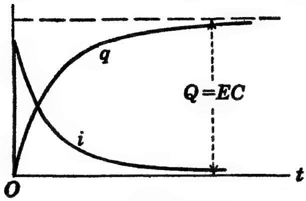


FIG. 30.—Curves of charge and current during charge of a condenser.

The current during the charging period starts out with its maximum value at the first instant (at least theoretically) and then decreases, at first rapidly, then more and more slowly until it is zero when the condenser is completely charged. Actually, it is not possible for the current, which is obviously zero before switch S is closed, to attain maximum value instantly after closing the switch. The apparent inconsistency is accounted for by the fact that a circuit made up exclusively of resistance and capacitance is a physical impossibility, since every circuit possesses some inductance, even though it may be very small. It will be seen later that, when the inductance is taken into account, the curve of current rises at first rapidly, reaches a maximum, and then decreases again; so the initial infinite rate of increase of current indicated in Fig. 30 is really the limiting case that corresponds to the assumed zero inductance.

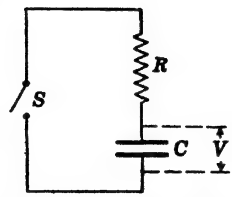


FIG. 31.—Discharging a condenser.

2. *Discharge.*—Assume that a condenser of capacitance C farads has been charged to a difference of potential of V volts and that at $t = 0$ it is discharged through a resistance R ohms (Fig. 31). The initial charge in the condenser is $Q = CV$, and at a general time t thereafter the charge is q and the current is i. By Kirchhoff's second law,

$$E = 0 = iR + \frac{q}{C} \tag{74}$$

and since $i = dq/dt$,

$$0 = R\frac{dq}{dt} + \frac{q}{C} \tag{75}$$

and

$$-\int_0^t \frac{dt}{RC} = \int_Q^q \frac{dq}{q}$$

or

$$q = Qe^{-t/RC} \tag{76}$$

and

$$i = \frac{dq}{dt} = -\frac{V}{R}e^{-t/RC} \tag{77}$$

The graphs of Eqs. (76) and (77) are shown in Fig. 32. If V is assumed to be the same as E in the preceding case, the curves of Fig. 32 will be the same as those of Fig. 30, but upside down.

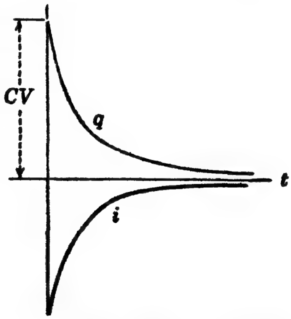


FIG. 32.—Curves of charge and current during discharge of a condenser.

In the exponential term $\epsilon^{-t/RC}$, the quantity $1/RC$ is called the *time constant* of the circuit and corresponds to L/R in the inductive circuit.

If both sides of Eq. (76) are multiplied by $1/C$, q/C represents the potential difference across the condenser at any instant, and $Q/C = V$ is the initial potential difference; or

$$v = Ve^{-t/RC}$$

and

$$R = \frac{t}{C \ln (V/v)} \tag{78}$$

Consequently, if a condenser of known capacitance is allowed to discharge slowly through a high resistance and simultaneous readings of v and t are taken at intervals, it is possible to compute R. This method is often used for the measurement of the high insulation resistance of cables and the like.

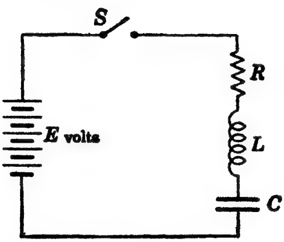


FIG. 33.—Charge of condenser, inductive circuit.

28. General Case of Condenser Charge and Discharge through Inductive Circuit.—Figure 33, which represents the case to be considered, differs from Fig. 29 only in the addition of the inductance L henrys. After closing the switch when $t = 0$, the general equation defining the condition of the circuit is

$$E = iR + L\frac{di}{dt} + \frac{q}{C} \tag{79}$$

where $i = dq/dt$, as before. Differentiating Eq. (79),

$$0 = \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} \tag{80}$$

a differential equation of the second order with constant coefficients; its solution is given by

$$i = Ae^{m_1t} + Be^{m_2t} \tag{81}$$

where A and B are constants of integration, and m_1 and m_2 are the roots of the equation

$$D^2 + \frac{R}{L}D + \frac{1}{LC} = 0 \tag{82}$$

Accordingly,

$$\begin{aligned} m_1 &= -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \\ m_2 &= -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \end{aligned} \tag{83}$$

Since the quantities R, L, and C are all essentially positive but may have any magnitudes, the radical $\sqrt{(R/2L)^2 - (1/LC)}$ may be real, zero, or imaginary depending upon whether

(1) $$\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$$

(2) $$\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$$

(3) $$\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$$

Case 1, $(R/2L)^2 > 1/LC$: *The Logarithmic Case.*—In this case, both m_1 and m_2, Eq. (83), are essentially negative, the current is represented by two exponential terms as in Eq. (81), and it remains only to evaluate the coefficients A and B; since there are two of them, two conditions must be given to determine their magnitudes. We may assume, for example, that when $t = 0$, both $i = 0$ and $q = 0$. From the first condition, there is obtained from Eq. (81) the relation

$$A + B = 0 \tag{84}$$

Differentiating Eq. (81) and substituting in the result $t = 0$,

$$\left.\frac{di}{dt}\right|_{t=0} = Am_1 + Br_2$$

Inserting this equation in Eq. (79), together with $q = 0$ and $i = 0$,

$$E = ALm_1 + BLm_2 \tag{85}$$

Solving Eqs. (84) and (85) for A and B,

$$\left.\begin{aligned} A &= \frac{E}{2L}\frac{1}{\sqrt{(R/2L)^2 - (1/LC)}} \\ B &= -\frac{E}{2L}\frac{1}{\sqrt{(R/2L)^2 - (1/LC)}} \end{aligned}\right\} \tag{86}$$

whence

$$i = \frac{E}{2L}\frac{1}{\sqrt{(R/2L)^2 - (1/LC)}}\left\{e^{\left[-\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}\right]t} - e^{\left[-\frac{R}{2L}-\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}\right]t}\right\} \tag{87}$$

To find the charge q in the condenser, we may substitute

$$i = Ae^{m_1t} + Be^{m_2t}$$

and

$$\frac{di}{dt} = Am_1e^{m_1t} + Bm_2e^{m_2t}$$

in Eq. (79), giving

$$q = CE - \frac{CE}{2\sqrt{R^2 - \frac{4L}{C}}}\left[R + \sqrt{R^2 - \frac{4L}{C}}\right]e^{\left[-\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}\right]t} + \frac{CE}{2\sqrt{R^2 - \frac{4L}{C}}}\left[R - \sqrt{R^2 - \frac{4L}{C}}\right]e^{\left[-\frac{R}{2L}-\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}\right]t} \tag{88}$$

The graphs of Eqs. (87) and (88) are shown in Figs. 34 and 35.

Case 2, $(R/2L)^2 = 1/LC$: *The Critical Case*.—With this relation between the coefficients of Eq. (80), the general solution has the form

$$i = e^{-Rt/2L}[A_1 + B_1t]$$

where A_1 and B_1 are constants of integration; but since $i = 0$

when $t = 0$, it follows that $A_1 = 0$, so that

$$i = B_1 t e^{-Rt/2L} \tag{89}$$

$$\frac{di}{dt} = B_1 e^{-Rt/2L}\left(1 - \frac{Rt}{2L}\right) \tag{90}$$

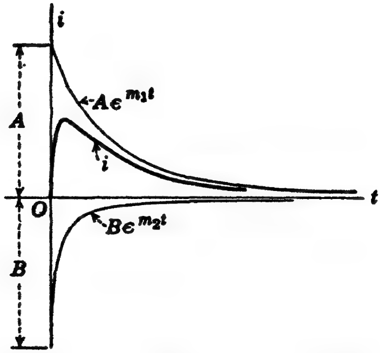


FIG. 34.—Current in circuit containing R, L, and C; logarithmic case.

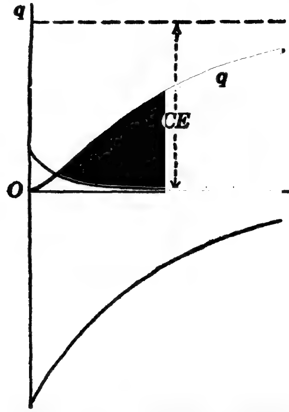


FIG. 35.—Condenser charge, logarithmic case.

Substituting Eqs. (89) and (90) in Eq. (79) and inserting the condition $q = 0$ when $t = 0$, we have the result

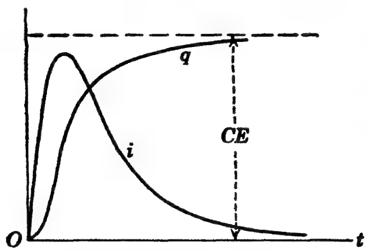


FIG. 36.—Variation of current and charge, critical case.

$$B_1 = \frac{E}{L}$$

and therefore

$$i = \frac{E}{L} t e^{-Rt/2L} \tag{91}$$

$$q = CE\left[1 - e^{-Rt/2L}\left(1 + \frac{Rt}{2L}\right)\right] \tag{92}$$

The graphs of Eqs. (91) and (92) are shown in Fig. 36.

Case 3, $(R/2L)^2 < 1/LC$: *The Oscillatory Case.*—This particular relation between R, L, and C, which is of course physically possible, makes the roots m_1 and m_2 [Eq. (83)] imaginary, but the solution must nevertheless be real.

The values of m_1 and m_2 may be written in the form

$$m_1 = -\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

$$m_2 = -\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

or, for the sake of brevity,

$$m_1 = -a + jb \tag{93}$$
$$m_2 = -a - jb \tag{94}$$

where

$$a = \frac{R}{2L} \tag{95}$$

$$b = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{96}$$

and $j = \sqrt{-1}$.

The solution of the general differential equation (80), obtained by using Eq. (81), is

$$i = e^{-at}(Ae^{jbt} + Be^{-jbt}) \tag{97}$$

But

$$e^{j\theta} = \cos\theta + j\sin\theta \tag{98}$$
$$e^{-j\theta} = \cos\theta - j\sin\theta \tag{99}$$

Hence,

$$i = e^{-at}[(A + B)\cos bt + j(A - B)\sin bt] \tag{100}$$

$$\frac{di}{dt} = e^{-at}\left\{\begin{matrix}[jb(A - B) - a(A + B)]\cos bt \\ -[ja(A - B) + b(A + B)]\sin bt\end{matrix}\right\} \tag{101}$$

Substituting Eqs. (100) and (101) in Eq. (79) and solving for q/C,

$$\frac{q}{C} = E - e^{-at}\left\{\begin{matrix}[(R - aL)(A + B) + jbL(A - B)]\cos bt \\ +[j(R - aL)(A - B) - bL(A + B)]\sin bt\end{matrix}\right\} \tag{102}$$

If in Eqs. (100) and (102) the conditions are now imposed that $i = 0$ and $q = 0$ when $t = 0$, the results are

$$A + B = 0$$
$$j(A - B) = \frac{E}{bL}$$

and when these values are inserted in Eqs. (100) and (102) the result is

$$i = \frac{Ee^{-Rt/2L}}{\sqrt{(L/C) - (R^2/4)}} \cdot \sin\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\,t \tag{103}$$

and

$$q = CE\left\{1 - e^{-Rt/2L}\left[\cos\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\,t + \frac{R}{\sqrt{(4L/C) - R^2}}\sin\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\,t\right]\right\} \quad (104)$$

The graphs of Eqs. (103) and (104) are shown in Fig. 37. Both the current and the charge in the condenser are oscillatory, q reaching its final steady value $Q = CE$ after a series of damped oscillations, which makes the charge first overshoot its final value, then fall below it, and so on. The current in the circuit is alternately positive and negative; that is, it flows first in one direction, then in the other.

The period of the oscillation is readily found by giving t a value T such that

$$\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\,T = 2\pi$$

whence

$$T = \frac{2\pi}{\sqrt{(1/LC) - (R/2L)^2}} \quad (105)$$

If R is so small as to be negligible, this reduces to

$$T_{R=0} = 2\pi\sqrt{LC} \quad (106)$$

a formula that occurs frequently in radio problems and in some transmission line cases.

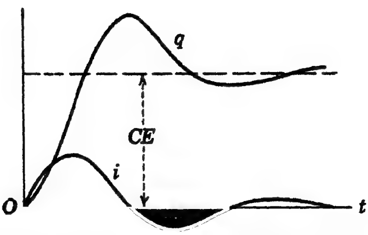


FIG. 37.—Current and charge, oscillatory case.

The physical reason for the oscillation of current and charge is apparent when it is remembered that a circuit like that of Fig. 33 is capable of storing energy in two ways: as potential energy, $\frac{1}{2}CE^2$, in the condenser; and as kinetic energy, $\frac{1}{2}LI^2$, in the inductance. The ordinary pendulum is an example of a mechanical system that has similar properties; any disturbance imposed upon it produces a mechanical oscillation, the energy being transferred back and forth from potential to kinetic form until it is finally dissipated in overcoming frictional resistance.

CHAPTER V

SYSTEMS OF ELECTRICAL UNITS

1. Units and Dimensions in General.—When the statement is made that the distance between two points, such as P_1 and P_2 (Fig. 1), is l m., it is implied that the entity called the length of the measured line P_1P_2 is l times that of a standard unit of length, called the meter, the magnitude of which may be designated as L. This unit length L is the distance between two points, say O_1 and O_2, on the standard meter bar maintained at

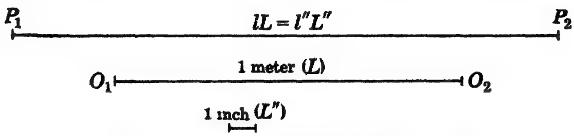


FIG. 1—Lengths and units of length

the International Bureau of Weights and Measures at Sèvres, France. The length of the line P_1P_2, which for convenience can be visualized as the length of a table or other physical object, is $l \times L$; or, otherwise stated,

$$\textit{Length } P_1P_2 = l \times \textit{length } O_1O_2$$

which is equivalent to the relation

$$l = \frac{\text{length } P_1P_2}{\text{length } O_1O_2} = \frac{\overline{P_1P_2}}{\overline{O_1O_2}} \tag{1}$$

It thus appears that the quantity l, ordinarily referred to as the measure of length, is really a pure numeric which expresses the number of times that the unit length $L = \overline{O_1O_2}$ is contained in the measured length $\overline{P_1P_2}$.

The same actual length $\overline{P_1P_2}$ can be expressed in inch units by writing

$$\overline{P_1P_2} = l''L''$$

where L'' is the magnitude of the inch, and l'' is the number of times the inch is contained in $\overline{P_1P_2}$. Since $\overline{P_1P_2}$, the length of

the table, is the same regardless of the unit selected for purposes of measurement,

$$lL = l''L''$$

or

$$\frac{l}{l''} = \frac{L''}{L} \tag{2}$$

The pure numbers l and l'' which are used to *express* the given length are *inversely proportional* to the *magnitudes*, or sizes, of the units themselves. In the illustration above it is known from actual comparison that $L/L'' = 39.37$, whence $l = (1/39.37)l''$; or the *number* required to express a given length in terms of a relatively large unit like the meter is smaller than the *number* required to express the same length in terms of the smaller unit, the inch.

Although these considerations appear to be so elementary as to make elaboration unnecessary, it is nevertheless a common experience to find that, in converting quantities from one system of units to another, these simple relations are either overlooked or incorrectly used, with results that may be serious. Examples of actual conversions are given later.

2. Fundamental Units of Mass, Length, and Time.—The quantities that enter into engineering calculations are expressed in terms of units derived from the fundamental concepts of mass, length, and time. The particular units in which these three fundamental quantities are themselves measured are wholly arbitrary, though in general each of them is based upon some fixed property observable in our physical environment, the choice being in most cases governed by the consideration that the unit thus determined shall be as nearly as possible removed from the possibility of capricious change.

In the c.g.s. system, for example, the unit of mass, the gram, represented by the symbol M, was originally intended to be the equivalent of the mass of a cubic centimeter of distilled water at the temperature at which its density is a maximum. Urey's discovery (in 1932) of the existence of heavy hydrogen and heavy water invalidated this assumed relationship, but the original gram (the one-thousandth part of the standard kilogram preserved at Sèvres) serves its purpose just as well as if the assumed ideal had been realized. In the British system, the unit of mass,

the pound, is taken as 7000 grains, the grain representing the average of grains taken from the middle of ears of wheat; there is obviously nothing in the nature of a grain of wheat that is fixed and immutable, yet the pound, embodied as a piece of platinum kept in London, serves just as well as a more rigorously defined unit could have done.

The centimeter, the unit of length in the c.g.s. system, represented by the symbol L, is the one-hundredth part of the meter, which was originally intended to be exactly one ten-millionth of the earth's quadrant. Actually, the meter is the distance between two marks on a certain bar kept at Sèvres, which has been found by careful measurement to be not quite equal to the ten-millionth part of the earth quadrant. It serves its purpose as a standard of length just as well as if the original intention had been fulfilled.

The unit of time, the second, represented by the symbol T, accepted in all systems of measurement, is $1/(60 \times 60 \times 24) = 1/86{,}400$ of the duration of the mean solar day, which is as nearly an invariant measure of time as is known.

Since the units of mass and length in the c.g.s. and English systems are entirely arbitrary, the relations between corresponding units in the two systems do not depend upon any absolute considerations, and their ratios can be found only by direct comparison. Thus, it has been found that

1 pound	≡	453.59	(453.6) grams
1 kilogram	≡	2.204	(2.2) pounds
1 foot	≡	30.4797	(30.48) centimeters
1 inch	≡	2.5399	(2.54) centimeters
1 meter	≡	3.28	feet ≡ 39.37 inches

Example.—If a certain mass contains 150 g., this fact is expressible, in accordance with Art. 1, as

$$\text{Mass} = m_g M_g$$

where $m_g = 150$, and M_g is the magnitude of the gram. The same mass, expressed as a definite number (m_p) of pounds is

$$\text{Mass} = m_p M_p$$

where M_p is the magnitude of the pound. The mass is the same in both cases; hence,

$$m_p = m_g\left(\frac{M_g}{M_p}\right) = 150 \times \frac{1}{453.6} = 0.3306$$

3. Dimensions of Area and Volume.—Consider a rectangle of breadth b and height h, both expressed in feet. In general, the area a is given by

$$a = \text{constant} \times bh$$

where the constant of proportionality depends upon the unit in which area is expressed. For instance, if a is in square miles, the constant is $1/(5280)^2$, but it is a simple numeric. For convenience, the unit area is usually selected as the area of a square the sides of which are equal to the unit of length; in that case the proportionality constant is unity, and

$$a = bh \tag{3}$$

If the dimensions of area are represented by $[A]$, the numerical relation $a = bh$ really means that

$$a[A] = (bL)(hL) \tag{4}$$

and when Eq. (4) is divided by Eq. (3) the result is

$$[A] \equiv L^2 \tag{5}$$

which states that the dimensions of area are of the nature length squared; or, otherwise, the measurement of an area involves two measurements of length.

In the same way it is readily seen that the dimensions of volume are

$$[V] \equiv L^3 \tag{6}$$

In the case of a circle of radius r units (any unit of length), the area in square units is

$$a = \pi r^2$$

whence

$$a[A] = \pi(rL)^2$$

and again

$$[A] \equiv L^2$$

showing that the numeric π has nothing to do with the basic dimensions.

4. Dimensions of Force.—Practically all the derived units that appear in mechanics and in engineering are based upon Newton's laws, and in particular upon the third law which states that

the force acting upon a body is proportional to the rate of change of its momentum. This is equivalent to the statement

$$\textit{Force} = \text{constant} \times \textit{rate of change of momentum}$$

or

$$f = \text{constant} \times \frac{d}{dt}(mv)$$

Since for all ordinary speeds mass is not appreciably affected by velocity, this relation is equivalent to

$$\begin{aligned} f &= \text{constant} \times m \times \frac{dv}{dt} \\ &= \text{constant} \times ma \end{aligned} \tag{7}$$

where a is the acceleration.

In Eq. (7) the proportionality constant may be made equal to unity, and at the same time be kept dimensionless, by assigning to the force f the particular dimensions that are consistent with this assumption. But when this assumption has once been made, subsequent proportionality constants that appear in other relations involving forces cannot be taken to be dimensionless without a careful check of the validity of such an assumption. With these reservations in mind, it may then be stated that

$$f = ma = m\frac{dv}{dt} \tag{8}$$

where f, m, v, and a represent the *numbers* which express how many units of force, mass, velocity, and acceleration are involved in a particular case.

It is obvious that velocity (length divided by time) has dimensions of the nature

$$[V] \equiv \frac{L}{T} = LT^{-1} \tag{9}$$

and that acceleration (change of velocity per unit time) has the dimensions

$$[A] \equiv \frac{[V]}{T} = LT^{-2} \tag{10}$$

Consequently, from Eq. (8), the dimensions of force are

$$[F] \equiv MLT^{-2} \tag{11}$$

which indicates that force is of such a nature that its measure-

ment involves one measurement of mass, one of length, and two of time. It is also clear from Eq. (11) that force and mass are inherently different in nature. This difference is explicitly recognized in the c.g.s. system, in which force is expressed in dynes and mass in grams; but in the English engineering units there is considerable confusion because of the practice of expressing *both* force and mass in pounds. For example, in the c.g.s. system, a mass of 1 g. is attracted by the earth with a force of $f = mg = 1 \times 980$ dynes, which is its true *weight*, if weight is interpreted to mean the gravitational pull upon a mass. In the English system the weight of a pound is, similarly, $f = mg = 1 \times 32.2$ units of force, and in the English absolute system this force is called 32.2 *poundals*. The word poundal is analogous to the word dyne. Unfortunately, the word poundal has not received general acceptance, and instead it is common practice to say that a mass of 1 pound "weighs" 1 pound. At high altitudes and at points inside the earth, where g is smaller than at sea level, the force of the earth's pull on a given mass is known to be less than at sea level. Consequently, the statement "a body weighs W lb." means merely that the body is attracted by the earth with the same force that the earth exerts upon a mass of W standard pounds placed at the same point.

These considerations have been discussed at some length because there has long existed a somewhat similar confusion between strength of magnetic field (H) and magnetic flux density (B); and between electric field strength (E) and dielectric flux density (D), with the result that many publications treat B and H as interchangeable. It will be shown later than B and H are dimensionally different and that E and D are dimensionally different, in a manner that is analogous to the dimensional difference between mass and weight.

From Eq. (8), $f = ma$, it follows that, if $m = 1$ and $a = 1$, the force f is equal to unity; or *unit force acting upon unit mass produces unit acceleration.* Thus, in the c.g.s. system, a force of 1 dyne acting upon a mass of 1 g. produces an acceleration of 1 cm. per sec., per sec. In the English system, a mass of 1 lb. will be accelerated at the rate of 1 ft. per sec., per sec., by a force of 1 poundal (equivalent to the "weight" of 1/32.2 lb.). In the meter-kilogram-second (m.k.s.) system, unit mass (1 kg. $\equiv$ 1000 g.) will undergo unit acceleration (1 m. per sec., per sec.; or

100 cm. per sec., per sec.) if the unit force is

$$f = ma = 1000 \times 100 \text{ dynes} = 10^5 \text{ dynes} \tag{12}$$

This new unit of force, equivalent to 10^5 dynes, has been given the name *newton*, or

Force (in newtons) = mass (in kg.)
× acceleration (in m. per sec., per sec.)

In all three of these systems, the dimensions of force are $[F] \equiv MLT^{-2}$, where M L, and T are the dimensions of the units of mass, length, and time. This formula may be used to find the numerical factor for converting measurements of force from one system to another; thus in the m.k.s. system let the fundamental dimensions of mass and length be

$$M_{kg} \equiv \text{dimension of the kilogram}$$
$$L_m \equiv \text{dimension of the meter}$$

whereas in the c.g.s. system the corresponding units are

$$M_g \equiv \text{dimension of the gram}$$
$$L_{cm} \equiv \text{dimension of the centimeter}$$

T_s representing the dimension of the second, common to both systems. It follows that

$$\frac{[F]_{newton}}{F_{dyne}} = \left(\frac{M_{kg}}{M_g}\right)\left(\frac{L_m}{L_{cm}}\right)\left(\frac{T_s}{T_s}\right)^{-2} = 1000 \times 100 = 10^5$$

the fact thus being checked that the newton is 10^5 times as large as the dyne.

5. Dimensions of Work or Energy.—When a force f acts over a distance l, the work done is proportional to the product of force and distance, so that in general

$$\text{Work} = \text{constant} \times \text{force} \times \text{distance}$$

In the c.g.s. system the proportionality constant is unity and dimensionless, and the resultant equation is

Work (in ergs) = force (in dynes) × distance (in centimeters)

Similarly, in the m.k.s. system, the proportionality constant is unity and dimensionless, and the unit of work, the joule, is such that

Work (in joules) = force (in newtons) × distance (in meters)

Dimensionally, work is of the nature

$$[W] \equiv [F] \cdot L \equiv ML^2T^{-2} \tag{13}$$

and hence the ratio

$$\frac{[\text{Joule}]}{[\text{Erg}]} = \left(\frac{M_{kg}}{M_g}\right)\left(\frac{L_m}{L_{cm}}\right)^2\left(\frac{T_s}{T_s}\right)^{-2} = 1000 \times 100^2 = 10^7$$

and thus the relation, previously given, that

$$1 \text{ joule} \equiv 10^7 \quad \text{ergs}$$

is checked.

6. Dimensions of Power.—Since power is defined as the rate at which work is done, the dimensions of power are

$$[P] \equiv \frac{[W]}{T} \equiv ML^2T^{-3} \tag{14}$$

In the c.g.s. system, unit power signifies that work is done at the rate of 1 erg per sec. In the m.k.s. system, unit power, called the *watt,* is equivalent to 1 joule per sec. Consequently,

$$\frac{[\text{Watt}]}{[\text{Erg per sec.}]} \equiv \left(\frac{M_{kg}}{M_g}\right)\left(\frac{L_m}{L_{cm}}\right)^2\left(\frac{T_s}{T_s}\right)^{-3} = 10^7$$

or

$$1 \text{ watt} \equiv 10^7 \text{ ergs per sec.} \equiv 1 \text{ joule per sec.}$$

Inspection of Eq. (14) shows that it may be written in the form

$$[P] \quad \left(\frac{ML}{T^2}\right)(L)\left(\frac{1}{T}\right)$$

which indicates that power is of the nature (force $\times$ distance) divided by time. In both the c.g.s. and m.k.s. systems, force, in dynes or newtons, is definitely distinguished from mass (in grams or kilograms). If, therefore, the metric units of power are to be compared with the English unit of power, such as the foot-pound per second, or the horsepower (550 ft.-lb. per sec.), it is essential to distinguish between the pound as unit of mass and the pound as unit of force. To do this, the word pound in the expression "foot-pound per second" must be converted into the equivalent force unit, namely, the poundal, so that the real unit of power in the English system is 32.2 poundal-feet per sec. Consequently, the ratio

$$\frac{\text{Horsepower}}{\text{Watt}} \equiv 550\left(\frac{32.2M_p}{M_{kg}}\right)\left(\frac{L_{ft}}{L_m}\right)^2\left(\frac{T_s}{T_s}\right)^{-3}$$

$$= 550 \times \frac{32.2}{2.2} \times \left(\frac{1}{3.28}\right)^2 = 746$$

or

$$1 \text{ hp.} \equiv 746 \quad \text{watts}$$

7. Consistency of Units in Equations. Dimensionless Equations.—Consider a mass M, Fig. 2, sliding down an inclined plane under the influence of gravity and friction. The component of the weight that is effective in producing acceleration down the incline is $Mg \sin \theta$, and this must overcome the inertial force Ma and the retarding force due to friction, $fMg \cos \theta$. The equation of motion is

$$Mg \sin \theta = fMg \cos \theta + Ma \tag{15}$$

Each term in Eq. (15) is of the nature mass times acceleration (that is, a force); or it is mass times acceleration times a pure numeric like $\sin \theta$ or $f \cos \theta$, since f is itself the coefficient of friction and therefore merely a ratio or abstract number. The equation is consistent throughout, in the sense that terms added together must be of like nature. This test must be satisfied by every equation if actual equality is to be correctly expressed.

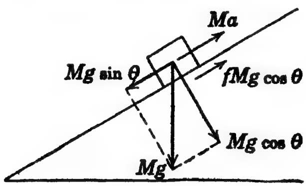


FIG. 2

If both sides of Eq. (15) are divided by Mg, there results

$$\sin \theta = f \cos \theta + \frac{a}{g} \tag{16}$$

in which all terms are pure numerics; equations such as (16) are therefore dimensionless.

In many types of engineering problems, particularly in cases where the behavior of a full-sized structure is to be studied by means of a small model, it is important that there should be complete similitude between the properties of the model and those of the structure itself. Similitude implies that the ratios of significant quantities, or combinations of quantities, influencing the behavior of the structure and of the model shall be dimensionless. If this kind of correspondence does not exist,

it is not legitimate to draw conclusions about the structure based upon the observed results with the model.

For example, in hydraulic installations, the pressure drop p in a pipe line of length l and diameter d will be affected not only by l and by d but also by the density of the fluid δ; the viscosity of the fluid, ν; and the velocity of the flow, v. One may either write that p is a function of these variables, as

$$p = f(l, d, \delta, \nu, v)$$

or that some function of all six variables is a dimensionless number, say N; that is,

$$N = \varphi(p, l, d, \delta, \nu, v) = (p)^a(l)^b(d)^c(\delta)^e(\nu)^f(v)^g \qquad (17)$$

where the exponents $a, b, c, \ldots$ remain to be determined by experiment.

Since pressure has the nature of force divided by area, its dimensions are $ML/T^2 \div L^2 = M/LT^2$; length and diameter both have the dimension of length L; density, which is mass per unit volume, has the dimensions ML^{-3}; and velocity has the dimensions LT^{-1}. The viscosity of a fluid is determined experimentally by placing a film of liquid of thickness t between two horizontal plates, each having area A, and then measuring the shearing force S required to make the upper plate slide parallel to the lower plate with a velocity V. Experiment shows that

$$S = \frac{\nu A V}{t}$$

or $\nu = St/AV$, from which the dimensions of ν are

$$[\nu] \equiv \frac{(ML/T^2)(L)}{(L)^2(L/T)} \equiv \frac{M}{LT}$$

Upon inserting these dimensions in the right-hand expression in Eq. (17), the result is

$$\left(\frac{M}{LT^2}\right)^a (L)^b (L)^c \left(\frac{M}{L^3}\right)^e \left(\frac{M}{LT}\right)^f \left(\frac{L}{T}\right)^g =$$
$$(M)^{a+e+f}(L)^{-a+b+c-3e-f+g}(T)^{-2a-f-g} \qquad (18)$$

and since this must be a numeric the exponents of M, L, and T must all be zero; or

$$a + e + f = 0$$
$$-a + b + c - 3e - f + g = 0$$
$$-2a - f - g = 0$$

There are six unknowns in these three equations, and hence it is not possible to evaluate them individually; but any three of them can be expressed in terms of the remaining three, as for example,

$$c = -b - f$$
$$e = -a - f$$
$$g = -2a - f$$

On substituting these values of c, e, and g in Eq. (17), the result is

$$N = (p)^a(l)^b(d)^{-b-f}(\delta)^{-a-f}(\nu)^f(v)^{-2a-f} = \left(\frac{p}{\delta v^2}\right)^a\left(\frac{l}{d}\right)^b\left(\frac{\nu}{\delta\, d\, v}\right)^f \qquad (19)$$

To obtain similitude between two systems of piping carrying different fluids, the quantities $p/\delta v^2$, l/d, and $\nu/\delta\, d\, v$ must be the same in both.

8. Dimensions of E.S. Units of Quantity and Current.—Coulomb's law for the force between two charges,

$$f = \frac{qq'}{\epsilon_0 r^2} \qquad (20)$$

establishes the relation between the abstract numbers that express force in dynes, charges in statcoulombs, distance in centimeters, and the absolute permittivity ϵ_0. Inserting the dimensions of each of these quantities, and taking $[\epsilon]$ as the dimension of permittivity without at this time inquiring further as to its nature,

$$f\left(\frac{ML}{T^2}\right) = \frac{(q[Q]_s)(q'[Q]_s)}{(\epsilon_0[\epsilon])(rL)^2}$$

whence

$$[Q]_s = [\epsilon]^{1/2}M^{1/2}L^{3/2}T^{-1} \qquad (21)$$

Since current in e.s. units (statamperes) is quantity per unit of time, the magnitude of the current unit (the statampere) is

$$[I]_s = \frac{[Q]_s}{T} = [\epsilon]^{1/2}M^{1/2}L^{3/2}T^{-2} \qquad (22)$$

Because of the fact that the absolute permittivity of free space is assumed to have the value unity in the e.s. system (that is, $\epsilon_0 = 1$), numerous authors have assumed that ϵ_0 may be omitted from Eq. (20). If this is done, Eqs. (21) and (22) do not include

$[\epsilon]$, and in that case $[Q]_s = M^{1/2}L^{3/2}T^{-1}$ and $[I]_s = M^{1/2}L^{3/2}T^{-2}$. But this procedure ignores the possibility that ϵ_0 may possess distinctive dimensions and leads to confusion. As an example of the faultiness of this procedure, consider a rectangle of length l, breadth b, and area $a = bl$ square units; it may easily happen that $b = 1$ unit of length, in which case $a = l$ (numerically), but it does not in the least follow that the unit of area $[A]$ has the dimension of length L.

9. Dimensions of Unit Magnet Pole.—Coulomb's law for the force between two magnet poles,

$$f = \frac{mm'}{\mu_0 r^2} \tag{23}$$

gives

$$f\left(\frac{ML}{T^2}\right) = \frac{(m[\bar{M}])(m'[\bar{M}])}{(\mu_0[\mu])(rL)^2}$$

where $[\mu]$ denotes the dimension of permeability and $[\bar{M}]$ denotes the dimension of the unit pole (which must not be confused with M that designates the dimension of mass). Consequently

$$[\bar{M}] = [\mu]^{1/2}M^{1/2}L^{3/2}T^{-1} \tag{24}$$

10. Dimensions of E.M. Units of Current and Quantity.—The fundamental relations that define current (in abamperes) are embodied in Ampère's law

$$df = \frac{i\,dl\,m}{r^2}\sin\theta \tag{25}$$

as discussed in Art. 2, Chap. III. If the dimensions of each quantity are introduced,

$$df\left(\frac{ML}{T^2}\right) = \frac{(i[\bar{I}])(dl \cdot L)(m[\mu]^{1/2}M^{1/2}L^{3/2}T^{-1})}{(rL)^2}\sin\theta$$

whence

$$[\bar{I}] = [\mu]^{-1/2}M^{1/2}L^{1/2}T^{-1} \tag{26}$$

The dimensions of $[\bar{Q}]$ (the abcoulomb) must by definition be of the nature $[\bar{I}]T$, or

$$[\bar{Q}] = [\mu]^{-1/2}M^{1/2}L^{1/2} \tag{27}$$

Because of the assumption that $\mu_0 = 1$ in the e.m. system, it is quite common to find Eq. (23) written in the form $f =$

mm'/r^2, which is, of course, quite correct so far as numerical results are concerned; but if μ_0 thus disappears from Eq. (23), $[\mu]$ will not appear in Eqs. (26), (27), which would then read $[\bar{I}] = M^{1/2}L^{1/2}T^{-1}$ and $[\bar{Q}] = M^{1/2}L^{1/2}$, the possibility that $[\mu]$ may possess dimensions being thus ignored.

11. Ratio between Dimensions of E.S. and E.M. Units of Current and Quantity.—In Art. 1, it is self-evident that a particular entity such as the distance between the points P_1 and P_2 remains unaltered whether it is measured in meters, inches, or any other arbitrary unit of length. All that happens when the inch (magnitude L'') takes the place of the meter (magnitude L) is that the *number* of inches (l'') in $\overline{P_1P_2}$ changes to the *number* of meters (l) in $\overline{P_1P_2}$; and in accordance with Eq. (2), $l/l'' = L''/L$. Since l/l'' is an abstract (dimensionless) number, consistency points to the conclusion that L''/L (or L/L'') is likewise an abstract number. In other words, the dimensions of any two units that are used to measure the *same thing* must inherently be the same, or their ratio must be an abstract number.

It is apparent that this conclusion must apply to the magnitudes of the e.s. and the e.m. units of quantity—the statcoulomb and the abcoulomb—and also to the corresponding units of electric current—the statampere and the abampere. For whether one system or the other is used, the thing that is being measured is in the one case electric charge, in the other electric current, and both of these are real entities that are unaffected by the standard selected to express them.

From Eqs. (21) and (27),

$$\frac{[Q]_s}{[\bar{Q}]} = \frac{[\epsilon]^{1/2}M^{1/2}L^{3/2}T^{-1}}{[\mu]^{-1/2}M^{1/2}L^{1/2}} = \sqrt{[\mu\epsilon]}\frac{L}{T} \tag{28}$$

and from Eqs. (22) and (26),

$$\frac{[I]_s}{[\bar{I}]} = \frac{[\epsilon]^{1/2}M^{1/2}L^{3/2}T^{-2}}{[\mu]^{-1/2}M^{1/2}L^{1/2}T^{-1}} = \sqrt{[\mu\epsilon]}\frac{L}{T} \tag{29}$$

from which it follows that $\sqrt{[\mu\epsilon]}$ must have the dimensions T/L, or the reciprocal of velocity, in order that the ratios in Eqs. (28) and (29) may be abstract numbers. It is obvious that, if the dimensions of $[\mu]$ and $[\epsilon]$ were ignored in the manner already referred to, the ratios $[Q]_s/[\bar{Q}]$ and $[I]_s/[\bar{I}]$ would both

have the dimensions of velocity; the literature of the subject abounds with statements to that effect, notwithstanding the plain fact that $[Q]_s$ and $[\bar{Q}]$ are both measures of the same thing, namely, electrical charge—and similarly with respect to $[I]_s$ and $[\bar{I}]$—and that their ratios must in the very nature of things be abstract numbers.

The conclusion that $1/\sqrt{[\mu\epsilon]}$ must have the dimensions of velocity is in strict accord with the electromagnetic theory first developed in mathematical form by Maxwell and later verified experimentally by Hertz. Maxwell's theory predicted that the velocity of an electromagnetic wave in free space must be

$$v = \frac{c}{\sqrt{\mu_0\epsilon_0}} = \frac{3 \times 10^{10}}{\sqrt{\mu_0\epsilon_0}} \text{ cm. per sec.}$$

provided that μ_0 and ϵ_0 have the unit values assigned to them in the c.g.s. system. A simple derivation of this formula is given in Art. 22; but if its validity is accepted for the time being, it is seen that the dimensions of $1/\sqrt{[\mu\epsilon]}$ must be LT^{-1}.

12. Ratio between Dimensions of E.S. and E.M. Units of Potential and E.M.F.—The difference of potential between two points being measured by the work required to move a unit charge from the one point to the other, it follows that the product of charge and potential difference must be proportional to the work done in moving the charge. The proportionality constant is made equal to unity, so that

Quantity (in statcoulombs) × potential difference (in statvolts) = work (in ergs)
Quantity (in abcoulombs) × potential difference (in abvolts) = work (in ergs)
Quantity (in coulombs) × potential difference (in volts) = work (in joules)

In every case, therefore, the dimensions of charge multiplied by the dimensions of potential difference must give the dimensions of work or energy, whence

$$[Q]_s[V]_s = ML^2T^{-2}$$
$$[\bar{Q}][\bar{V}] = ML^2T^{-2}$$

On substituting for $[Q]_s$ from Eq. (21) and for $[\bar{Q}]$ from Eq. (27),

$$[V]_s = [\epsilon]^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1} \tag{30}$$

and

$$[\bar{V}] = [\mu]^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2} \tag{31}$$

and the ratio of these dimensions is

$$\frac{[V]_s}{[\bar{V}]} = \frac{1}{\sqrt{[\mu\epsilon]}} \cdot \frac{T}{L} \tag{32}$$

Comparison of this ratio with Eq. (28) shows that it is the reciprocal of the ratio in that equation; this result checks with the statements in Chap. I to the effect that

$$1 \text{ abcoulomb} \equiv 3 \times 10^{10} \quad \text{statcoulombs}$$

and that

$$3 \times 10^{10} \text{ abvolts} \equiv 1 \quad \text{statvolt}$$

13. Dimensions of E.S. and E.M. Units of Resistance.—Since Ohm's law states that the resistance of a passive conductor is equal to the difference of potential between its terminals divided by the current, it is possible to write at once, from Eqs. (22) and (30), that the dimensions of the absolute e.s. unit of resistance (the statohm) are given by

$$[R]_s = \frac{[V]_s}{[I]_s} = \frac{[\epsilon]^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}}{[\epsilon]^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}} = \frac{1}{[\epsilon]}\frac{T}{L} \tag{33}$$

Similarly, from Eqs. (26) and (31), the dimensions of the abohm are

$$[\bar{R}] = \frac{[\bar{V}]}{[\bar{I}]} = \frac{[\mu]^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}}{[\mu]^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}} = [\mu]\frac{L}{T} \tag{34}$$

whence

$$\frac{[R]_s}{[\bar{R}]} = \frac{1}{[\mu\epsilon]}\frac{T^2}{L^2} \tag{35}$$

This ratio is the square of the ratio that appears in Eq. (32); hence, it may be concluded that

$$9 \times 10^{20} \text{ abohms} \equiv 1 \quad \text{statohm}$$

It is interesting to note that if the dimensions $[\mu]$ and $[\epsilon]$ were disregarded the statohm would be of the nature of reciprocal of velocity, whereas the abohm would be of the nature of velocity.

14. Dimensions of E.S. and E.M. Units of Inductance.—The defining equation for inductance being

$$e = -L\frac{di}{dt}$$

or

$$L = -\frac{e}{di/dt}$$

it follows that, dimensionally, the unit of inductance in the e.m. system must be of the nature $[\bar{L}] = \frac{[\bar{V}]}{[\bar{I}]/T}$; therefore, from Eqs. (26) and (31),

$$[\bar{L}] = [\mu]L \tag{36}$$

If the dimensions of $[\mu]$ are ignored, the dimension of inductance is of the nature of length; it is this consideration which accounts for the universal use of the symbol L to denote inductance.

The e.s. unit of inductance, though rarely referred to, may be evaluated by using Eqs. (22) and (30), so that

$$[L]_s = \frac{1}{[\epsilon]}\frac{T^2}{L} \tag{37}$$

whence

$$\frac{[L]_s}{[\bar{L}]} = \frac{1}{[\mu\epsilon]}\frac{T^2}{L^2} \tag{38}$$

It will be recalled that in Art. 29, Chap. III, the ratio L/R was called the time constant of a circuit (L and R being based upon e.m.u.); it is seen from Eqs. (34) and (36) that $[\bar{L}]/[\bar{R}] = T$, the use of the phrase being therefore justified.

15. Dimensions of E.S. and E.M. Units of Capacitance.—Since capacitance is in general the quotient of charge divided by potential difference, the dimensions of capacitance in the e.s. system are obtained from Eqs. (21) and (30), giving

$$[C]_s = \frac{[Q]_s}{[V]_s} = [\epsilon]L \tag{39}$$

In the e.m. system, by using Eqs. (27) and (31),

$$[\bar{C}] = \frac{[\bar{Q}]}{[\bar{V}]} = \frac{1}{[\mu]}\frac{T^2}{L} \tag{40}$$

so that

$$\frac{[C]_s}{[\bar{C}]} = [\mu\epsilon]\frac{L^2}{T^2} \tag{40a}$$

With reference to Art. 27, Chap. IV, it will be observed that the ratio t/RC occurs as a numeric; accordingly, RC must be of the nature of time, and on multiplying Eqs. (34) and (40),

$$[\bar{R}][\bar{C}] = [\mu]\frac{L}{T} \times \frac{1}{[\mu]}\frac{T^2}{L} = T$$

It is to be noted that, if the dimensions of $[\epsilon]$ in Eq. (39) are ignored, capacitance in the e.s. system has the dimension length. This checks with the fact that the capacitance of an isolated sphere in free space (see Art. 9, Chap. IV), expressed in statfarads, is numerically equal to its radius in centimeters. It also checks with the other formulas for capacitance (in e.s. units) given in Chap. IV.

16. Practical System of Units.—The units commonly used——volt, coulomb, ampere, ohm, henry, farad—and their decimal multiples and submultiples are derived from the e.m. system, each of these practical units being related to its electromagnetic prototype by a positive or negative integral power of 10; the particular integer is in each case so selected that the resulting practical unit has a convenient magnitude. Table I has been arranged to show the ratio of the *magnitude* of each of these practical units to the *magnitude* of the corresponding unit in both the e.m. and e.s. systems.

TABLE I.—RATIO: $\dfrac{\text{MAGNITUDE OF PRACTICAL UNIT}}{\text{MAGNITUDE OF ABSOLUTE UNIT}}$

Item	Name of Unit	Ratio to e.m.u.	Ratio to e.s.u.
Charge	Coulomb	10^{-1}	3×10^9
E.m.f. or potential	Volt	10^8	$1/300$
Current	Ampere	10^{-1}	3×10^9
Resistance	Ohm	10^9	$1/(9 \times 10^{11})$
Inductance	Henry	10^9	$1(9 \times 10^{11})$
Capacitance	Farad	10^{-9}	9×10^{11}

It must be clearly understood that each factor in Table I states the number of times that a particular absolute unit is contained

in the corresponding practical unit. For example, there are 10^8 abvolts in 1 volt, so that the volt is a much larger unit than the abvolt; otherwise stated, if a particular value of potential difference is to be expressed in volts, the *number* of volts will be much smaller than the corresponding *number* of abvolts.

As a matter of convenience, Table II has been prepared to show the factors by which any given *number* of absolute units must be multiplied in order to convert it to the corresponding *number* of the equivalent practical unit.

TABLE II.—CONVERSION FACTORS

Item	To obtain the number of	Multiply the number of absolute units by	
		E.m. system	E.s. system
Charge	Coulombs	10	$1/(3 \times 10^9)$
E.m.f. or potential	Volts	10^{-8}	300
Current	Amperes	10	$1/(3 \times 10^9)$
Resistance	Ohms	10^{-9}	9×10^{11}
Inductance	Henrys	10^{-9}	9×10^{11}
Capacitance	Farads	10^9	$1/(9 \times 10^{11})$

In Tables I and II the quotients obtained by dividing any factor in the electrostatic column by the corresponding electromagnetic factor are 3×10^{10}, $(3 \times 10^{10})^2$, or the reciprocals of these numbers. The *number* 3×10^{10} is involved in the velocity of light* and of electromagnetic radiation in general, provided that velocity is expressed in centimeters per second.

17. Other Systems of Units. The M.K.S. System.—The absolute e.s. and e.m. systems of units, which are discussed in the preceding material, are based upon the fundamental units, the centimeter, the gram, and the mean solar second. The practical system of units, related to the e.m. system by integral powers of 10, is therefore likewise dependent upon a c.g.s. system of reference. All three systems grew piecemeal out of the original pioneer studies of the subject, and the practical system in particular has been modified from time to time in the sense that the

* The velocity of light is actually 2.99776×10^{10} cm. per sec., and all the ratios in Tables I and II, must conform to this actual value if strict accuracy is required.

magnitudes of some of the units have been altered to meet changing conditions in practical work.

For example,* the first resistance boxes produced in Germany in 1848 were calibrated in terms of the linear resistance of particular sizes of telegraph wire. Gauss and Weber, about 1850, showed how to measure certain electric and magnetic quantities in absolute measure, using a millimeter-milligram-second (m.m.s.) system. In 1861 the British Association (B.A.) for the Advancement of Science established a committee to report upon standards of resistance, and after trying a foot-grain-second (f.g.s.) system advocated an absolute system based upon the meter-gram-second (m.g.s); it was this committee which worked out the theoretical and practical aspects of several units, especially that of resistance, for which Latimer Clark suggested the name ohm. This m.g.s. unit of resistance was so extremely small in comparison with the resistances of telegraph lines, which at that time were the chief concern of "electricians," that a practical unit 10^7 times as large was recommended, since 10 such "ohms" would be approximately the resistance of a mile of ordinary telegraph wire. Moreover, because the m.g.s. unit of e.m.f. was also extremely small, 10^5 such units were proposed as the equivalent of 1 "volt" (then so named for the first time) since it then became nearly the same as the e.m.f. of the Daniell cell, at that time in common use as a source of e.m.f. The decimal factors 10^7 and 10^5 were purely arbitrary; but once they were adopted, it became desirable to adjust the units so that a unit current through a unit resistor would correspond to unit potential drop, and this unifying tendency extended to the other units, which are no longer arbitrary when any three are fixed. Thus, what was originally called the farad in the B.A. recommendations was equivalent to what is now called the microfarad, and the change to the present magnitude of the farad was made about 1870.

Beginning in 1868, a second B.A. committee undertook a systematic appraisal of the c.g.s., m.g.s., f.g.s., and m.m.s. systems with a view to the formulation of a comprehensive system adapted to all branches of science. Its report of 1873 decided in favor of the c.g.s. system, chiefly for the reason that of the four systems considered it was the only one in which unit

* Based upon an article by A. E. KENNELLY, I.E.C. Adopts M.K.S. System of Units, *Elec. Eng.*, December, 1935; *Trans. A.I.E.E.*, **54**, 1373, 1935.

density (mass per unit volume) came out the same as that of distilled water* (1 g. per cc.) at the temperature corresponding to its maximum density. The practical units already in use, the ohm, volt, and farad, then acquired the multiplying factors 10^9, 10^8 and 10^{-9}, as these exist today, in place of the factors 10^7, 10^5, and 10^{-7} as they had been in the m.g.s. system.

Final acceptance of the c.g.s. system did not occur until 1881, when the First Electrical Congress, meeting at Paris, officially adopted the c.g.s. (e.m.) system as fundamental and also the five derived practical units, the ohm, volt, ampere, coulomb, and farad. Three other practical units have been adopted at subsequent international congresses: the joule (1889), the watt (1889), and the henry (1893). An additional practical unit, the weber (equivalent to 10^8 maxwells), was adopted by the I.E.C. at Paris in 1933 and was confirmed in 1935 at Brussels.

In 1881 Maxwell pointed out† that the practical series of e.m. units might be looked upon as a completely consistent e.m. system based upon the earth quadrant (10^9 cm.) as unit of length, 10^{-11} g. (eleventh-gram) as unit of mass, and the mean solar second as unit of time. Considerably later, in 1904, Ascoli‡ showed that there is the possibility of an indefinitely large number of such systems, all consistent with the accepted practical units, such that if the length unit were 10^l cm. and the mass unit 10^m g.,

$$2l + m = 7$$

so that in Maxwell's system $l = 9$ and $m = -11$. In the m.k.s. system already referred to, $l = 2$ and $m = 3$.

In 1901 and again in 1904, Giorgi§ pointed out that, if the permeability of free space is taken as 10^{-7} (see Art. 20) instead of unity as in the c.g.s. system, the series of practical units would be characteristic of an absolute system in which unit length is the meter, unit mass is the kilogram, and unit time the mean

* Since the discovery of heavy hydrogen and heavy water the unit density of water has lost whatever significance it may have had.

† "A Treatise on Electricity and Magnetism."

‡ On the Systems of Electrical Units, *Proc. International Elec. Congress*, 1. 130–135, 1904.

§ Unita razionali di elletromagnetismo, *Atti dell' A.E.I.*, 1901; Proposals concerning Electrical and Physical Units, *Proc. International Elec. Congress*. St. Louis, 1, 136–141, 1904.

solar second, a *meter-kilogram-second* (*m.k.s.*) system thus being developed.

It is a matter of great importance to all students of electrical engineering that the Giorgi, or m.k.s., system of units was officially adopted as the world's standard at a meeting of the I.E.C. held at Scheveningen-Bruxelles in June, 1935. Of the 25 countries that constitute the I.E.C., 15 were represented at this meeting, and the adoption of the system was unanimous. It is therefore a legal system in our own country.*

18. Magnetic Flux and Flux Density in M.K.S. Units.—It has been shown in Chap. III that the the c.g.s. formula for e.m.f. in abvolts is

$$\bar{E} = -N\frac{d\Phi}{dt}$$

or

$$\bar{E} = Blv$$

whereas if e.m.f. is to be expressed in volts it is necessary to write

$$E = -N\frac{d\Phi}{dt} \times 10^{-8}$$

or

$$E = Blv \times 10^{-8}$$

Similarly, the force in dynes on a current-carrying wire placed perpendicular to a magnetic field is

$$f = Bl\bar{I}$$

if $\bar{I}$ is in abamperes, whereas

$$f = BlI \times 10^{-1}$$

if the current I is in amperes.

Inspection of these equations shows that if all the quantities that appear in any one of them belong to the *same* system the

* For information concerning the deliberations that led to the adoption of the m.k.s. system, the reader is referred to the following articles by A. E. KENNELLY: Adoption of the Meter-kilogram-mass-second (M.K.S.) Absolute System of Practical Units by the International Electrotechnical Commission, Bruxelles, June, 1935, *Proc. Nat. Acad. Sci.*, **21** (No. 10), 579–583, October, 1935; I.E.C. Adopts M.K.S. System of Units, *Elec. Eng.*, December, 1935; The M.K.S. System of Giorgi as Adopted by the International Electrotechnical Commission in June, 1935, *Jour. Eng. Education*, **27** (No. 4), December, 1936.

implied proportionality constant in the equation is unity. But when some of the quantities are in c.g.s. units and others are in practical units, that is, when units are mixed, the resultant equation contains a proportionality constant, or conversion factor, that is not equal to unity. The conversion factors that have already been encountered are for the most part integral powers of 10, such as 10^1, 10^7, 10^8, 10^9, or their reciprocals; but in some of the formulas in Chap. IV the factors are $(3 \times 10^{10})^2$ and $(9 \times 10^{20}) \times 10^{-9}$.

The m.k.s. system is in every way an absolute and consistent series of units just as is the c.g.s. system, and it includes, as the c.g.s. system does not, the practical units commonly used in electrical engineering—the volt, ampere, coulomb, ohm, henry, and farad. It is to be expected, therefore, that if m.k.s. units are used consistently in all formulas, the proportionality constants, or conversion factors, will in every case become unity, the necessity of remembering a whole series of such factors thus being obviated. It is this simplification that gives to the m.k.s. system its chief value but, as will be seen later, there arises the necessity of assigning to μ_0 and ϵ_0 entirely new values which differ radically from their c.g.s. values of unity.

It has already been shown in Chap. III that the e.m.f. formulas in the m.k.s. system are

$$E = -N\frac{d\Phi_1}{dt} \quad \text{volts}$$

and

$$E = B_1 l_1 V_1 \quad \text{volts}$$

where Φ_1 is in webers (the weber being 10^8 maxwells), $B_1 \equiv B \times 10^{-4}$ is in webers per square meter, $l_1 \equiv l \times 10^{-2}$ is in meters, and $v_1 \equiv v \times 10^{-2}$ is in meters per second. Consider next the formula for the force on a current-carrying wire perpendicular to a magnetic field; in c.g.s. units, this is

$$f = BlI \times 10^{-1} \quad \text{dyne}$$

But if m.k.s. units are substituted for B and l,

$$f = (B_1 \times 10^4)(l_1 \times 10^2)I \times 10^{-1} = B_1 l_1 I \times 10^5 \quad \text{dynes}$$

and since 10^5 dynes are equivalent to 1 newton, the formula becomes

$$f = B_1 l_1 I \quad \text{newtons} \tag{41}$$

in which all the quantities are in m.k.s. units and the proportionality constant is unity.

In the formula $E = -N(d\Phi_1/dt)$, the coefficient N, the number of turns, is a pure numeric; hence, the dimensions of flux have the nature volts times time, or

$$[\Phi_1] = [\bar{V}]T = [\mu]^{\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1} \tag{42}$$

Accordingly, the weber has the nature of a *volt-second.* In mechanics, the product of force and time is called the impulse of a force; hence, flux may be regarded as the impulse of an e.m.f.

The dimensions of flux density, or flux per unit area, are

$$[B_1] = \frac{[\Phi_1]}{L^2} = [\mu]^{\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1} \tag{43}$$

This result may be checked by substituting Eq. (26) in the relation

$$[F] = MLT^{-2} = [B_1]L[\bar{I}]$$

which gives

$$MLT^{-2} = ([\mu]^{\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1})(L)([\mu]^{-\frac{1}{2}}M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1})$$

which is seen to be an identity.

19. M.K.S. Unit of Magnetic Field Strength.—In Chap. III it has been shown that H, the strength of a magnetic field in c.g.s units, is measured by the force in dynes on a unit magnet pole in general, the force acting on a pole of m units is

$$f = mH \quad \text{dynes}$$

so that

$$H = \frac{f}{m}$$

and the dimensions of H are therefore

$$[H] = \frac{MLT^{-2}}{[\bar{M}]}$$

Substituting for $[\bar{M}]$ from Eq. (24),

$$[H] = [\mu]^{-\frac{1}{2}}M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1} \tag{44}$$

and on comparing this with Eq. (43) it is seen that

$$\frac{[B_1]}{[H]} = [\mu] \tag{45}$$

Thus, it is shown that B and H are dimensionally different, provided that $[\mu]$ has dimensions. Since $[\mu\epsilon]$ has the dimensions T^2/L^2, it is apparent that either $[\mu]$ or $[\epsilon]$ may be considered to be dimensionless, in which case the other will have the dimensions T^2/L^2; but it is much more reasonable to assign dimensions to both $[\mu]$ and $[\epsilon]$, subject to the condition that the products of these dimensions must be T^2/L^2.

In the c.g.s. system, H in oersteds is also determined by the relation

$$H = \frac{2\bar{I}}{r} = \frac{2I}{10r}$$

which gives the field strength at a distance r cm. from an indefinitely long straight wire carrying a current of $\bar{I}$ abamp., or I amp. (NOTE: The factor 10 appears because of the use of mixed units.) In m.k.s. units, this formula becomes

$$H_1 = \frac{2I}{r_1}$$

where H_1 is the new value of field intensity, and r_1 is in meters. Since $r \equiv r_1 \times 10^2$,

$$H = \frac{2I}{10r} = \frac{2I}{10(r_1 \times 10^2)} = \frac{2I}{r} \times 10^{-3} = H_1 \times 10^{-3} \quad (46)$$

from which it may be concluded that *the m.k.s. unit of field strength is the one-thousandth part of the c.g.s. oersted.*

Since in the c.g.s. system the force on a pole m in a field of H oersteds is $f = mH$ dynes, it is to be expected that, in the m.k.s. system,

$$f_1 = m_1 H_1 \quad \text{newtons}$$

where m_1 is the pole strength in m.k.s. units. Taking the quotient of these equations,

$$\frac{m_1}{m} = \frac{f_1}{f} \cdot \frac{H}{H_1}$$

or, dimensionally,

$$\frac{[\bar{M}_1]}{[\bar{M}]} = \frac{[\text{newton}][H]}{[\text{dyne}][H_1]} = 10^5 \times 10^3 = 10^8 \quad (47)$$

That is,

1 m.k.s unit pole $\equiv 10^8$ *c.g.s. unit poles*

This result checks with the consideration that the flux from a pole of strength m c.g.s. units is $\Phi = 4\pi m$ maxwells, whereas the flux from a pole of m_1 m.k.s. units is

$$\Phi_1 = 4\pi m_1 \text{ webers} \equiv 4\pi m_1 \times 10^8 \quad \text{maxwells}$$

Since the flux density at a distance r cm. from a point pole of m c.g.s. units is $B = m/r^2$, it is seen that, if B is replaced by $B_1 \times 10^4$, m by $m_1 \times 10^8$, and r by $r_1 \times 10^2$, the result is

$$B_1 = \frac{m_1}{r_1^2} \tag{48}$$

the parallelism between m.k.s. and c.g.s. formulas thus being shown again.

20. Absolute Permeability in M.K.S. Units.—In the c.g.s. system, B in gausses and H in oersteds are related by the equation $B = \mu_0 H$, on the assumption that the magnetic field under consideration is in free space. In the m.k.s. system, $B_1 = (\mu_0)_1 H_1$, where $(\mu_0)_1$ is the m.k.s. value of space permeability. On taking the quotient of these two equations and rearranging terms, it is seen that

$$\frac{(\mu_0)_1}{\mu_0} = \left(\frac{B_1}{B}\right)\left(\frac{H}{H_1}\right) = 10^{-4} \times 10^{-3} = 10^{-7}$$

or

$$(\mu_0)_1 = \mu_0 \times 10^{-7} \tag{49}$$

from which *the absolute permeability of free space in the m.k.s. system has the numerical value* 10^{-7}.

The consistency of this value of $(\mu_0)_1$ may be checked by the consideration that Coulomb's law, in m.k.s. units, should give

$$f_1 = \frac{(m_1)^2}{(\mu_0)_1 r_1^2} \quad \text{newtons}$$

as the force between two equal point poles of strength m_1 m.k.s. units when they are placed r_1 m. apart. If $m_1 = 1$, and $r_1 = 1$, the result is

$$f_1 = \frac{1 \times 1}{10^{-7} \times 1} = 10^7 \quad \text{newtons}$$

Since $m_1 = 1$ is equivalent to a pole of which the strength is 10^8 c.g.s. units and since 1 m. is equivalent to 100 cm., the force

between the poles is

$$f = \frac{10^8 \times 10^8}{\mu_0 \times (100)^2} \text{ dynes} = 10^{12} \text{ dynes} \equiv 10^7 \quad \text{newtons}$$

which agrees with the previous result.

The fact that $(\mu_0)_1$ is equal to 10^{-7} may also be checked from the consideration that the flux density at a distance r_1 m. from an infinitely long straight wire carrying a current of I amp. is

$$B_1 = \frac{2(\mu_0)_1 I}{r_1} \quad \text{webers per sq. m.}$$

whereas, in c.g.s. units,

$$B = \frac{2\mu_0 I}{10r} \quad \text{gausses per sq. cm.}$$

where r is in centimeters. Dividing one equation by the other and rearranging,

$$(\mu_0)_1 = \frac{\mu_0}{10}\left(\frac{B_1}{B}\right)\left(\frac{r_1}{r}\right) = \frac{\mu_0}{10} \times 10^{-4} \times 10^{-2} = \mu_0 \times 10^{-7}$$

21. Dielectric Flux Density, Electric Field Strength, and Absolute Permittivity in M.K.S. Units.—Expressed in c.g.s. units, Coulomb's law for the force between two point charges in free space is

$$f = \frac{q_1 q_2}{\epsilon_0 r^2} \quad \text{dynes}$$

where q_1 and q_2 are in statcoulombs and r is in centimeters, while $\epsilon_0 = 1$. The dimensions of the statcoulomb are therefore given by

$$[Q_s]^2 = [\epsilon]\frac{M_g L_{cm}^2}{T_s^2}$$

In m.k.s. units, Coulomb's law reads

$$f_1 = \frac{(q_1)_1 (q_2)_1}{(\epsilon_0)_1 r_1^2} \quad \text{newtons}$$

where $(q_1)_1$ and $(q_2)_1$ are in coulombs, r_1 is in meters, and $(\epsilon_0)_1$ is the modified value of space permittivity. The dimensions of the coulomb are therefore

$$[Q]^2 = [\epsilon_1] \frac{M_{kg} L_m^3}{T_s^2}$$

and, on taking the quotient of these two equations,

$$\left(\frac{[Q]}{[Q_s]}\right)^2 = \frac{[\epsilon_1]}{[\epsilon]}\frac{M_{kg}}{M_g}\left(\frac{L_m}{L_{cm}}\right)^3$$

Since the coulomb is 3×10^9 times as large as the statcoulomb, this relation reduces to

$$\frac{[\epsilon_1]}{[\epsilon]} = 9 \times 10^{18} \times \frac{1}{10^3} \times \frac{1}{10^6} = 9 \times 10^9$$

hence the *unit* in which m.k.s. permittivity is measured is 9×10^9 times as large as the unit in which c.g.s. permittivity is measured. Consequently, since $\epsilon_0 = 1$ in the c.g.s. system, *the numerical value of m.k.s. space permittivity is* $1/(9 \times 10^9)$, or

$$(\epsilon_0)_1 = \frac{\epsilon_0}{9 \times 10^9} \tag{50}$$

At a distance r cm. from a point charge of q statcoulombs the dielectric flux density is $D = q/r^2$, from which it is seen that D has the dimensions statcoulombs per square centimeter, or

$$[D] = \frac{[Q_s]}{L_{cm}^2}$$

In the m.k.s. system, dielectric flux density must be

$$D_1 = \frac{q_1}{r_1^2} \tag{51}$$

where q_1 is in coulombs and r_1 is in meters; it has the dimensions coulombs per square meter, or

$$[D_1] = \frac{[Q]}{L_m^2}$$

hence

$$\frac{[D_1]}{[D]} = \frac{Q}{[Q_s]}\left(\frac{L_{cm}}{L_m}\right)^2 = 3 \times 10^9 \times \frac{1}{10^4} = 3 \times 10^5$$

Thus, if $D = 10$ in c.g.s. units, $D_1 = 10/(3 \times 10^5)$.

Strength of electric field is given by $E = D/\epsilon_0$ in the c.g.s. system, and by

$$E_1 = \frac{D_1}{(\epsilon_0)_1} \tag{52}$$

in the m.k.s. system.

22. Velocity of Propagation of Electromagnetic Waves.—It is generally understood, even by those who have only an elementary knowledge of the principles of radio communication, that the vertical antenna of a simple transmitting station is traversed by a current which surges up and down at high frequency; or a charge of electricity moves up and down in such a way that the free end of the antenna is charged alternately positively and negatively. Lines of electric force issuing from a positive charge on the antenna terminate on the negatively charged surface of the surrounding earth, and at each reversal of charge these lines of electric force also change their direction; their general arrangement in the vicinity of the antenna is shown in Fig. 3. As the positive charge accumulates on the antenna, the zone of negative charge surrounding its grounded base spreads out; and later, when the antenna becomes negatively charged, the original looped lines of force snap away from the antenna and continue to move outward into space.

The moving charges on the antenna constitute an alternating current and are encircled by lines of magnetic force which change direction with each reversal of current. The magnetic field spreads out with the electric field in such fashion that at every point in the advancing wave, at a sufficient distance from the antenna source, *the vector representing the electric field intensity is at right angles to the vector representing the magnetic field intensity*, and the two fields move together, at the same speed.

The flux-cutting law [Eq. (27), Chap. III], expressed in m.k.s. units, states that, if a magnetic field of flux density B_1 webers per sq. m. sweeps perpendicularly across a straight wire of length l_1 m. at a speed of v_1 m. per sec. the e.m.f. developed in the wire is $B_1l_1v_1 = (\mu_0)_1H_1l_1v_1$ volts. The factor $(\mu_0)_1$ implies that the wire is in free space. If it is assumed that the wire is not connected to any other conducting circuit, this e.m.f. is equal to the difference of potential between its ends, or

$$V = B_1l_1v_1 = (\mu_0)_1H_1l_1v_1 \quad \text{volts}$$

By definition, the potential difference between the ends of the wire is equal to the work (in joules) required to carry 1 coulomb from one end to the other, and this implies that in the direction of the axis of the wire there exists an electric field of such strength $E_1 = D_1/(\epsilon_0)_1$ that E_1l_1 must be equal to V. Consequently,

$$B_1l_1v_1 = (\mu_0)_1H_1l_1v_1 = E_1l_1 = \frac{D_1l_1}{(\epsilon_0)_1}$$

and after canceling the common factor l_1 the result is

$$B_1v_1 = (\mu_0)_1H_1v_1 = \frac{D_1}{(\epsilon_0)_1} \tag{53}$$

Equation (53) has been derived by assuming the existence of an actual wire which is cut by the moving magnetic field; but it will be observed that the length of the wire does not appear in the final equation (53) which relates B_1, D_1, and v_1. This result can have no other meaning than that a magnetic field B_1 moving in space at velocity v_1 automatically induces an electric field $E_1 = D_1/(\epsilon_0)_1$ at points where the magnetic flux density is B_1, pro-

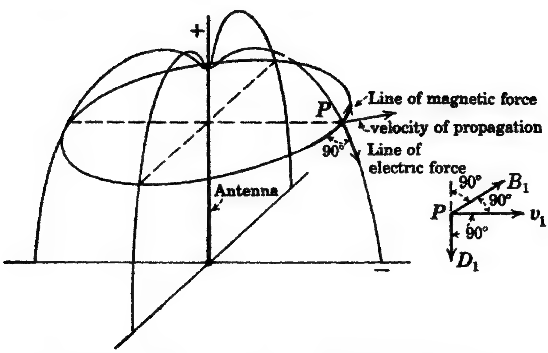


FIG. 3.—Lines of electric and magnetic force surrounding antenna. Electric field produced by moving magnetic field.

vided that the vectors representing B_1, v_1, and D_1 are mutually perpendicular. The result is therefore applicable to a point in free space, such as P (Fig. 3), where there is no wire but where the directions of the vectors B_1, v_1, and D_1 are related in the same way as if a conductor were present.

Though Eq. (53) establishes a relation between B_1, D_1, and v_1 in a traveling wave, it does not explicitly determine the magnitude of the velocity. But an additional independent relation may be obtained from a consideration of Fig. 4, in which Q (in coulombs) represents a positive point charge moving to the right with any arbitrary velocity, say v m. per sec., which will be assumed to be so small that inertial effects are negligible. At the instant represented in Fig. 4, an observer at point P, distant r_1 m. from Q, would be aware of an electric field directed along r_1,

such that the dielectric flux density at P is equal to $D_1 = Q/r_1^2$ coulombs per sq. m.

Since the charge Q is in motion relative to the observer at P, it is equivalent to an electric current flowing toward the right in the diagram, and consequently the line l_1 is encircled by lines of magnetic induction, one of which is shown in Fig. 4. This effect is in accord with the famous experiment of Rowland, who was the first to demonstrate that a moving electrostatic charge is equivalent to a current.

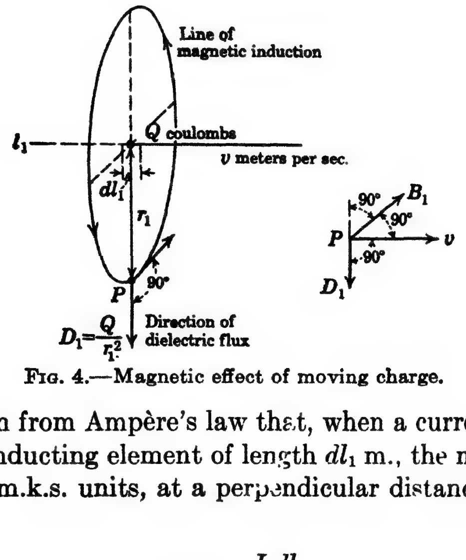


FIG. 4.—Magnetic effect of moving charge.

It is known from Ampère's law that, when a current of I amp. flows in a conducting element of length dl_1 m., the magnetic field strength, in m.k.s. units, at a perpendicular distance r_1 m. from dl_1 is

$$dH_1 = \frac{I\,dl_1}{r_1^2}$$

But $I = dQ/dt$, so that

$$dH_1 = \frac{(dQ/dt)\cdot dl_1}{r_1^2} = \frac{dQ(dl_1/dt)}{r_1^2}. \tag{54}$$

In Eq. (54), dl_1/dt is clearly equal to the velocity v (in meters per second) with which the charge dQ, carrying its electric field with it, is moving in the direction dl_1. If the charge is Q coulombs, instead of dQ, the field strength becomes H_1 instead of dH_1; hence,

$$H_1 = \frac{Qv}{r_1^2} \tag{55}$$

but since $Q/r_1^2 = D_1$, Eq. (55) becomes

$$H_1 = \frac{B_1}{(\mu_0)_1} = D_1 v \tag{56}$$

Inspection of Eq. (56) shows that it is does not contain Q as a factor, so that the result is independent of the magnitude of the charge with which the analysis started, just as in Eq. (53) the result was independent of the presence of the conductor initially introduced into the argument. It may therefore be concluded from Eq. (56) that, if an electric field of flux density D_1 moves with *any* velocity v, an observer would be aware of a magnetic field of flux density $B_1 = (\mu_0)_1 D_1 v$ which moves with D_1. Consequently, if D_1 moves with the *same* velocity v_1 which appears in Eq. (53), that is, with the velocity of a wave in free space, it must be true that, at the same point where the dielectric flux density is D_1, there is automatically engendered a magnetic field the strength of which is

$$H_1 = \frac{B_1}{(\mu_0)_1} = D_1 v_1 \tag{57}$$

Moreover, the space relations between H_1, D_1, and v, as shown in Fig. 4, are seen to be exactly the same as in Fig. 3; and from this fact it may be concluded that whereas in Fig. 3 the magnetic field sets up the electric field, while in Fig. 4 the electric field sets up the magnetic field, these separate effects must operate simultaneously in an electromagnetic wave traveling freely in space—in other words, the two fields mutually sustain each other. It is therefore legitimate to treat Eqs. (53) and (57) as simultaneous; and, on dividing one by the other and rearranging, the result is

$$v_1^2 = \frac{1}{(\mu_0)_1(\epsilon_0)_1} \tag{58}$$

or

$$v_1 = \frac{1}{\sqrt{(\mu_0)_1(\epsilon_0)_1}} \tag{59}$$

On substituting $(\mu_0)_1 = \mu_0 \times 10^{-7}$ and $(\epsilon_0)_1 = \epsilon_0/(9 \times 10^9)$ the last equation becomes

$$v_1 = \frac{3 \times 10^8}{\sqrt{\mu_0 \epsilon_0}} \tag{60}$$

or, since $\mu_0 = \epsilon_0 = 1$, the velocity of propagation is 3×10^8 m. per sec., which may be written

$$v = \frac{3 \times 10^{10}}{\sqrt{\mu_0 \epsilon_0}} \text{ cm. per sec.} \tag{61}$$

23. Resistance, Inductance, and Capacitance in M.K.S. System.—It has already been indicated that the use of the m.k.s. system retains without modification the practical units of e.m.f., quantity, current, resistance, inductance, and capacitance. But it is important to note that in making computations in the m.k.s. system, complete consistency requires that lengths must be expressed in meters, area in square meters, volume in cubic meters, mass in kilograms, force in newtons, magnetic flux in webers, flux density in webers per square meter, and so on. The following examples illustrate these considerations.

1. *Resistance.*—The resistance of a conductor is given in general by the relation $R = \rho l/a$, so that, if the length l is in meters and area a in square meters, ρ must be expressed in ohms per meter-cube. Since the resistance of a centimeter-cube of annealed copper is 1.7241×10^{-6} ohm at 20°C., the corresponding resistivity in the m.k.s. system is $\rho_1 = 1.7241 \times 10^{-8}$ ohm per meter-cube, whence

$$R = \rho_1 \frac{l_1}{a_1} \tag{62}$$

2. *Inductance.*—The inductance of a circuit is defined by the relation $L = (N\Phi/I) \times 10^{-8}$; that is, it is the flux linkages (in maxwells) per ampere divided by 10^8. But if flux is expressed in webers,

$$L \text{ (in henrys)} = \frac{\text{flux linkages (in webers)}}{\text{current (in amperes)}}$$

or the factor 10^{-8} disappears.

Expressed in terms of the dimensions and permeability of the magnetic circuit, inductance is given by

$$L = \frac{4\pi}{10^9} \frac{N^2 \mu_a A}{l} = \frac{4\pi}{10^9} \frac{N^2 \mu \mu_0 A}{l}$$

where A is in square centimeters, l is in centimeters, and $\mu_0 = 1$. If A_1 and l_1 are the corresponding dimensions based upon the

meter, this equation becomes

$$L = \frac{4\pi}{10^9} \frac{N^2[(\mu_0)_1 \times 10^7](A_1 \times 10^4)}{l_1 \times 10^2} = \frac{4\pi N^2 \mu(\mu_0)_1 A_1}{l_1} \tag{63}$$

so that the conversion factor 10^9 disappears.

3. *Capacitance.*—If the capacitance of a parallel-plate condenser is taken as typical, Eq. (12) of Chap. IV shows that

$$C_s = \frac{K\epsilon_0 A}{4\pi t} \quad \text{statfarads}$$

or

$$C = \frac{K\epsilon_0 A}{4\pi t \times 9 \times 10^{11}} \quad \text{farads}$$

Substituting $\epsilon_0 = (\epsilon_0)_1 \times 9 \times 10^9$, $A = A_1 \times 10^4$, and $t = t_1 \times 10^2$,

$$C = \frac{K[(\epsilon_0)_1 \times 9 \times 10^9] \times (A_1 \times 10^4)}{4\pi(t_1 \times 10^2) \times 9 \times 10^{11}} = \frac{K(\epsilon_0)_1 A_1}{4\pi t_1} \tag{64}$$

or the original simple form that gives C_s in statfarads now gives C in farads, the conversion factor having again disappeared.

For concentric cylinders,

$$\begin{aligned} C_s &= \frac{K\epsilon_0}{2 \ln (R_2/R_1)} \quad \text{statfarads per cm.} \\ &= \frac{K\epsilon_0}{2 \ln (R_2/R_1)} \times 10^2 \quad \text{statfarads per m.} \end{aligned}$$

so that

$$C = \frac{K(\epsilon_0)_1 \times 9 \times 10^9}{2 \ln (R_2/R_1)} \times \frac{10^2}{9 \times 10^{11}} = \frac{K(\epsilon_0)_1}{2 \ln (R_2/R_1)} \quad \text{farads per m.} \tag{65}$$

again restoring the original simple form.

One additional illustration involves the force of attraction between the plates of an absolute electrometer, as in Eq. (54), Chap. IV. The force is

$$f = \frac{V_s^2 A \epsilon_0}{8\pi t^2} \quad \text{dynes}$$

where V_s, in statvolts, is equal to $V/300$, if V is in volts. Replacing A, t, and ϵ_0 by A_1, t_1, and $(\epsilon_0)_1$,

$$\begin{aligned} f &= \frac{(V/300)^2(A_1 \times 10^4)(\epsilon_0)_1 \times 9 \times 10^9}{8\pi(t_1 \times 10^2)^2} \\ &= \frac{V^2A_1(\epsilon_0)_1}{8\pi t_1^2} \times 10^5 \quad \text{dynes} \\ &\equiv \frac{V^2A_1(\epsilon_0)_1}{8\pi t_1^2} \quad \text{newtons} \end{aligned} \tag{66}$$

which again agrees in form with the original simple form of the c.g.s. equation.

24. Rationalized M.K.S. Units.—About 1881, Oliver Heaviside* suggested that, if electric and magnetic forces were standardized in terms of rectangular coordinates, such as arise naturally from consideration of parallel-plate condensers and of parallel pole faces, the factor 4π, which appears in many formulas because of the radial and spherical distributions about point charges and point poles, would disappear from some of the formulas. Thus, in the c.g.s. system, a unit pole emits 4π lines of magnetic flux, and a unit charge emits 4π lines of electric flux. It would be a simple matter, requiring merely a change in the definition of the units, to assume that a unit charge emits *one* line of electric flux and that a unit pole emits *one* line of magnetic flux. This suppression of the factor 4π is described as *rationalizing* the units, and the existing systems, in which 4π appears explicitly, are referred to as *nonrationalized.*

The particular method advocated by Heaviside for rationalizing the c.g.s. units required changing the accepted practical units (the volt, ampere, ohm, etc.) by awkward numerical factors involving 4π, and no one was willing to face the confusion that would have resulted had Heaviside's suggestion been adopted; the cure would have been worse than the disease. But it is possible to rationalize the units, especially in the m.k.s. system, by procedures that do not affect the commonly used practical units, while at the same time the factor 4π is made to disappear from many, but not all, of the formulas already derived in preceding chapters. It must be remembered that 4π is a factor which enters into calculations because it represents a fundamental property of space, and if it is suppressed in one place it is certain to reappear elsewhere; for it is self-evident that the properties of space cannot be denied their appropriate

* "Collected Papers," Vol. II.

expression. It is no more possible completely to dispense with π or 4π from electrical and magnetic calculations than it is to "square the circle."

Consider, for example, the equation that expresses the law of the magnetic circuit

$$\Phi = \frac{(4\pi/10)NI}{l/\mu_a A} = \frac{(4\pi/10)NI}{l/\mu\mu_0 A} \quad \text{maxwells}$$

where l and A are in centimeters and square centimeters. On changing to m.k.s. units, the equation becomes

$$\Phi_1 = \frac{4\pi NI}{l_1/\mu(\mu_0)_1 A_1} \quad \text{webers} \tag{67}$$

where l_1 and A_1 are in meters and square meters and $(\mu_0)_1 = 10^{-7}$. There is left in Eq. (67) the incommensurable number 4π; but by transforming the expression into

$$\Phi_1 = \frac{NI}{l_1/\mu[4\pi(\mu_0)_1]A_1} \quad \text{webers} \tag{68}$$

the factor 4π disappears, provided that the value of space permeability is assigned the new value

$$(\mu_0)_1' = 4\pi(\mu_0)_1 = \frac{4\pi}{10^7} \tag{69}$$

This procedure was first suggested by Giorgi. It is perfectly legitimate, because the choice of the numerical value of space permeability is in any case entirely arbitrary. All that is necessary is to make such additional adjustments in other relations as are required to maintain unaltered the practical units, such as the volt, ampere, coulomb, ohm, henry, and farad. Whatever advantage inheres in the process of rationalization is due to the simplification of equations like (67) that take forms like (68); the factor 4π no longer appears in the numerator, which is expressed in terms of the widely used unit, the *ampere-turn,* instead of the somewhat more artificial unit, the gilbert. But it is worthy of note that, although the formula is simplified as to outward appearance, there is no change whatever in the actual calculations that must be made in any given problem; for the factor 4π has merely been transferred from the numerator of Eq. (67) to the subdenominator of Eq. (68) and 4π must be used whether we like it or not.

The rationalized value of space permeability affects the formula for inductance [Eq. (63)], which becomes

$$L = \frac{N^2\mu(\mu_0)_1'A_1}{l_1} \text{ henrys} \tag{70}$$

the factor 4π that appears in Eq. (63) having been absorbed into the magnitude of $(\mu_0)_1'$.

It is evident that a change in the value assigned to space permeability calls for a compensating change in the value of space permittivity, since their product must continue to be $1/(9 \times 10^{16})$, in accordance with Eqs. (49) and (50). Consequently, the rationalized value of space permittivity is

$$(\epsilon_0)_1' = \frac{1}{4\pi \times 9 \times 10^9} = \frac{(\epsilon_0)_1}{4\pi} \tag{71}$$

Reference to Eq. (64), which gives the capacitance of a parallel-plate condenser, shows that if $(\epsilon_0)_1$ is replaced by $(\epsilon_0)_1'$ the formula becomes

$$C = \frac{K(\epsilon_0)_1'A_1}{t_1} \text{ farads} \tag{72}$$

and is thereby rationalized; but the capacitance of a pair of concentric cylinders [Eq. (65)] becomes

$$C = \frac{2\pi K(\epsilon_0)_1'}{\ln (R_2/R_1)} \text{ farads per m.} \tag{73}$$

and it is thus demonstrated that the ubiquitous π cannot be universally suppressed.

25. Dielectric Flux in Rationalized M.K.S. Units.—It has been shown in Art. 21 that the force of repulsion between two point charges q_1 and q_1' coulombs, separated by a distance x_1 m., can be computed from the formula

$$f_1 = \frac{q_1q_1'}{(\epsilon_0)_1x_1^2} \text{ newtons}$$

provided that $(\epsilon_0)_1 = 1/(9 \times 10^9)$, this being the unrationalized m.k.s. value of space permittivity. It must be remembered that the force of repulsion between the charges is a fundamental physical fact and that the formula is merely put together in such a way that the factors which enter into it are expressed in units

which are consistent with experimental observations. If, therefore, this form of Coulomb's law is to be expressed in rationalized m.k.s. units, in which space permittivity is $(\epsilon_0)_1' = (\epsilon_0)_1/4\pi$, it follows that

$$f_1 = \frac{q_1 q_1'}{4\pi(\epsilon_0)_1' x_1^2} \quad \text{newtons} \tag{74}$$

If in Eq. (74) there is substituted $q_1' = 1$ coulomb, the resultant value of f_1 is the strength of the electric field at a distance x_1 m., namely,*

$$E_1 = \frac{1}{4\pi(\epsilon_0)_1'} \frac{q_1}{x_1^2}$$

Since flux density is in general equal to field intensity multiplied by permittivity, so that $D_1 = (\epsilon_0)_1' E_1$, it follows that the dielectric flux density is

$$D_1 = \frac{1}{4\pi} \frac{q_1}{x_1^2} \tag{75}$$

and this value being the same at all points on a sphere of radius x_1 m. surrounding q_1 the total dielectric flux is

$$\psi_1 = 4\pi x_1^2 \times \frac{1}{4\pi} \frac{q_1}{x_1^2} = q_1 \tag{76}$$

Thus, in the rationalized m.k.s. system, the dielectric flux issuing from a given charge is numerically equal to the charge in coulombs and may be expressed directly in coulombs. It is also evident from Eq. (75) that dielectric flux density is expressible in coulombs per square meter. The factor 4π appears, however, in the value of space permittivity.

The discussion of rationalized units in Arts. 24 and 25 has been included for the sake of completeness, but thus far there has been no international agreement concerning their official adoption. Attention may therefore be concentrated on the unrationalized m.k.s. system.

26. Need for a Fourth Fundamental Unit.—Inspection of the dimensions of the various units derived in Arts. 4, 5, 6, 8, 9, 10, 11 shows that in every case there appears the dimension of either permeability or permittivity ($[\mu]$ or $[\epsilon]$). In the absence of any

* In accordance with the original definition that field strength is measured by the force (now in newtons) exerted upon a unit charge.

other consideration, there is no way to assign to either $[\mu]$ or $[\epsilon]$ a definite dimension, since all that is known about them is that the reciprocal of their product has the nature of velocity squared. This difficulty would be overcome and $[\mu]$ and $[\epsilon]$ thus eliminated from the dimensions of the units if a fourth fundamental unit, in addition to mass, length, and time, were to be agreed upon. It is, in fact, essential to a consistent set of derived electrical units that a fourth fundamental unit should be adopted, for the nature of electricity and magnetism makes it impossible completely to define their properties in terms of the three fundamental units (mass, length, and time) that are sufficient for mechanical units. The attempt to limit the fundamental units to mass, length, and time has led to the absurdities involved in dropping functions of $[\mu]$ and $[\epsilon]$ from the dimensions of the electrical units; for when $[\mu]$ and $[\epsilon]$ are dropped, the ratios of corresponding units in the e.s. and e.m. systems turn out to be of the nature of velocity, or the square of velocity, or the reciprocals of these dimensions, and this result is manifestly ridiculous.

Any one of the practical electrical units—volt, ampere, coulomb, ohm, henry, farad—might serve as the needed fourth unit, at least from a theoretical point of view. At the present time, though no official action has been taken, the weight of opinion favors the coulomb and the ohm. The reasons advanced in support of the coulomb appear to depend upon the fact that the charge on an electron is a fixed natural constant, though it is true that there is no simple relation between the electron charge and the coulomb; for example, the charge on an electron is 4.803×10^{-10} statcoulomb, which is equivalent to 1.601×10^{-19} coulomb; that is, there are 6.246×10^{18} electron charges in a coulomb. On the other hand, there are good reasons* for selecting the ohm as the fourth fundamental unit, for resistance can be measured with extreme precision, and copies of standard resistors can be made with comparative ease; moreover, a standard ohm "kept at Sèvres" along with the standard kilogram and the standard meter appears to offer advantages that would certainly not be attainable if the coulomb were the fourth unit.

In what follows it will be assumed: (1) that the coulomb plays the role of the fourth unit and that its dimension, designated

*See G. E. M. JAUNCEY and A. S. LANGSDORF, "M.K.S. Units and Dimensions."

as Q, may be considered coordinate in rank with M, L, and T; (2) that the dimension of the ohm, designated as R, may be used in a similar manner.

For convenience, there are assembled in the following table the dimensions of the basic electrical units as derived in Arts. 6 to 11:

TABLE III.—DIMENSIONS OF ELECTRICAL UNITS IN TERMS OF $[\mu]$ AND $[\epsilon]$

Item	E.s. system	E.m. system	Ratio e.m./e.s.
Quantity	$[\epsilon]^{1/2}M^{1/2}L^{3/2}T^{-1}$	$[\mu]^{-1/2}M^{1/2}L^{1/2}$	$[\mu\epsilon]^{-1/2}L^{-1}T$
Current	$[\epsilon]^{1/2}M^{1/2}L^{3/2}T^{-2}$	$[\mu]^{-1/2}M^{1/2}L^{1/2}T^{-1}$	$[\mu\epsilon]^{-1/2}L^{-1}T$
E.m.f	$[\epsilon]^{-1/2}M^{1/2}L^{1/2}T^{-1}$	$[\mu]^{1/2}M^{1/2}L^{3/2}T^{-2}$	$[\mu\epsilon]^{1/2}LT^{-1}$
Resistance	$[\epsilon]^{-1}L^{-1}T$	$[\mu]LT^{-1}$	$[\mu\epsilon]L^2T^{-2}$
Inductance	$[\epsilon]^{-1}L^{-1}T^2$	$[\mu]L$	$[\mu\epsilon]L^2T^{-2}$
Capacitance	$[\epsilon]L$	$[\mu]^{-1}L^{-1}T^2$	$[\mu\epsilon]^{-1}L^{-2}T^2$

1. If the dimension of quantity in the e.m. system is designated as Q,

$$[\mu]^{-1/2}M^{1/2}L^{1/2} = Q$$

or

$$[\mu] = MLQ^{-2} \tag{77}$$

and since

$$[\mu\epsilon] = T^2L^{-2}$$

it follows that

$$[\epsilon] = \frac{T^2L^{-2}}{[\mu]} \equiv M^{-1}L^{-3}T^2Q^{-2} \tag{78}$$

When these dimensions of $[\mu]$ and $[\epsilon]$ are substituted in the expressions in Table III, the results take the forms shown in Table IV.

TABLE IV.—DIMENSIONS OF ELECTRICAL UNITS IN TERMS OF M, L, T, Q

Item	E.s. system	E.m. system	Ratio
Quantity	Q	Q	1
Current	QT^{-1}	QT^{-1}	1
E.m.f	$ML^2T^{-2}Q^{-1}$	$ML^2T^{-2}Q^{-1}$	1
Resistance	$ML^2T^{-1}Q^{-2}$	$ML^2T^{-1}Q^{-2}$	1
Inductance	ML^2Q^{-2}	ML^2Q^{-2}	1
Capacitance	$M^{-1}L^{-2}T^2Q^2$	$M^{-1}L^{-2}T^2Q^2$	1

The fact that the dimensions of any given unit are now the same in both the e.s. and the e.m. systems should not be surprising, for that is what they ought to be. Not only are they the same in these two systems, but they will remain unchanged in any other consistent system, such as the m.k.s. system. In case the coulomb is later officially selected as the fourth fundamental unit, the system might well be called the meter-kilogram-coulomb-second (m.k.c.s.) system.

2. If the dimension of resistance in the e.m. system is designated as R,

$$[\mu]LT^{-1} = R$$

or

$$[\mu] = L^{-1}TR \tag{79}$$

$$[\epsilon] = \frac{T^2L^{-2}}{[\mu]} = L^{-1}TR^{-1} \tag{80}$$

Upon substituting these dimensions of $[\mu]$ and $[\epsilon]$ in the expressions in Table III, the results take the forms shown in Table V.

TABLE V.—DIMENSIONS OF UNITS IN TERMS OF M, L, T, R

Item	E.s. system	E.m. system	Ratio
Quantity	$M^{1/2}LT^{-1/2}R^{-1/2}$	$M^{1/2}LT^{-1/2}R^{-1/2}$	1
Current	$M^{1/2}LT^{-3/2}R^{-1/2}$	$M^{1/2}LT^{-3/2}R^{-1/2}$	1
E.m.f	$M^{1/2}LT^{-3/2}R^{1/2}$	$M^{1/2}LT^{-3/2}R^{1/2}$	1
Resistance	R	R	1
Inductance	RT	RT	1
Capacitance	$R^{-1}T$	$R^{-1}T$	1

A system thus including the ohm as a fourth fundamental unit might be called the meter-kilogram-ohm-second (m.k.o.s.) system.

CHAPTER VI

THE DYNAMO

1. Dynamo, Generator, and Motor.—A dynamo-electric machine, or more briefly a *dynamo*, may be defined as a machine for converting mechanical energy into electrical energy, in which case it is called a *generator;* or, conversely, it may convert electrical energy into mechanical energy, in which case it is called a *motor*. Otherwise stated, the word dynamo is a generic term which includes the other two; a dynamo is a reversible machine capable of operating either as a generator or as a motor.

The design and construction of all types of generators and motors are based upon the facts: (1) that an e.m.f. is developed in a conductor (or set of conductors) subjected to the action of a magnetic field in such manner that the conductor either cuts across the lines of magnetic force or is cut by them; and (2) that a conductor (or a set of conductors), suitably placed in a magnetic field, is acted upon by a mechanical force when it is made to carry an electrical current supplied by some outside source. The first fact is characteristic of generator action, the second of motor action.

Every generator consists of a set of suitably connected conductors so disposed that e.m.fs. may be induced in them by the influence of a magnetic field moving in space relative to the conductors. Current will be produced when these active conductors form part of a circuit that is closed through an external circuit. On the other hand, motor action will be produced by the same set of conductors, suitably located in a magnetic field, when current from an external source is passed through them.

In the case of *generator action*, each active conductor is the seat of an induced e.m.f.

$$\left.\begin{aligned} \bar{E} &= Blv \quad \text{abvolts} \\ \text{or} \qquad E &= Blv \times 10^{-8} \equiv B_1 l_1 v_1 \quad \text{volts} \end{aligned}\right\} \qquad (1)$$

where B is the flux density (in gausses, or lines per square centi-

meter) of the magnetic field through which it is moving; l is its length in centimeters; v the relative velocity, in centimeters per second, in a direction mutually perpendicular to that of the field and that of the wire; and B_1, l_1, and v_1 are, respectively, the same quantities, but in m.k.s. units (B_1 in webers per square meter, l_1 in meters, v_1 in meters per second).

On closing the circuit there will flow a current of, say $\bar{I}$ abamp. $\equiv I$ amp., the magnitude of which will depend upon the resistance of the circuit as a whole, in accordance with Ohm's law. The conductor will be acted upon by a force

$$\left.\begin{aligned} F &= Bl\bar{I} \quad \text{dynes} \\ \text{or} \qquad F_1 &= B_1l_1I \quad \text{newtons} \end{aligned}\right\} \tag{2}$$

in a direction opposite to its motion; hence, to maintain the action, an equal and opposite driving force must be applied to the conductor and work must be done at the rate of $Fv = Bl\bar{I}v = \bar{E}\bar{I}$ ergs per sec., or $F_1v_1 = B_1l_1Iv_1 = EI$ joules per sec. The mechanical power (Fv or F_1v_1) thus reappears in the form of the equivalent power $\bar{E}\bar{I}$ ergs per sec., or EI watts.

In the case of *motor action*, each conductor is caused to carry a current of I amp., so that it is acted upon by a lateral thrust of

$$F_1 = B_1l_1I \quad \text{newtons}$$

because of which motion of the wire results; and under the influence of the field intensity B_1 and velocity v_1 there is induced in the wire an e.m.f.

$$E = B_1l_1v_1 \quad \text{volts}$$

in a direction opposite to the current. To maintain the current flow there must be impressed an e.m.f. of sufficient magnitude to balance this counter-generated e.m.f., and work is done by the electrical source of supply at the rate of $EI = B_1l_1v_1I = F_1v_1$ joules per sec. In this case the electrical power EI watts reappears as mechanical power F_1v_1.

Thus, in the case of a *generator*, the flow of current produced by the generated e.m.f. is accompanied by the appearance of a counter-torque that opposes the torque of the prime mover; in a *motor*, the rotation produced by the reaction between current and magnetic field sets up a counter e.m.f. that opposes the voltage originally impressed. Both effects are in accord with Lenz's law.

In this discussion ideal conditions have been tacitly assumed, namely, that all the energy supplied reappears as useful energy after the conversion process has been completed. As a matter of fact this condition is never realized in practice; the energy supplied must be greater than that usefully converted by an amount equal to the loss of energy inevitable in the conversion.

The *armature* of a dynamo is the part in which the e.m.f. is generated in the case of a generator or the part which carries the working current in the case of a motor. The *field* member is the part that produces the magnetic field. The relative motion of one structure with respect to the other is most easily obtained by making one or the other rotate, so that in general the two have concentric cylindrical forms. Either may be the rotating member; if the armature rotates the machine is called a revolving-armature machine, whereas if the field rotates it is called a revolving-field machine.

There are two distinct types of dynamo-electric machines, according to the nature of the e.m.f. and current produced; they are (1) *a-c* machines, and (2) *d-c* machines. The first type, when used as a generator, is called an *alternator* and produces an e.m.f. that acts alternately in opposite directions, so that when the armature circuit is completed the current in the external circuit flows first in one direction and then in the other. The second type produces a current through the external circuit that flows in one direction only. ✓A direct current, though characterized by constancy of direction, may, however, vary in magnitude from instant to instant, that is, it may be pulsating—or it may be constant in magnitude as well as in direction. In the former case the current is said to be a *direct current;* in the latter case the current is said to be a *continuous current.* ✓

The a-c generator or motor is the simplest form of dynamo. Reduced to the most elementary type, it consists of a loop of wire, *abcd,* Fig. 1, rotating in a magnetic field that passes across from pole N to pole S. It is understood that the pole pieces N and S are the extremities of the field structure and that the excitation of the magnets is effected by a direct current from some suitable source circulating in coils wound on the field structure. The ends of the armature coil are attached to the insulated *slip rings* r_1, r_2. In the position shown in the figure, wire *ab* will have generated in it an e.m.f. directed from front to back, whereas the

e.m.f. in cd will be directed from back to front; the collecting brush touching ring r_1 will therefore be positive, and that touching r_2 will be negative. After half a revolution it will be seen that the polarity of the terminals reverses, so that each terminal is alternately of opposite polarity.

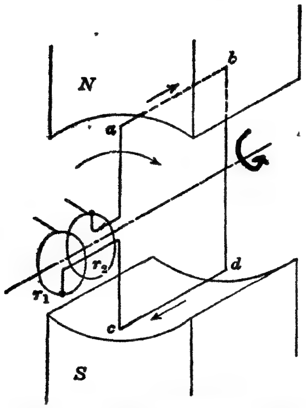


FIG. 1.—Elementary alternator.

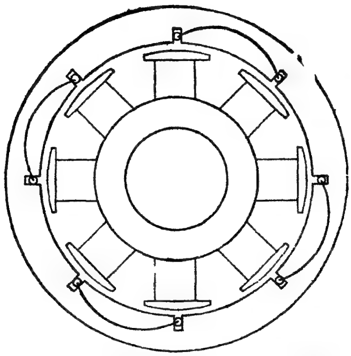


FIG. 2.—Multipolar revolving-field alternator.

Alternating-current machines usually have more than the two poles shown in Fig. 1, in which case they are *multipolar*. The winding consists of a number of coils connected in series in such manner that the e.m.fs. of the individual coils add together. Figure 2 represents diagrammatically an eight-pole revolving-field machine with the winding of the stationary armature

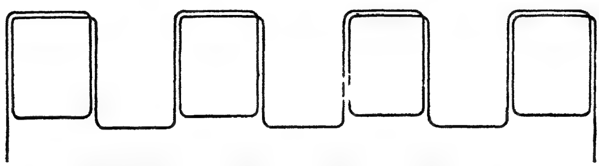


FIG. 3.—Developed armature winding of alternator of Fig. 2.

arranged in eight slots. Figure 3 is a development of this particular type of winding as it would appear if the cylindrical surface of the armature were rolled out into a plane.

With the exception of the homopolar machine described in Art. 21, all standard forms of d-c generators and motors consist of a wire- or bar-wound armature arranged to rotate between inwardly projecting poles of alternate polarity, in the manner

illustrated in Fig. 14. Each of the armature conductors is, therefore, the seat of an alternating e.m.f. which changes its direction each time the conductor moves from the influence of one pole to that of the adjacent pole. It is the function of the commutator to convert this internal alternating e.m.f. into a unidirectional e.m.f. in the external circuit; but so far as the armature winding itself is concerned, *every d-c machine* (with the exception of the homopolar machine) *is essentially an a-c machine*, and hence it is important to analyze the development of the e.m.f. in an alternator in order to understand thoroughly what is happening in the case of the d-c machine.

2. E.M.F. of Elementary Alternator.—Consider first the elementary alternator of Fig. 4, the armature winding of which

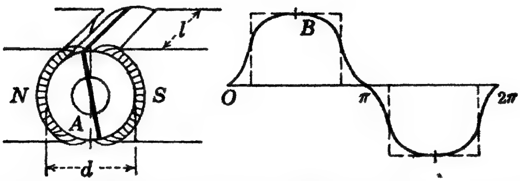


FIG. 4.—Distribution of flux density around armature.

consists of a concentrated coil having Z conductors (or $N = Z/2$ turns) on the external periphery of the armature core A. If the airgap between the pole faces N, S and the armature core is uniform, as is usual in d-c machines (except at the pole tips), the lines of force of the magnetic field will tend to cross the gap on radial lines, and the field strength will have practically uniform strength everywhere under the poles; at the pole tips the lines of force will "fringe," the spreading apart of the lines indicating that the strength of the field tapers off more or less gradually to zero value midway between the poles. These facts are represented in Fig. 4, where the line marked B is so drawn that its ordinates represent the *radial* component of the flux density in the airgap all around the periphery of the armature, the latter being developed, that is, rolled out into a plane. If it were not for the fringing of the flux at the pole tips, the flux distribution would be represented by the rectangular diagram shown in broken lines.

If the diameter of the armature is d cm. and its active length is l cm. and if it is driven at a speed of n r.p.m., the tangential

or peripheral velocity of the conductors is

$$v = \pi d \frac{n}{60} \quad \text{cm. per sec.}$$

and at the instant when the conductor is cutting through the magnetic field where the flux density has a radial component of B gausses, the generated e.m.f. per conductor is

$$e = Blv \times 10^{-8} = \pi \, dl \frac{n}{60} B \times 10^{-8} \quad \text{volts} \tag{3}$$

Because of the assumed concentration of the Z conductors in a diametral plane and because of the further assumption that the flux distribution is symmetrical around the armature, the e.m.f. is the same in all of the conductors at the same instant; hence, the total instantaneous value of the generated e.m.f. is

$$e = \pi \, dl \frac{n}{60} ZB \times 10^{-8} \quad \text{volts} \tag{4}$$

Since all the terms on the right-hand side of this expression are constant with the exception of B, it follows that the variation of e.m.f. with respect to *time* is identical in form with the curve B of Fig. 4, which shows the *space* distribution of the flux density. The e.m.f. is zero at the instant when the active sides of the coil are passing through the neutral axis midway between the poles, rises sharply as the active coil sides pass under the poles, remains fairly constant as they pass under the pole faces, falls again to zero when the coil is again in the neutral plane, and then goes through these changes with a reversal of direction. Thereafter, the same cycle of changes is repeated indefinitely.

If the field intensity in the airgap were truly uniform and radial, with no fringing at the pole tips, the distribution would be represented by the rectangle shown in dashed lines in Fig. 4. The wave form of the corresponding e.m.f. would likewise be rectangular, the altitude of the rectangle being given by Eq. (4). If the e.m.f. is plotted with time as argument, the angular abscissas of Fig. 4 must be replaced by time, such that T, the time of a complete revolution of the armature (in seconds), is equivalent to the angle 2π; but $T = 60/n$; and if the poles embrace ψ per cent of the pole pitch, the duration of the active

part of a half wave will be $\psi(T/2)$. The *average* e.m.f. during a half wave is therefore equal to the area of the rectangle divided by the interval corresponding to a half cycle, or

$$E_{aver.} = \frac{e\psi(T/2)}{T/2} = \pi\, dl\psi BZ\frac{n}{60} \times 10^{-8}$$

But $\pi\, dl\psi$ is the area of both pole faces, and $\pi\, dl\psi B$ is equal to 2Φ, where Φ is the flux per pole. The average e.m.f. is, therefore,

$$E_{aver.} = 2\Phi Z\frac{n}{60} \times 10^{-8}$$

which agrees with the fact that each conductor cuts 2Φ lines per revolution, or $2\Phi(n/60)$ lines per sec.

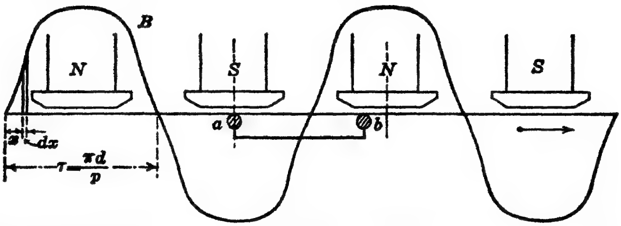


FIG. 5.—Multipolar alternator, non-sinusoidal flux distribution.

3. General Case of the E.M.F. of an Alternator.—The discussion of Art. 2 is based upon the assumption of a bipolar field structure and a full-pitch armature coil, that is, a coil spanning the arc from center to center of poles. Generally, however, there are more than a single pair of poles, and the coil spread may be greater or less than the pole pitch.

Let Fig. 5 represent a partial development of an alternator having p poles (like Fig. 2), and let the space distribution of flux at the armature surface be represented by curve B; further, let the armature have a diameter of d cm. and the conductors an active length of l cm. in a direction parallel to the shaft, and let the speed of rotation be n r.p.m. The instantaneous e.m.f. generated in each conductor is the same as that given by Eq. (3),

$$e = Blv \times 10^{-8} = Bl\pi d\frac{n}{60} \times 10^{-8}$$

and the graph of this e.m.f. is a curve which is the same as that showing the flux distribution, except for a change in scale. The

average e.m.f. per conductor is

$$E_{aver.} = \frac{1}{\tau}\int_0^{\tau = (\pi d/p)} e\,dx = \frac{1}{\tau}\frac{n}{60}\pi d\int_0^{\tau} Bl\,dx \times 10^{-8} = p\frac{n}{60}\Phi \times 10^{-8} \tag{5}$$

where $\Phi = \int_0^{\tau} Bl\,dx$ is the flux per pole. Equation (5) might have been anticipated from the fact that the average e.m.f. is equal to the number of lines of force cut per second, divided by 10^8; thus, each conductor in one revolution cuts Φ lines per pole. or $p\Phi$ lines per revolution, and hence $p\Phi(n/60)$ lines per sec. It is interesting to note that Φ is the integral of the B function; conversely, B is the first derivative of the flux function.

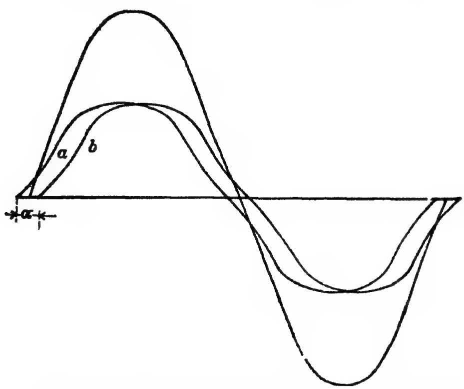


FIG. 6.—E.m.f. in coil of fractional pitch.

If the armature is wound with Z conductors, all connected in series as in Fig. 3, all the coils being of full pitch, the total average e.m.f. is

$$E_{aver.} = p\Phi Z\frac{n}{60} \times 10^{-8} \tag{6}$$

It has been pointed out that each conductor is the seat of an e.m.f. whose variation from instant to instant is represented graphically by a curve identical (except for a change of scale) with the curve B of flux distribution (Fig. 5). If the conductors are arranged as in Fig. 2 so that the coil spread is the same as the pole pitch, the e.m.f. in all conductors will be simultaneously in the same phase of the variation, and the total instantaneous e.m.f. will be Z times that of a single conductor. But if the coil

spread differs from the pole pitch, as indicated by coil *ab*, Fig. 5, the instantaneous e.m.fs. of the two sides of the coil will differ in phase, that of coil side *a* following curve *a*, Fig. 6, and that of coil side *b* following curve *b*, where the displacement α between the curves corresponds to the amount by which the sides of the coil *ab* of Fig. 5 fall short of being a pole pitch apart. The total instantaneous e.m.f. of the coil is obtained by adding the ordinates of the individual e.m.f. curves. It is evident from Fig. 6 that the maximum e.m.f. of such a "fractional-pitch" winding is less than that of a full-pitch winding of the same number of conductors and that the shape of the resultant wave form of e.m.f. differs from that of its component parts.

4. Rectification of an Alternating E.M.F.—If the terminals *a* and *c* of the elementary alternator of Fig. 1 are connected,

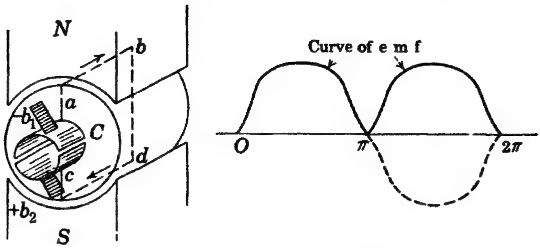


FIG. 7 — Two-part commuator. Rectification of e.m.f.

respectively, to the two insulated segments of a *commutator C*, as in Fig. 7, and stationary brushes b_1, b_2 are mounted so as to make sliding contact with the revolving commutator segments, the plane of the brushes being coincident with that through the shaft and the polar axis, the brush b_1 will always be of negative polarity and brush b_2 will always be of positive polarity. The reversal of the e.m.f. of the coil takes place simultaneously with the passage of the brushes across the gaps between the segments of the commutator. If the flux distribution is like that of curve *B*, Fig. 4, the brush voltage will vary in the manner shown in Fig. 7; the negative half loop of the original alternating voltage is reversed, so far as the external circuit is concerned, and hence the voltage in the external circuit connected to the brushes b_1 and b_2 will be unidirectional, but its magnitude will pulsate between zero and a maximum value.

If the *Z* peripheral conductors constituting the loop of Fig. 7 are replaced by a winding like that of Fig. 8, the latter likewise

having Z peripheral conductors, the generated e.m.f. will remain the same in magnitude and in the manner of its variation. The winding in Fig. 7 is of the *drum* type; that of Fig. 8 is of the *ring* type. It will be noted that in the drum winding there is one complete turn for each pair of conductors, whereas in the ring type there is a turn for each conductor; but in the ring winding the wires inside the core play no part in generating e.m.f., for they do not cut lines of force. Consequently, for equal numbers of peripheral conductors in the two types of winding, the electrical characteristics are identical except for minor differences due to unequal length of wire and therefore of resistance.

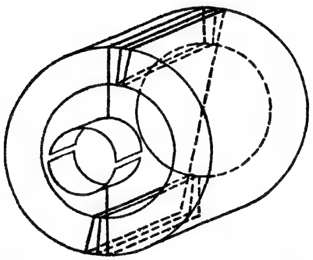


FIG. 8.—Elementary ring-wound armature.

5. Effect of Distributed Winding.—An e.m.f. pulsating as in Fig. 7 is not desirable, and means must be found to make it more nearly continuous. The large amplitude of the pulsation in Fig. 7 is due to the fact that the entire armature is inactive each time the active coil sides pass through the neutral zone between the poles (that is, each time commutation takes place); if the

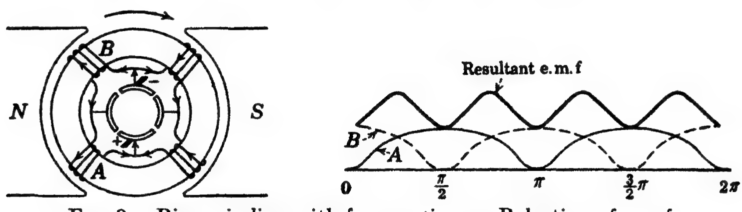


FIG. 9.—Ring winding with four sections. Pulsation of e.m.f

winding can be so disposed that small sections, each consisting of relatively few turns, undergo commutation successively, the pulsations will become insignificant when the number of such winding sections is sufficiently great. Thus, Fig. 9 is a diagrammatic sketch of the armature of Fig. 8, but with the original Z conductors arranged in four equidistant groups of concentrated coils; the end of each coil is connected to the beginning of the next, and there is a connection between each of these junction points and a segment of a four-part commutator. A study of the directions of the e.m.fs. generated in the coils shows that the brushes

must now be placed along an axis perpendicular to the polar axis, in order that current may be delivered effectively to the external circuit.

It is clear that, although the four individual coils form a closed ring so far as the internal armature circuit is concerned, they are connected to the external circuit by way of the brushes in such manner that with respect to the external circuit the armature winding consists of two equal halves in parallel with each other; each half of the armature winding consists of a pair of winding sections connected in series. In each half of the armature winding the generated e.m.f. is equal to that of the other half, on the assumption that the flux distribution is the same under both poles and that the winding is symmetrical; but, with respect to the closed circuit of the armature winding itself, these two e.m.fs. are opposite in direction, and hence they balance each other exactly and there is no tendency to set up a circulating current around the armature winding. (The case of asymmetrical flux distribution is reserved for discussion in a subsequent chapter.)

Since the winding consists of two equal parts in parallel, the voltage at the brushes will be equal to that of either half alone. Considering the particular half of the armature winding made up of sections A and B, it will be observed that section A generates a wave of e.m.f. similar to that of Fig. 7, but of only one-fourth the amplitude since coil A has only one-fourth as many active conductors as the coil of Fig. 8. Similarly, section B generates a wave exactly like that of section A, but the two waves differ in phase by 90 deg., as shown in Fig. 9. The resultant brush voltage at any instant will be obtained by adding the corresponding ordinates of these two component curves, as shown. There are now four pulsations instead of the original two, but the range from minimum to maximum is much reduced.

Carrying the subdivision of the winding one step further by arranging the Z peripheral conductors in eight equidistant groups of coils as shown in Fig. 10, each half of the resultant ring winding will consist of four sections in series. In each section there will be generated an e.m.f. which will vary from instant to instant in the manner shown by curve B of Fig. 7, but the maximum value will be only one-eighth as great as in that figure; the e.m.fs. in the four coils which at any moment are in series in one-half of

the ring differ in phase from each other by 45 deg., as shown in Fig. 10, so that the resultant e.m.f. between the brushes is at any moment the sum of the ordinates of these four curves. It will be seen that there are now eight pulsations instead of the original two, and the amplitude of the pulsations is decidedly less than before.

Figure 10 shows the armature in two different positions: in part (*a*), the four coils marked *A*, *B*, *C*, *D*, are directly in series, each contributing a definite e.m.f. to the total; in part (*b*), rotation of the armature has carried coil *D* to the position where commuta-

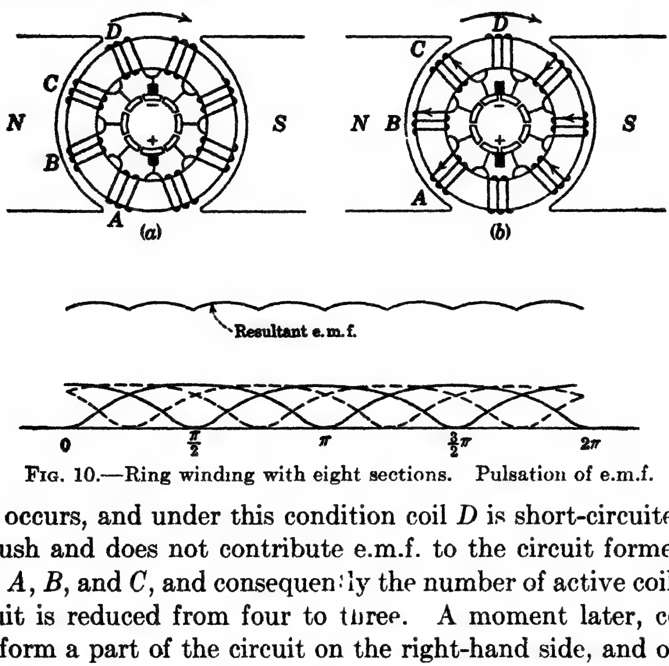


FIG. 10.—Ring winding with eight sections. Pulsation of e.m.f.

tion occurs, and under this condition coil *D* is short-circuited by a brush and does not contribute e.m.f. to the circuit formed by coils *A*, *B*, and *C*, and consequently the number of active coils per circuit is reduced from four to three. A moment later, coil *D* will form a part of the circuit on the right-hand side, and one of the coils from the right-hand side will have been transferred to the left-hand side; thereafter the process is repeated indefinitely. In position (*a*) the total e.m.f. developed by the four sections in series is a maximum; in position (*b*) the total e.m.f. is a minimum. Clearly, therefore, the variation will become less and less as the angular separation between adjacent coils is reduced, that is, as the number of sections is increased.

The smoothing effect upon the resultant e.m.f. produced by increasing the number of winding sections is analogous to the

effect upon the torque of a gas engine produced by increasing the number of cylinders, provided that the successive crankpins are uniformly displaced around the crankshaft.

6. Average E.M.F. of D-c Armature.—If a ring winding of the type discussed in the preceding article is rotated in a multipolar field structure, as indicated in Fig. 11, the e.m.fs. generated in the individual winding sections will be directed in the manner shown by the small arrowheads. In the diagram the e.m.fs. are so directed that the entire winding is divided into four belts, one per pole, in each of which the individual e.m.fs. are additive. In order to take full advantage of this distribution, brushes must be placed at each neutral point: half the brushes will be of positive polarity, the other half negative. If all the positive brushes are connected to one another and to the positive side of the external circuit and if the negative brushes are likewise connected to one another and to the negative side of the external circuit, then, with respect to the external circuit, the armature winding consists of four paths (in the case illustrated) which are in parallel with one another. The e.m.f. will be the same in each of these paths provided that the entire armature winding is symmetrical and that the flux from each of the poles is the same in magnitude and in distribution; and the total e.m.f. of the machine as a whole will be the same as that in any one of the paths.

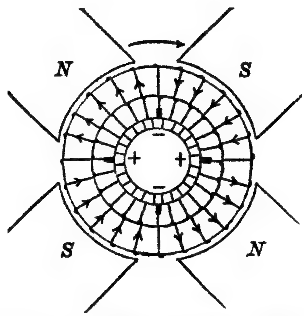


FIG. 11—Ring-wound armature in multipolar field

In the winding illustrated in Fig. 11 the number of paths is equal to the number of poles, this equality being characteristic of all simple ring windings of the type illustrated. But it will be shown in Chap. VII that the number of paths, a, is not necessarily equal to the number of poles, p, in all armature windings, and that by suitably connecting the individual coils of the winding the number of paths may be made any even number, from two up.

It is clearly of fundamental importance to be able to compute the average e.m.f. developed in the armature winding of a d-c machine. Thus, let it be required to find the average e.m.f. generated in a winding having the following data:

Z = total number of peripheral conductors.
a = number of parallel paths through the armature.
p = number of poles.
Φ = flux per pole, in maxwells.
n = revolutions of armature per minute.

Each conductor cuts $p\Phi$ lines per revolution, or $p\Phi(n/60)$ lines per sec., so that the average e.m.f. per conductor is $p\Phi(n/60) \times 10^{-8}$ volts. Since the entire number of conductors is divided into a paths connected in parallel with one another, the number of *conductors in series per path* is Z/a. The average e.m.f. per path, and therefore of the armature as a whole, is

$$E = \frac{Z}{a} p\Phi \frac{n}{60} \times 10^{-8} = \frac{p}{a} \frac{\Phi Z n}{60 \times 10^{8}} \text{ volts} \tag{7}$$

This is the *general equation for the generated e.m.f. of a d-c machine, provided* that the brushes are so placed that the winding sections of any one group, as shown in Fig. 11, are simultaneously under the influence of one pole. Thus, if the brushes of the armature of Fig. 12 are so placed that commutation takes place in coils opposite the middle of the pole shoes, the potential difference between the brushes will be zero; for in that case each path through the armature is made up of conductors half of which are subjected to the inductive action of one pole and the other half to the influence of a pole of opposite polarity, with the result that the e.m.fs. generated in the two halves of each path are equal and opposite.

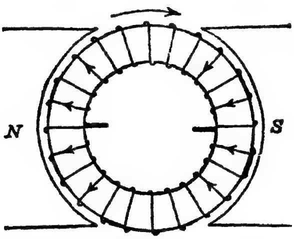


FIG. 12.—Brushes displaced from proper position.

7. Magnitude of E.M.F. Pulsations.—Comparison of the curves of Figs. 7, 9, and 10 with the corresponding windings brings out very clearly that subdividing the winding into relatively few distributed sections causes a marked reduction in the magnitude of the pulsations of e.m.f. above and below the average value. Further subdivision of the winding produces a still further suppression of the pulsations, though beyond a certain point the smoothing out of the wave of e.m.f. proceeds at a greatly reduced rate. In practice the number of winding sections is determined by such considerations as the attainment

of satisfactory commutation, and the number so fixed is sufficiently great to make the pulsations of e.m.f. of minor importance. Nevertheless, it is of interest to investigate the relation between the number of winding sections and the magnitude of the voltage fluctuation.

In order to simplify the analysis, it will be assumed that the machine has two poles, like that of Fig. 10; that the winding consists of Z conductors divided into s sections having Z/s turns each; and further that the flux distribution, instead of having the form shown in Fig. 4, is sinusoidal. Therefore, the radial component of flux density at any point in the airgap is proportional to the sine of the angle measured from the neutral axis to the point in question; thus, if B_m is the radial component of flux density under the middle of the pole face, the flux density at any point displaced θ deg. from the neutral axis is

$$B = B_m \sin \theta \tag{8}$$

In one of the winding elements which at a given instant is displaced θ deg. from the neutral axis the instantaneous value of the generated e.m.f. is

$$e_1 = \frac{Z}{s}Blv \times 10^{-8} = \pi\frac{Z}{s}B_m ld\frac{n}{60} \sin \theta \times 10^{-8} = E_m \sin \theta \tag{9}$$

where

$$E_m = \pi\frac{Z}{s}B_m ld\frac{n}{60} \times 10^{-8} \tag{10}$$

is the maximum value of the e.m.f. generated in the winding element of Z/s turns at the particular instant when $\theta = \pi/2$, that is, when the coil is passing under the middle of the pole face. Since there are s winding sections uniformly distributed around the armature, the angle between adjacent sections is $2\pi/s$; and at the instant when the coil above referred to occupies the position determined by the angle θ, the next coil ahead of it occupies the position $\left(\theta + \frac{2\pi}{s}\right)$, the next one beyond occupies the position $\left(\theta + 2\frac{2\pi}{s}\right)$, and so on. It follows that the instantaneous values of e.m.f. in the $s/2$ successive winding sections in series with each other are

$$e_1 = E_m \sin \theta$$
$$e_2 = E_m \sin\left(\theta + \frac{2\pi}{s}\right)$$
$$e_3 = E_m \sin\left(\theta + 2\frac{2\pi}{s}\right)$$
. . . .
$$e_{s/2} = E_m \sin\left[\theta + \left(\frac{s}{2} - 1\right)\frac{2\pi}{s}\right] = E_m \sin\left[\theta + \left(\pi - \frac{2\pi}{s}\right)\right]$$

Thus, in a winding like that shown in Fig. 10 where $s = 8$, the e.m.fs. in the successive winding sections are 45 deg. apart, and

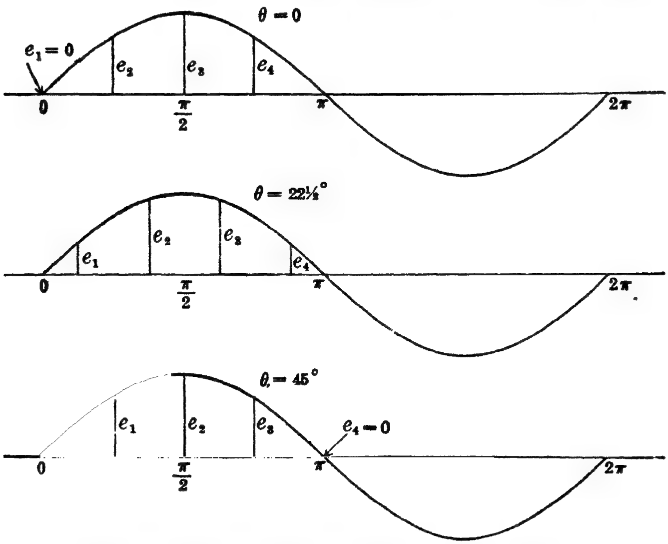


FIG. 13.—Successive phases of e.m.f. in eight-coil ring winding, sinusoidal flux distribution.

their values are indicated in Fig. 13 for three different values of the angle θ, corresponding to three successive positions of the armature as the latter rotates. The instantaneous value of the total e.m.f. contributed by all the $s/2$ sections is

$$\Sigma e = e_1 + e_2 + e_3 + \cdots + e_{s/2}$$
$$= E_m\left\{\sin\theta + \sin\left(\theta + \frac{2\pi}{s}\right) + \cdots + \sin\left[\theta + \left(\pi - \frac{2\pi}{s}\right)\right]\right\} \quad (11)$$

The minimum value of this expression occurs when $\theta = 0$, as was pointed out in connection with Fig. 10, and the maximum value occurs when $\theta = \pi/s$; it follows that

$$E_{min.} = E_m\left[\sin\frac{2\pi}{s} + \sin\frac{4\pi}{s} + \cdots + \sin\left(\pi - \frac{2\pi}{s}\right)\right]$$
$$= E_m \cot\frac{\pi}{s} \tag{12}$$

and

$$E_{max.} = E_m\left[\sin\frac{\pi}{s} + \sin\frac{3\pi}{s} + \cdots + \sin\left(\pi - \frac{\pi}{s}\right)\right]$$
$$= E_m \operatorname{cosec}\frac{\pi}{s} \tag{13}$$

The percentage variation from minimum to maximum, in terms of the minimum value, is

$$\frac{\operatorname{cosec}\frac{\pi}{s} - \cot\frac{\pi}{s}}{\cot\frac{\pi}{s}} \times 100$$

and the magnitude of this quantity, for various values of s, is shown in the following table:

s	Per Cent Variation
2	∞
4	41.00
6	15.40
10	5.17
20	1.24
30	0 56
60	0.13

The table shows that, if the winding is divided into 30 or more sections (the field structure being bipolar), the fluctuations are quite insignificant.

Since the instantaneous value of the total e.m.f. varies from a minimum when $\theta = 0$ to a maximum when $\theta = \pi/s$ and since thereafter the e.m.f. falls symmetrically to the original minimum when $\theta = 2\pi/s$, the average e.m.f. is

$$E_{aver.} = \frac{1}{\pi/s}\int_0^{\pi/s} (e_1 + e_2 + e_3 + \cdots + e_{s/2})d\theta$$

$$= ZB_mld\frac{n}{60} \times 10^{-8}\int_0^{\pi/s}\left\{\sin\theta + \sin\left(\theta + \frac{2\pi}{s}\right) + \cdots + \sin\left[\theta + \left(\pi - \frac{2\pi}{s}\right)\right]\right\}d\theta$$

$$= ZB_mld\frac{n}{60} \times 10^{-8} \tag{14}$$

If the flux density is so distributed that it follows the equation

$$B = B_m \sin\theta$$

its average value over each half of the armature surface is

$$B_{aver.} = \frac{1}{\pi}\int_0^{\pi} B_m \sin\theta\, d\theta = \frac{2}{\pi}B_m \tag{15}$$

and the flux per pole is

$$\Phi = B_{aver.} \times \text{area of one-half of armature surface}$$

$$= B_{aver.} \times \frac{\pi d}{2} \times l = B_m\, dl \tag{16}$$

Hence,

$$E_{aver} = \Phi Z\frac{n}{60} \times 10^{-8} \tag{17}$$

which agrees with Eq. (7) since in the case here considered $p = 2$ and $a = 2$.

8. Resistance of Armature Winding.—In an armature having a paths, the total armature current I_a will divide equally between them, provided that all paths have the same resistance. If the total series resistance of all the wire on the armature is R_t ohms, the resistance per path is R_t/a ohms; and since these a paths are connected in parallel, the actual resistance of the armature, as measured between brushes, is

$$R_a = \frac{R_t}{a^2} \quad \text{ohms}$$

The drop of potential due to the entire current I_a flowing through the resistance R_a, or I_aR_a volts, is equal to the drop of potential through any one of the paths, or

$$\frac{I_a}{a} \times \frac{R_t}{a} = I_aR_a$$

9. Construction of Dynamos.—The dynamo consists essentially of a magnetic circuit and an electric circuit placed in inductive relation to each other. The magnetic circuit is made up of

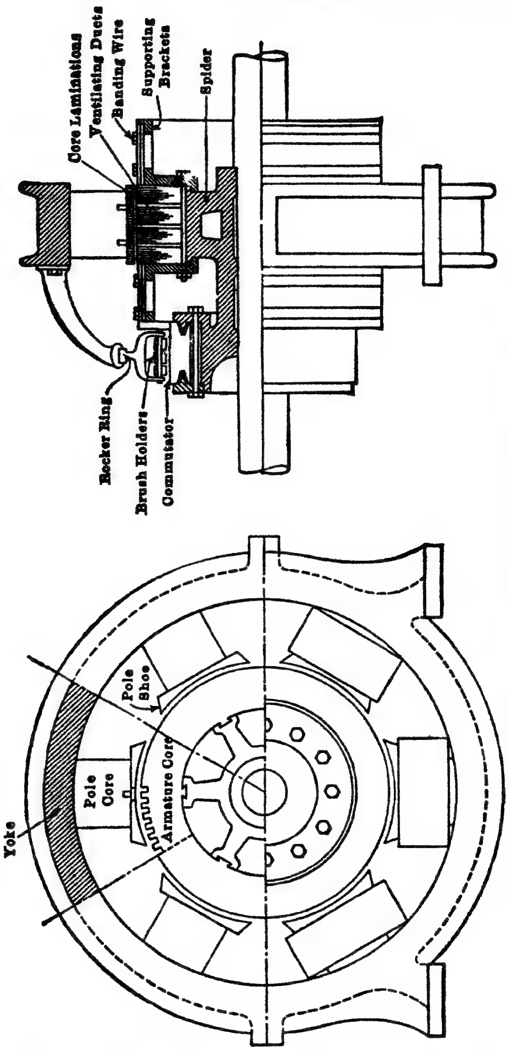


FIG. 14.—Diagrammatic view of multipolar dynamo.

the *yoke, pole cores* and *pole shoes*, and the *armature core.* The electric circuit consists of the *armature winding* and its associated *commutator*, together with the *field winding* which provides the excitation for the magnetic circuit. The annular space between

the revolving armature and the stationary field structure is called the *airgap*. Other parts of the machine are the *field winding*, the *brushes*, *brush holders* and the *rocker arm*, the *armature spider*, and the *bearings*.

The parts named in the preceding paragraph are indicated in Fig. 14; this illustrates an *open machine* of the *self-ventilated* type, in which the air is circulated by means integral with the machine and without restriction to its flow other than that necessitated by mechanical construction.

The operating conditions to which motors may be subjected are often so severe that they must be protected by partly or

(*a*) (*b*)

FIG. 15.—(*a*) Totally enclosed, fan-cooled motor. (*b*) Semi-enclosed d-c motor. (*Courtesy Westinghouse Electric and Manufacturing Company.*)

wholly enclosing the external frame.* In such cases the machine may be either self-ventilated or separately ventilated.

Thus, an *enclosed self-ventilated* machine has openings for the admission and discharge of the ventilating air, which is circulated by means integral with the machine; but the machine is otherwise totally enclosed, and the openings are so arranged that the inlet and outlet ducts may be connected to them.

An *enclosed separately ventilated* machine has openings for the admission and discharge of the ventilating air, which is circulated by fans or blowers not a part of the machine; the machine is otherwise totally enclosed, and the openings are so arranged that inlet and outlet ducts may be connected to them.

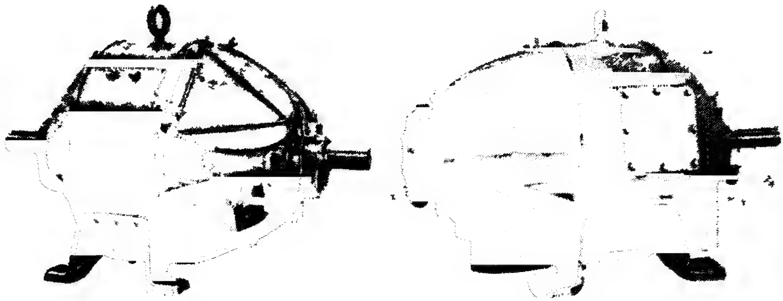
Totally enclosed machines (Fig. 15*a*) are so constructed as to prevent exchange of air between the inside and the outside of the

* See American Standards for Rotating Electrical Machinery, Part C50, A.I.E.E. Standards.

enclosing case but are not sufficiently enclosed to be termed air-tight. A totally enclosed machine is said to be *fan-cooled* if it is equipped for exterior cooling by means of a fan integral with the machine, but external to the enclosing parts.

The degree of protection afforded by the enclosing case is indicated by the following classifications:

Protected machine [formerly called semi-enclosed (Fig. 15*b*)]: Ventilating openings in the frame are protected with wire-screen, expanded-metal, or perforated covers, the openings in which do not exceed ½ sq. in. in area, and which will not permit the passage of a rod larger than ½ in. in diameter; except that where the distance of exposed live parts from the guard is more than 4 in.



(*a*) (*b*)

FIG 16 – (*a*) Drip-proof d-c motor (*b*) Splash-proof d-c motor (*Courtesy Westinghouse Electric and Manufacturing Company*)

the openings may be ¾ sq. in. in area, and not permit the passage of a rod larger than ¾ in. in diameter.

Semi-protected machine: Part of the ventilating openings in the frame, usually in the top half, are protected as in the "protected machine," but the others are left open.

Drip-proof machine (Fig. 16*a*): Ventilating openings are so constructed that drops of liquid or solid particles, falling at any angle not greater than 15 deg. from the vertical, cannot enter the machine either directly or by striking and running along a horizontal or inwardly inclined surface.

Splash-proof machine (Fig. 16*b*): This has protection similar to that of the drip-proof machine, except that the angle of approach is 100 deg. from the vertical.

Explosion-proof machine: The enclosing case is constructed to withstand an explosion of a specified gas or dust that may occur

within it, and to prevent the ignition of the specified gas or dust surrounding the machine, by sparks, flashes, or explosions of the specified gas or dust that may occur within the machine casing.

Water-proof machine: A totally enclosed machine that will exclude water applied as a stream not less than 1 in. in diameter, under a head of 35 ft. and from a distance of about 10 ft.; except that leakage which may occur around the shaft may be considered permissible, provided that it is prevented from entering the oil reservoir and provision is made for automatically draining the machine.

Dust-tight machine: One so constructed that the enclosing case will exclude dust.

Resistant (used as a suffix), as moisture-resistant, fume-resistant, etc.: One so constructed that the machine will not be readily injured when subjected to the specified material, as follows:

Moisture, in very damp or humid atmosphere.

Fumes, as specified.

Acid, as specified.

Gas, as specified.

Alkali, as specified.

Submersible machine: Constructed so that it will operate successfully when submerged in water under specified conditions of pressure and time.

10. Bipolar and Multipolar Machines.—Although for the sake of simplicity much of the preceding discussion has been based upon the assumption of a bipolar field structure, this type of field is seldom used except in machines of the smallest size. The actual number of poles generally varies from 4 to a maximum (in d-c machines) of 20 to 24, the number increasing with the power rating, though not at all regularly. The explanation of the principles underlying the choice of the number of poles must be deferred to a later section; in general, however, the choice of the number of poles depends upon the consideration that the magnetic reaction of the armature, when carrying current, cannot exceed definite limits without impairing the operating characteristics of the machine. Further, with an armature core of given dimensions and with pole pieces that cover a definite percentage of the armature surface, usually about 70 per cent, the field frame becomes more compact, up to a certain limit, as the number of poles is increased beyond two.

The optimum limit occurs when the peripheral spread of the pole faces is approximately equal to the axial length of the pole face. A compact field frame is advantageous in that comparatively little of the field flux leaks from pole to pole without entering the armature core.

11. Commutator.—The commutator is built up of wedge-shaped segments of drop-forged or hard-drawn copper insulated from one another by accurately gaged thin sheets of insulating material such as mica. The process of assembling a large number of segments into a rigid structure is an interesting one. The segments, separated from one another by the mica insulation, are placed around the inner periphery of a sectored steel ring A, as in Fig. 17, and the copper segments S are then wedged together to form a rigid circular arch by means of cap screws C tapped radially through the outer steel ring B. The V-shaped grooves D are then machined out, and the commutator spider is bolted into place; then the auxiliary steel-ring clampings are removed, and the external surface is machined to true cylindrical form.

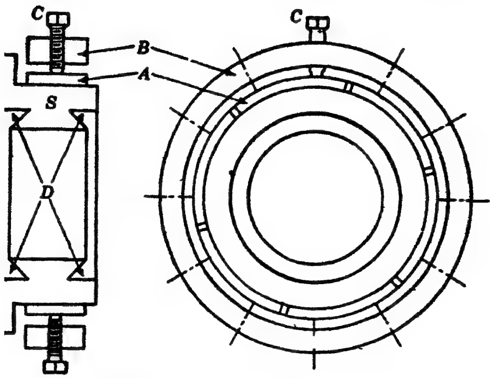


FIG 17.—Construction of commutator.

The insulation between the commutator and the supporting hub consists of molded mica cones and cylinders, as in Fig. 18. The completed commutator must be given a high-voltage test to ensure the thorough insulation of each segment from the others and from the spider.* The insulation between adjacent segments does not have to be as heavy as that between the segments and the commutator spider, for the latter must withstand the full

* See Art. 18, Chap. XIII

terminal voltage of the machine whereas the former is called upon to withstand only the smaller voltage between segments. The average voltage between adjacent segments should not exceed 10 to 15 volts in lighting and railway generators that are not of the commutating-pole type, and 20 to 25 volts in the case of railway motors. These limiting values of average voltage between segments are imposed by the requirements of satisfactory commutation, and they determine the minimum number of segments in the completed commutator. For example, if a six-pole, 600-volt railway generator is to have not more than 10 volts between adjacent segments, there must be at least 60 segments between adjacent brushes of opposite polarity, or not less than 360 segments in the entire commutator. The minimum diameter of the commutator is then determined if the minimum peripheral width of a segment is known; this minimum width is rarely less than $\frac{3}{16}$ in. for two reasons: (1) because the taper of the segments would result in too thin a section at the inner periphery if a smaller external width were used; (2) because some allowance in the radial depth of the segments must be made to permit turning down the surface in case of pitting, blistering, or wear.

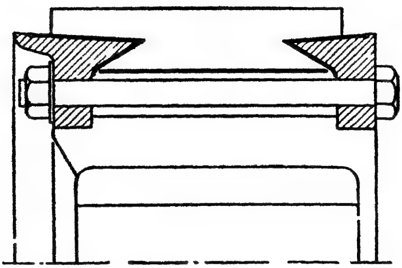


FIG. 18.—Insulation of commutator.

The thickness of the insulation between segments varies from 0.02 in. in low-voltage machines up to about 0.06 in. in high-voltage machines. Amber mica is largely used for this purpose, partly because its rate of mechanical wear is substantially the same as that of the copper bars, but mainly because of its excellent characteristics as an insulator under extreme conditions of temperature. As a matter of fact, the wearing down of the copper of the commutator is due only in small part to mechanical friction, the chief cause being an electrolytic transfer of the metal when current flows between the segments and the brushes. For this reason commutators are usually built so that the insulation does not come quite flush with the surface, the necessity of selecting the material for a definite rate of wear being thus obviated. The undercutting of the insulation is accomplished by a milling operation after the commutator has been assembled.

Commutators must be designed to have a sufficient amount of exposed peripheral surface to radiate the heat caused by brush friction and by the brush-contact resistance. The design must provide sufficient mechanical strength to withstand the centrifugal force. In the case of turbo-generators running at high speed the diameter is limited by the consideration that the peripheral velocity shall not exceed about 8000 ft. per min.; hence, to secure sufficient radiating surface the commutator must have considerable axial length. To prevent springing of the segments they are held in place by steel rings shrunk over the segments and thoroughly insulated therefrom, as shown in Fig. 28.

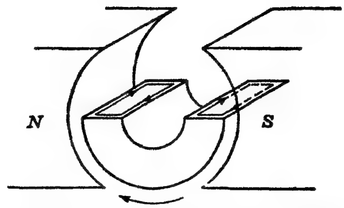
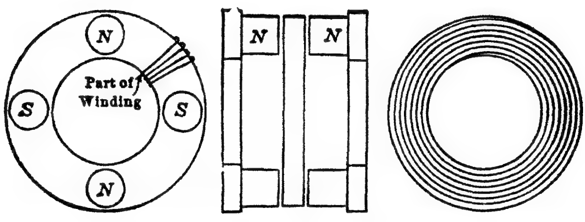


FIG. 19.—Eddy current paths, solid armature core.

12. Armature Core.– The armature core not only carries the magnetic flux from pole to pole but revolves through it in exactly the same manner as the conductors of the armature winding.



Disk Machine

Spirally Laminated Armature

FIG. 20.—Lamination of disk armature.

If the core were solid, it might be thought of as made up of a very large number of metallic filaments running parallel to the armature conductors and all connected; each filament would be the seat of a generated e.m.f., and currents would circulate in the mass of the core in the manner sketched in Fig. 19. The e.m.f. is obviously greatest near the surface where the peripheral velocity and the active component of the flux are likewise greatest. To minimize these *eddy* or *Foucault currents*, which, if unchecked, would result in excessive heating and loss of power, the core must be *laminated* in such a manner as to preserve the continuity of the flux path and to break up the current paths. The plane of the laminations must be perpendicular to the direction of the

generated e.m.f., or, by Fleming's rule, parallel to the direction of the flux and to the direction of motion. Accordingly, in machines of the usual radial-pole type (Fig. 14), the armature core is built up of thin sheet-steel punchings insulated from each other. Sometimes the insulation consists of a coating of varnish on one side of each disk, but generally the oxide, or scale, that forms on the sheets is relied upon to provide the necessary insulation; in some designs a layer of paper is inserted at intervals of an inch or two. Laminating the core does not completely eliminate eddy currents, but the loss due to them decreases as the square of the thickness of the sheets; the sheet steel ordinarily used in armature cores is 14 to 25 mils thick. Armatures of the now obsolete disk type (Fig. 20), with active conductors arranged radially, had cores built up of concentric hoops or of thin strap iron wound as a flat spiral.

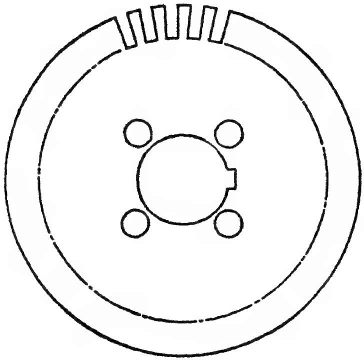


FIG. 21.—One-piece armature punching.

Core punchings up to a diameter of about 16 in. are generally made in one piece, as in Fig. 21. The disks are first blanked out, and the slots are then punched by an indexing punch press which cuts one or more slots at a time. Core punchings of this sort are generally keyed directly to the shaft and are sometimes provided with holes near the shaft to form longitudinal ventilating passages. Cores of large diameter are built up of segments that are attached to the spider by means of a dovetail joint, as in Fig. 14; the joints between segments are staggered from layer to layer in order to preserve the continuity of the magnetic circuit. The core punchings are held together by end flanges which, in small machines, are supported by lock nuts screwed directly to the shaft; in larger machines the end plates are held together by bolts passing through the laminations but insulated therefrom, and the end plates are shaped to provide a support for the end connections of the armature winding (see Fig. 14).

Radial ventilating ducts through the core are formed by means of spacing pieces placed at intervals of 2 to 4 in. along the axis of the core. The spacing pieces are generally made by riveting

brass strips, on edge, to a punching of heavy sheet steel, as illustrated in Fig. 22; or they may be made by pressing spherical depressions into a thick punching or by spot-welding steel strips, on edge, to the sheet-steel punching. The ventilating ducts vary in width from $\frac{1}{4}$ to $\frac{3}{8}$ in. The spacing pieces should be so designed as to support the teeth as well as the body of the core, in order to prevent vibration and humming.

If the core laminations are stacked so that the slots are parallel to the shaft and also parallel to the axial edge of the pole shoes, objectionable noise may result because of vibrations set up in the teeth. The tension along the lines of magnetic force that pass

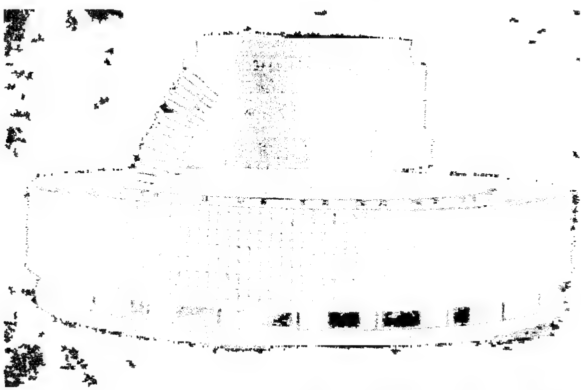


FIG. 22.—Armature core assembly, showing spacing pieces.

between the pole face and the teeth tends to pull the tooth in one direction as the tooth approaches a pole and in the other direction as it leaves. The tooth, therefore, tends to vibrate in the manner of a short reed clamped at one end, and this effect is accentuated because the teeth are narrower at the base than at the tip. The condition may become aggravated if the natural period of vibration of the tooth coincides with the frequency of the magnetic blows it undergoes on entering and leaving the pole face. To prevent this action, the slots are sometimes skewed by a slight angular displacement of successive laminations; or else the axial edge of the pole face is slightly off parallelism with the shaft.

13. Shape of Teeth and Slots.—Figure 23 illustrates typical forms of teeth and slots for d-c machines. Smooth-core armatures are used only in special machines. Open slots with parallel walls are generally used, except in very small machines, for they

permit the use of insulated, formed coils that can readily be slipped into place. Where semiclosed slots are used, the coils may be formed on a winding jig, but the wires of each side of a coil must be slipped into the slot one at a time. The coils are held in place in open slots by steel or bronze banding wires, or by wooden or fiber wedges driven into the recesses at the tips of the teeth, or by both methods.

The embedding of the armature winding in the slots serves a double function; the airgap, or distance from the pole face to the iron of the armature core, is less than it would be in a smooth-core construction having the same amount of armature copper, and so reduces the amount of field copper necessary to produce the flux; and the armature conductors are supported by the teeth when subjected to the tangential forces caused by the reaction of the armature current upon the field flux. When the armature con-

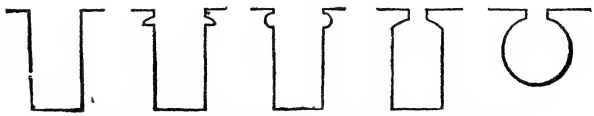


FIG 23.—Typical shapes of teeth and slots

ductors are thus embedded in the slots, they are apparently shielded from the inductive effect of the field flux, since the latter in large measure passes around the slots by way of the teeth. The flux density in the slots is much smaller than in the iron of the adjoining teeth. At first sight, therefore, it seems surprising that the fundamental equation for the generated e.m.f. is the same for a slotted armature as for a smooth-core armature. It may be considered, however, that a line of force, which at a given instant crosses the airgap from the pole face to a given tooth tip, must later, by reason of the motion of the armature, be transferred from this tooth tip to the following tooth. The line of force holds on, as it were, to the first tooth in the manner of a stretched elastic thread, until the increasing tension causes it to snap back suddenly to the next tooth. The increased velocity of cutting of the lines of force by the conductors due to this rapid movement of the lines of force exactly compensates for the reduced value of the field intensity in the slot.

14. Insulation of Armature Winding.—The coils comprising the armature winding are usually made of single or double cotton-

covered copper wire of which the cross-section may be circular, square, or rectangular; enameled wire is often used in small machines. If the coil consists of two or more turns, these are held together with a half-lapped cotton tape, and two or more such coils may then be taped together so that they will form a unit adapted to slip into the slots in the manner shown in Fig. 22, Chap. VII. Coils thus formed are usually dipped in an insulating compound and then baked to give them firmness and increased dielectric strength.

Though the walls of the slots are always filed to remove burrs at the edges of the laminations, a slot lining must be used to protect the coils from mechanical injury while they are being inserted and to guard against chafing under operating conditions. The slot lining is commonly made of fish paper, pressboard, or fiber that is itself lined with empire cloth, or varnished cambric or muslin. The armature windings of most d-c machines have two layers, and a strip of hard fiber or its equivalent is inserted between the layers, which have between them the full difference of potential of the machine.

Insulation of the type just described has definite temperature limitations because of its organic nature. Where higher temperatures are encountered, an inorganic insulating material, such as mica, is often used. The conductors—usually rectangular in section under these circumstances—are wrapped individually with mica tape, and the bundle of such conductors which lie side by side in a slot are wrapped first with a mica sheet and then with cotton tape; the unit so formed is then dipped and baked. Within recent years, insulation made of woven spun-glass fiber has come into use. The material is flexible and has high dielectric strength; and because it withstands high temperature without deterioration, machines in which it is used can be made materially smaller than would otherwise be the case.

15. Pole Cores and Pole Shoes.—The pole cores are generally made of cast steel. When cast steel is used, the poles usually have a circular cross-section because this results in minimum length and weight of the copper wire in the field winding. Laminated poles require a rectangular cross-section. Solid poles are commonly bolted to the yoke. Laminated poles may be secured in place either by a dovetail joint or by being cast into the yoke.

The flux density in the body of the pole core runs as high as 110,000 lines per sq. in., a value considerably greater than is economical in the airgap. The average flux density in the airgap should not exceed 62,000 lines per sq. in.; hence, the pole faces must have greater area than the pole cores. This increased area is secured by means of pole shoes bolted or dovetailed to the core in the case of solid poles or by means of projecting tips or horns punched integrally with the sheets composing a laminated pole. The pole faces or shoes are almost always laminated, even when solid poles are used, in order to reduce the loss and heating due to eddy currents set up in the pole faces by the armature teeth; for, as shown in Fig. 24, the flux passing between the pole face and armature core tends to tuft opposite the teeth, and as the teeth move across the pole face these tufts are drawn tangentially in the direction of rotation until the increasing tension along the lines of force causes them to drop back to the next following tooth. The tufts of flux are therefore continuously swaying back and forth, and if the pole face is considered as built up of thin filaments, as at *P* in Fig. 24, each of the filaments will be cut by these swaying tufts first in one direction, then in the other, an alternating e.m.f. being thus induced, directed parallel to the shaft. To minimize the flow of current the pole face must be laminated in planes parallel to those of the armature laminations, though the laminae of the pole shoes do not have to be made as thin as those of the armature core. These pole-face eddy-current losses will obviously be reduced by so proportioning the dimensions of teeth and slots as to prevent appreciable lack of uniformity in the distribution of the flux along the pole face. The determining factors in this proportioning are the ratio of slot opening to airgap and the length of the airgap itself.

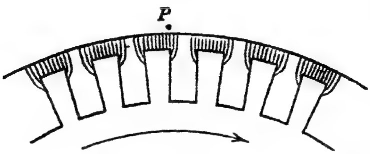


FIG. 24.—Tufting of flux at tips of teeth

16. Yoke.—The yoke is that part of the field structure which carries the flux from pole to pole and at the same time serves as a mechanical support for the pole cores. It is made of cast iron in small machines and of cast or rolled steel in larger sizes, or whenever saving in weight is important. In some machines made by the Westinghouse Electric and Manufacturing Com-

pany, the yoke is made from flat steel slabs which are first heated and then rolled into circular form, the butt joint between the two free ends being placed on the axis of one of the pole cores so that the break may not introduce additional reluctance into one of the magnetic circuits. In machines of moderate or large size the yoke is usually split on a horizontal diameter for convenience in assembling and repairing. In machines of moderate size the yoke is cast as an integral part of the bedplate; in larger sizes it is cast separately, but with lugs for bolting to the bedplate.

17. Brushes, Brush Holders, and Rocker Ring.—The connection between the revolving armature and the external circuit is made through the *brushes*, which are usually made of graphitic carbon, except in low-voltage machines where they may consist of copper or copper gauze. In automobile lighting generators and starting motors the brushes are generally made of a mixture of carbon and metallic copper. Carbon brushes are made of varying degrees of hardness to suit the requirements of commutation, as discussed in Chap. XII. The graphite in the brush serves to lubricate the commutator, which, when fitted with brushes of the proper composition, takes on a polished surface of dark-brown color. The width of the brush in the tangential direction is generally three to five times the width of a commutator segment, so that several armature coils are simultaneously short-circuited. The carbon brush must have sufficient resistance to limit the current in the short-circuited coils to a value below that which would result in sparking when the short-circuits are opened.

The brushes are commonly set at a trailing angle with respect to the direction of rotation, though in machines designed to run in both directions, such as railway motors, they are set radially.

When the tangential width of the brush has been decided upon, the total axial length of the brushes constituting a set is determined by the consideration that there must be a contact area of 1 sq. in. for every 30 to 50 amp. to be carried by the brush set, though this current density may be exceeded in the case of commutating-pole machines. Each individual brush of a set must not be too large in cross-section, or otherwise there would be difficulty in making and maintaining good contact over its entire contact surface. The subdivision of the set offers the additional advantage of allowing the individual brushes to be

trimmed one at a time without interfering with the operation of the machine when under load. Single brushes are used only in machines of small current output.

The individual brushes are supported in metal *brush holders* (Fig. 25), which are in turn supported by studs attached to, but insulated from, the *rocker ring*, as illustrated in Fig. 26. The brush holders serve as guides for the brushes and should allow the brush to slide freely in order that the brush may follow irregularities in the commutator surface. The construction of the brush holders must be such that there will be no vibration of the brushes, this being a common cause of sparking. The brushes are held against the commutator surface by adjustable springs attached to the holder, but in such manner that the springs do not carry any current. The tension of the springs is adjusted until the brush presses against the commutator with a force of 1.5 to 2 lb. per sq. in. of contact area. Increasing the brush pressure above this limit does not materially lower the contact resistance but increases the sliding friction and therefore results in increasing loss of power and heating of the commutator. The connection between the brush and brush holder is made through a flexible lead of braided copper wire, called a *pigtail*, or *shunt*, which is attached to the outer end of the brush by means of a metal band clamped tightly around the carbon. The carbon is generally copper-plated at its outer end to ensure good contact.

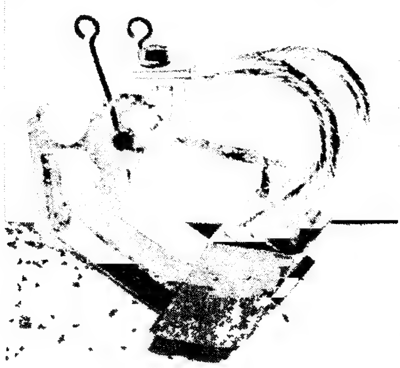


FIG. 25.—Brush holder for split brush. (*Westinghouse Electric and Manufacturing Company.*)

18. Motor Generator. Dynamotor.—It is frequently necessary to convert direct current at one voltage into direct current at some other voltage, higher or lower than the first. For this purpose a *motor generator* is used. As ordinarily constructed, a motor-generator set consists of two separate machines, a motor and a generator, directly connected to each other and mounted on a common bedplate, as illustrated in Fig. 26. Motor genera-

tors are also used to convert direct current into alternating current, or vice versa. This type of machine has the advantage that the voltage of the generator may be controlled independently of that of the motor of the outfit. The over-all efficiency of the set is equal to the product of the efficiencies of the motor and generator. The power rating of the motor must in general be sufficiently greater than that of the generator to allow for the losses that occur in the double transformation of the energy.

FIG. 26.—Motor-generator set, showing rocker ring. (*Westinghouse Electric and Manufacturing Company.*)

Instead of using two separate machines, as in a motor-generator set, to convert the current from one voltage to another, it is possible to combine the two into a single unit, called a *dynamotor*, having a single field structure and a single armature core. By providing the armature of such a machine with two separate windings, each with its own commutator and brushes, current may be introduced into one of the windings, thereby causing motor action, while the other winding will then generate an e.m.f. This type of machine is built in small and moderate sizes only. It is open to the objection that the voltage at the generator terminals cannot be independently regulated but is fixed by the voltage impressed upon the motor terminals. The truth of this statement can be seen from the following reasoning: If the voltage impressed upon the motor terminals is V_m, the rotation of the

armature through the field flux Φ will generate in the motor-armature winding an approximately equal and opposite e.m.f.; if there were no losses in the motor, this counter e.m.f. would be equal to V_m, and hence, by Eq. (7),

$$V_m = \frac{p\Phi Z_m n}{a_m \times 60 \times 10^8}$$

Since the generator winding rotates through the same field as the motor winding and at the same speed, the generator e.m.f. is

$$E_g = \frac{p}{a_g} \frac{\Phi Z_g n}{60 \times 10^8}$$

or

$$\frac{E_g}{V_m} = \frac{a_m}{a_g} \cdot \frac{Z_g}{Z_m} = \text{constant} \tag{18}$$

The disadvantage of the fixed ratio of voltage transformation is offset by the reduced cost of construction made possible by the single armature and field structure. The dynamotor has in addition a higher efficiency than a motor generator and is practically free from trouble due to armature reaction.

Examples of the use of dynamotors are found in the 3000-volt electrification of the Delaware, Lackawanna and Western Railway, where the motor of a motor-generator set, in addition to driving a 40-volt generator for the lighting circuits, supplies 1500 volts for operating the air compressor.*

19. Turbogenerators.—Generators for direct connection to steam turbines must be designed for high speed of rotation since the steam turbine develops its maximum efficiency under this condition. The high rotative speed calls for special design to withstand the centrifugal forces and to provide satisfactory commutation. The end connections of the armature winding are held in place by metal end shells in place of the usual banding wires, and the commutator segments are prevented from springing by a steel ring or rings shrunk over them. To provide for satisfactory commutation these machines are provided with *commutating poles*, or *interpoles* (Art. 20), the function of which is to generate in the coils undergoing commutation an e.m.f. of the proper magnitude and direction to reverse the current in the

* J. C. AXTELL, Auxiliaries for High-voltage Direct-current Multiple-unit Cars, *Jour. A.I.E.E.*, June, 1930, p. 523.

short time required for the segments to pass across the brush. Figure 27 represents a turbogenerator set manufactured by the General Electric Company. Figure 28 shows the steel ring around the commutator of a high-speed machine.

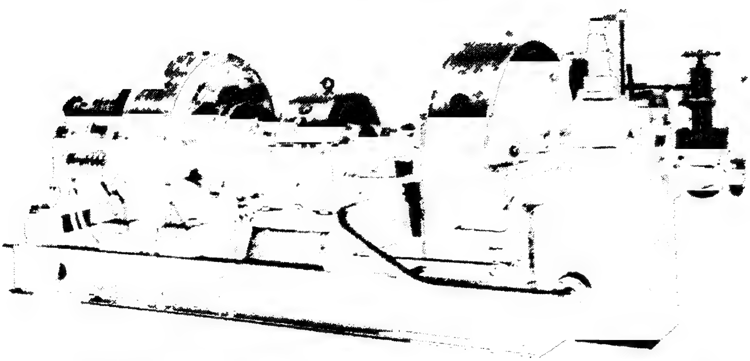


FIG. 27.—Geared turbogenerator set. (*General Electric Company.*)

Turbogenerators require a high grade of brushes to insure satisfactory commutation. The brushes wear down quite rapidly and must be adjusted with great care.

20. Commutating-pole Machines.—A full discussion of the function of commutating poles must be deferred to a later chap-

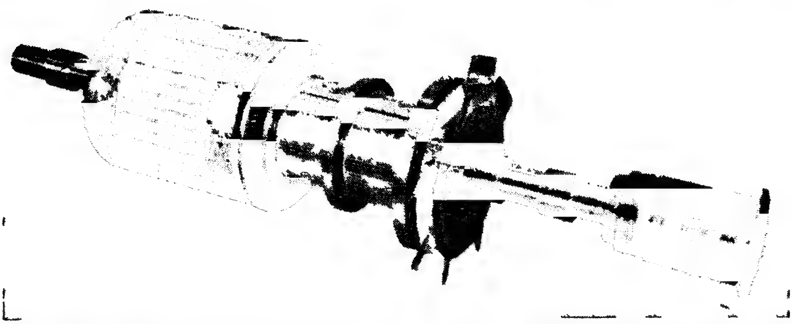


FIG. 28.—Commutator of a high-speed generator. (*Westinghouse Electric and Manufacturing Company.*)

ter. The commutating poles are small poles placed midway between the main poles; they are wound with coils through which the armature current, or a fractional part thereof, is made to flow. Commutating poles are used in machines where satisfactory commutation would otherwise be difficult or impossible of attainment,

as in turbogenerators and adjustable-speed motors having a wide range of speed.

21. Unipolar, or Homopolar, Machine.—In the armatures described in the preceding sections, the individual coils have generated in them alternating e.m.fs. that are rectified by the commutator, which plays much the same part as the valves of a double-acting reciprocating pump. In the centrifugal pump, on the other hand, the developed pressure acts continuously in one direction, the necessity for the rectifying valves being thus obviated, and the electrical analogue of the centrifugal pump is found in the so-called *unipolar*, or *homopolar*, or *acyclic* generator

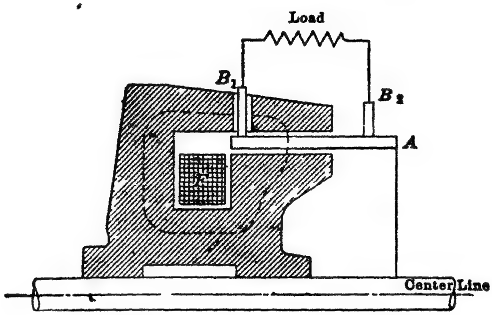


FIG. 29.—Homopolar or acyclic generator.

shown in section in Fig. 29. In principle, this machine consists of a conductor so disposed in a magnetic field that the cutting of the lines of force is continuously in one direction; it is a true continuous-current machine. The armature consists of a metal cylinder A of low resistance, insulated from the shaft and upon the edges of which the two sets of brushes B_1 and B_2 make sliding contact. The armature rotates in a magnetic field produced by the exciting winding F, the path of the flux being indicated by the dashed line. The lines of force pass radially across the airgap all around its periphery.

If the intensity of the magnetic field in the airgap is B lines per sq. cm., the axial length of the active part of the cylinder l cm., and its peripheral velocity v cm. per sec., the generated e.m.f. is $E = Blv \times 10^{-8}$ volt. The maximum e.m.f. obtainable with this type of machine is determined mainly by the consideration that B and v may not exceed definite limits; the length l is likewise limited by such mechanical features as rigidity and freedom from vibration. At the high rate of rotation required

for any reasonable value of e.m.f., difficulty is experienced in securing good brush contact. Thus, if $B = 15{,}500$ (100,000 lines per sq. in.), $l = 60$ cm. (about 2 ft.), and $v = 5000$ cm. per sec. (about 10,000 ft. per min.), $E = 46.5$ volts. Because of the fact that the armature consists of a single conductor of large cross-section, the machine is adapted for heavy currents at relatively low voltage. Unfortunately, however, the magnetizing action of the large armature current when the machine is under load so weakens the field produced by the exciting coil F that the voltage drops considerably.

The analogy between the homopolar machine and the centrifugal pump suggests the idea that, just as high pump pressures may be obtained by using several stages, higher voltages may be obtained in homopolar machines by using several inductors in series. Such a machine has been built by the General Electric Company* for 300 kw. at 500 volts and 3000 r.p.m.; and the Westinghouse Electric and Manufacturing Company† has built one for 2000 kw. and 260 volts, running at 1200 r.p.m.

22. Field Excitation of Dynamos.—In every dynamo-electric machine the generation of the armature e.m.f. depends upon the motion of the armature inductors through a magnetic field. In the earliest types of machines this magnetic field was produced by permanent magnets; such machines are called magneto-electric machines or, briefly, magnetos. Their use is now confined to small machines intended for ringing call bells in small telephone systems, for gas-engine igniters and for testing purposes. The field excitation of all other generators and motors is accomplished by means of electromagnets. The following types of field excitation may be recognized:

$$\begin{array}{l} \text{Separate excitation.} \\ \text{Self-excitation}\left\{\begin{array}{l}\text{Series excitation.}\\ \text{Shunt excitation.}\\ \text{Compound excitation.}\end{array}\right. \end{array}$$

23. Separate Excitation.—In this type of field excitation the field winding is traversed by a current supplied from a source external to the machine itself, such as a storage battery or another generator. The most prominent examples of this type

* J. E. NOEGGERATH, *Trans. A.I.E.E.*, **24**, p. 1, 1905.

† B. G. LAMME, *Trans. A.I.E.E.*, **31** (Part II), p. 1811, 1912.

are a-c generators and certain kinds of low-voltage, d-c generators used for electroplating. Figure 30 represents diagrammatically the connections of such a machine.

24. Self-excitation.—The use of electromagnets, separately excited for the production of the magnetic field, was introduced by Wilde in 1862. A great improvement was made in 1867 when Ernst Werner von Siemens discovered the principle of self-excitation, whereby the armature current, in whole or in part, was made to traverse the field winding, the machine thus being caused to develop its own magnetic field. Self-excited machines may be divided into three classes, depending upon the connections of the field winding to the other parts of the circuit; these classes are *series* excitation, *shunt* excitation, and *compound* excitation.

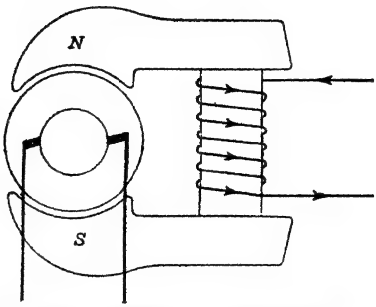


FIG. 30.—Diagram of separately excited machine.

25. Series Excitation.—In Fig. 31, *A* represents the armature and *N*, *S*, *N*, *S*, the field structure of a four-pole machine. All the current taken by the external circuit passes through the field winding and the armature, since all these parts of the circuit are in series. The arrows indicate the direction of the current in the case of generator action and for clockwise direction of rotation of the armature.

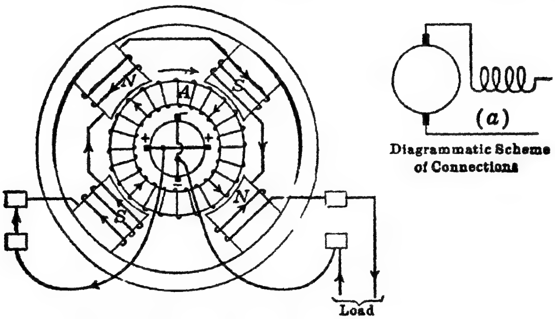


FIG. 31.—Connections of series generator.

If the field structure of such a machine is originally unmagnetized, rotation of the armature cannot generate e.m.f. and

hence there can be no current in the circuit. In order that the machine may self-excite, it is necessary that there be some residual magnetism in the field poles due to previous operation or, in the case of a new machine, produced by sending current through the field winding from some suitable external source. On the assumption then, that residual magnetism is present, a small e.m.f. will be generated when the armature is rotated, and, upon closing the external circuit through the load, a small current will flow. This current will further excite the field structure, more e.m.f. being thus developed and a still greater current, and so on. This gradual increase of both e.m.f. and current will continue until a condition of equilibrium is reached, this being determined by the degree of saturation of the field magnet and by the resistance of the circuit, in a manner that will be discussed in Chap. X.

It is important to note, however, that if the field terminals are reversed the machine will refuse to "build up" as described above. For in this case the generated e.m.f. will send a current through the circuit in such a direction as to neutralize the remanent magnetism. Further, if the resistance of the circuit exceeds a critical value, the resultant flow of current may be insufficient to produce the requisite magnetizing force.

From the foregoing description of the process of building up of a series generator, it is obvious that such a machine when running on open circuit (the receiving circuit disconnected) will develop only the small e.m.f. caused by residual magnetism; and that with increasing current, as the external resistance is lowered, the generated e.m.f. likewise increases, though not, in general, proportionally.

The field winding of series machines consists of relatively few turns of coarse wire. Since the entire current I delivered by the machine to the receiver circuit must flow through the resistance R_f of the field winding, there occurs a loss of power equal to I^2R_f watts in this part of the circuit. This loss must be kept as small as possible in order that the efficiency of the dynamo may not be seriously impaired; and since the magnitude of the current I is fixed by considerations of the load to be supplied, it follows that R_f must be kept as small as possible. Hence the conclusion that the wire of the field winding must have large cross-section and moderate length.

Another way of looking at the matter is as follows: The definite choice of the cross-section of the conductors comprising the field winding depends upon the factors of Eq. (26) (page 308), one of these factors being the number of ampere-turns per pair of poles (or per pole) required to produce the magnetic flux necessary to develop the desired e.m.f.; this number of ampere-turns depends upon the magnitude of the flux, as well as upon the dimensions and materials of the magnetic circuit, in the manner treated in detail in Chap. IX. If, then, a given armature and field frame are to be assembled to produce a specified voltage, the number of

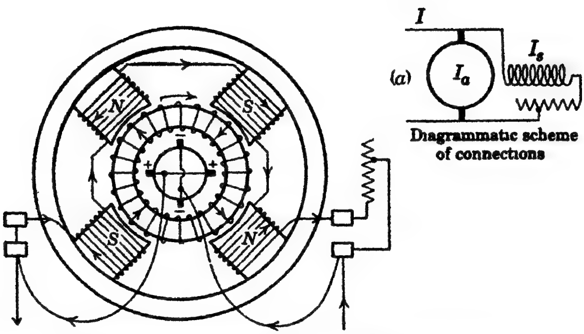


FIG. 32.—Connections of a shunt generator.

field ampere-turns will be the same no matter what type of excitation is to be used; from which it follows, in accordance with Eq. (26), that the cross-section of the wire of the field winding must be in inverse proportion to the drop of potential through that winding. Since this drop must be small in the case of series windings in order to minimize the loss of power, the cross-section must be correspondingly large.

26. Shunt Excitation.—Figure 32 shows the same armature and field frame as Fig. 31, but provided with a shunt field winding. Figure 32*a* represents the connections in a simple diagrammatic manner. It is evident that the exciting current now depends upon the difference of potential at the brushes and upon the ohmic resistance of the field winding; it is not dependent upon the resistance of the receiver circuit in the same sense as in the previous case, but only to the extent that variations of the external resistance affect the terminal voltage. If the external circuit is entirely disconnected, the remaining connection between

armature, shunt winding, and field-regulating rheostat is precisely the same as that of a series generator. On open circuit, therefore, a shunt generator will build up just as a series generator does under load conditions; if it fails to do so, it is usually because of one or the other of the reasons discussed in the preceding section.

It is seen, therefore, that a shunt generator, unlike the series generator, develops full terminal voltage on open circuit, that is, when no current is being supplied to the receiver circuit. Suppose, now, that the external circuit is closed through a considerable resistance so that a small load current I is drawn from the generator. The armature current, which was originally equal to I_s alone, now becomes $(I + I_s)$, and the effect of this increased current through the ohmic resistance of the armature is to cause a drop of terminal voltage; this in turn results in a decrease of the exciting current I_s, and consequently also of the magnetic flux and the generated e.m.f. As the load current becomes greater and greater, the terminal voltage therefore becomes less and less. It is readily seen that the drop of voltage will be minimized if the resistance of the armature is kept low. The drop of voltage under load conditions is also affected by armature reaction and by the degree of saturation of the magnetic circuit. A complete discussion of these effects is given in Chap. X.

The field winding of a shunt machine consists of numerous turns of fine wire, for the following reason: If the terminal voltage of the machine is V volts, the shunt field current I_s will be V/R_s, and the power loss in the winding will be $I_s^2R_s = V^2/R_s$; since V is fixed by other considerations, it follows that R_s must be as large as is practicable in order to keep down I_s and the loss of power, and hence it is necessary to use wire of small cross-section and considerable length.

In Eq. (26), it will be seen that the cross-section of the field winding must be inversely proportional to the voltage consumed in the field winding; and as this is of the order of the full terminal voltage in the case of a shunt machine, it is easy to see why the field winding must be made of much smaller wire than in the case of a series winding, where the drop through the field winding is only a small percentage of the terminal voltage.

The relation between the armature current I_a, the line current I, and the shunt field current I_s, in the case of *generator* action, is

given by

$$I_a = I + I_s \tag{19}$$

In the case of *motor* action the relation is

$$I = I_a + I_s \tag{20}$$

It should be remembered that the armature and field currents of a shunt motor do not divide in the inverse ratio of their respective resistances, for the reason that the armature, when running, is the

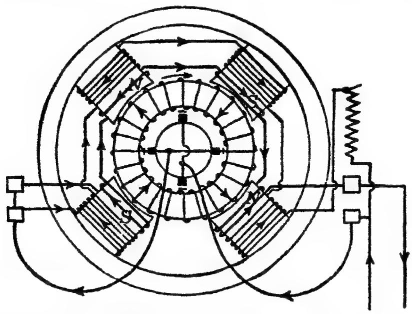


FIG. 33.—Connections of a compound generator.

seat of a counter-generated e.m.f. The field current is given by $I_s = V/R_s$, but the armature current is $I_a = (V - E_a)/R_a$, where E_a is the counter e.m.f.

27. Compound Excitation.—In some of the most important applications of d-c generators it is necessary to maintain a constant difference of potential between the supply mains no matter what the load may be. Since the center of the load is usually at

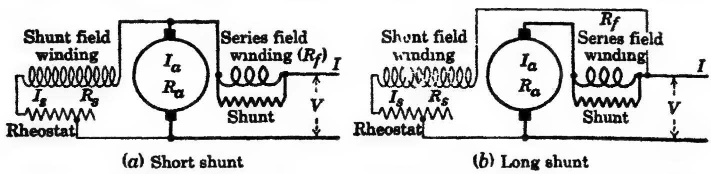


FIG. 34.—Diagrammatic scheme of connections of compound machines.

a distance from the generator, it follows that the potential difference between the generator terminals should rise as the external current increases, in order to compensate for the drop of potential in the supply mains. Field windings adapted to give this characteristic are called *compound* windings, illustrated in Fig. 33 and in diagrammatic form in Fig. 34. They are combinations of shunt and series field windings. Connections made in accordance

with Fig. 34a result in a *short-shunt* winding, and those of Fig. 34b in a *long-shunt* winding. The shunt winding of itself would produce a "drooping" characteristic, that is, one in which the terminal voltage falls with increasing current, as explained in the preceding section; but the series winding contributes field excitation which increases with increasing current, and hence the resultant effect will depend upon the relative magnitudes and directions of the magnetizing actions of the two field windings. By properly proportioning them, the voltage-current curve may rise, in which case the machine is said to be *over-compounded;* or the voltage may remain very nearly constant for all permissible values of current, as in a *flat-compounded* machine; or it may fall at a greater or lesser rate than with the shunt winding alone, in which case the machine is *under-compounded.*

In the short-shunt, compound-wound generator the relation between armature current I_a, line current I, and shunt-field current I_s is

$$I_a = I + I_s$$

The terminal voltage V and the generated e.m.f. E_a are related by the equation

$$E_a = V + IR_f + I_aR_a \tag{21}$$

and the shunt-field current is given by

$$I_s = \frac{E_a - I_aR_a}{R_s} = \frac{V + IR_f}{R_s} \tag{22}$$

In the long-shunt, compound-wound generator these relations become

$$I_a = I + I_s$$
$$E_a = V + I_aR_f + I_aR_a \tag{23}$$
$$I_s = \frac{V}{R_s} \tag{24}$$

28. Construction of Field Windings.—In designing the field windings of shunt, series, and compound machines, the selection of the correct number of turns and the cross-section of the conductors follows from a knowledge of the number of ampere-turns per pole required to produce the flux Φ and from the dimensions of the pole core. The calculation of these quantities depends upon principles that are discussed in detail in Chap. IX. On

the assumption that the number of ampere-turns per pole and the dimensions of the pole core are known, the determination of the size of wire to be used in the shunt field winding is as follows:

Let I_s = current in shunt winding.

V = terminal voltage at no load.

V_r = voltage consumed in regulating rheostat, varying from 10 to 20 per cent of V.

The purpose of the field rheostat is to permit an increase of I_s by cutting out a part or all of the regulating resistance, thereby raising the generated e.m.f.

The resistance of the winding per pole is

$$R'_s = \frac{V - V_r}{I_s p} = \rho \frac{\frac{1}{2} N_s L_t}{A} \tag{25}$$

where ρ = specific resistance of copper at the working temperature of the winding (about 75°C.).

N_s = number of shunt turns per pair of poles.

L_t = mean length of a turn.

A = area of cross-section of conductor.

If lengths are expressed in feet and cross-sections in circular mils, ρ = 12.6 at 75°C. Hence,

$$A = \frac{6.3(N_s I_s) L_t p}{V - V_r} \text{ cir. mils} \tag{26}$$

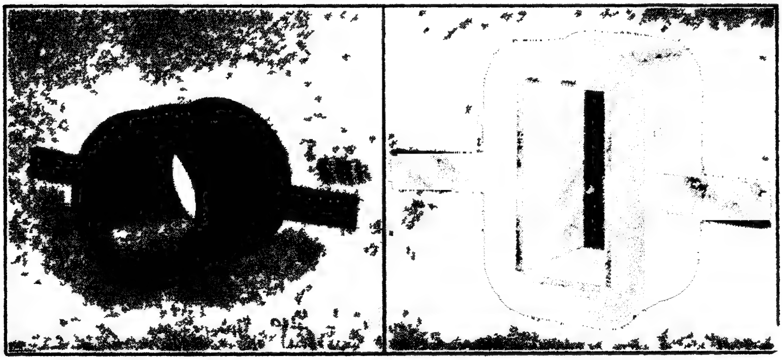
In general, without regard to whether the field winding is to be of the series or shunt type, the relation between the size of wire and the other determining factors is given by

$$A = \frac{6.3 \times \textit{amp.-turns per pair of poles} \times \textit{mean length of turn}}{\textit{drop of potential per pole}}$$

The mean length of a turn L_t is found by assuming a depth of winding of 1 to 3 in. If the cross-section of the pole core is rectangular, L_t will be approximately equal to the perimeter of the core plus four times the winding depth; if the pole core is circular, of diameter d_c, $L_t = \pi(d_c + \textit{winding depth})$. The winding depth must not exceed a definite limit; otherwise the heat generated in the interior of the core cannot readily be conducted to the surface. As a check on the calculations, it must be ascertained that the power lost in the coil ($I_s^2 R'_s$) does not

exceed approximately $\frac{2}{3}$ watt per sq. in. of exposed radiating surface, with ordinary conditions of ventilation.

Shunt coils are usually made of cotton-covered wire, of either round or rectangular section. Sometimes they are wound on



(a) (b)

Fig 35 —Ventilated field coils

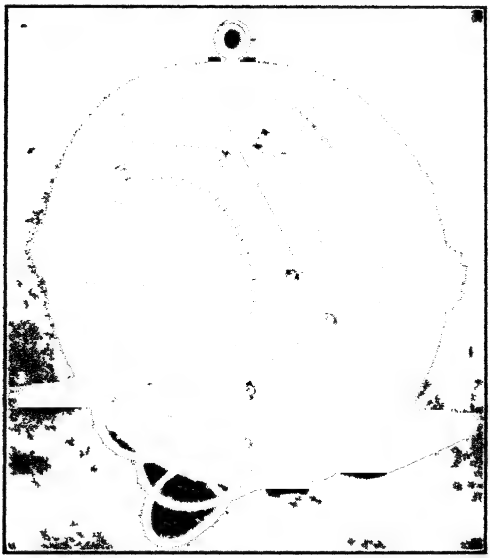


Fig. 36.—Interpole machine, edge-wound copper-strap coils (*Westinghouse Electric and Manufacturing Company*)

metal frames arranged to slip onto the pole cores; sometimes they are wound on removable winding forms, the coils being held in shape by suitable insulating materials and dipped in, or painted with, moisture-proof varnish. When metal frames are used, they

are frequently made with a double wall to allow the circulation of air between pole core and winding, as shown in Fig. 35. The coils of series-wound railway motors are usually impregnated with insulating compound, then taped and varnished. The series coils of compound and commutating-pole machines are often made of copper strap, wound on edge, the turns being separated by distance pieces of insulating material, as shown in Fig. 36.

In order that connections may be made easily between the coils of adjoining poles, the terminals of the coils are brought out on opposite sides, so that the number of turns per coil is an integer plus one-half.

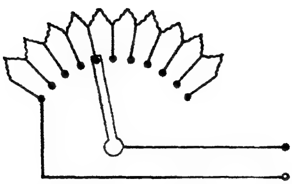


FIG. 37.—Diagram of connections of field rheostat.

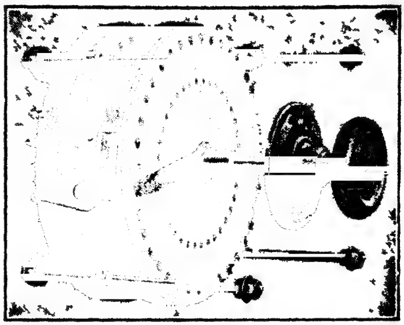


FIG 38.—Field rheostat, back connection (General Electric Company)

29. Field Rheostats.—To permit regulation of the voltage of shunt and compound generators, the current in the shunt field winding must be under control. To this end a variable resistance, or *field rheostat*, is inserted in series with the shunt winding, as shown diagrammatically in Figs. 32 and 33. This resistance is arranged in the manner shown in Fig. 37, taps being brought out from the high-resistance wire or ribbon composing the resistor to a series of insulated studs over which moves an adjustable contact arm. The terminals are always brought out in such a way that clockwise rotation of the regulating handle increases the resistance in circuit and so throttles the current in the manner of an ordinary valve.

Field rheostats are generally arranged to be mounted on the back of the switchboard, with the regulating handle on the front of the board, as in Fig. 38. Field rheostats for machines of large capacity are made of cast-iron grids, as shown in Fig. 39.

In shunt and compound generators of large capacity the energy stored in the magnetic field is very considerable, amounting to $\frac{1}{2}L_sI_s^2$, where L_s is the inductance of the shunt winding. The inductance may have a value of several hundred henrys. For instance, if $L_s = 600$ and $I_s = 4$, the energy stored in the field is

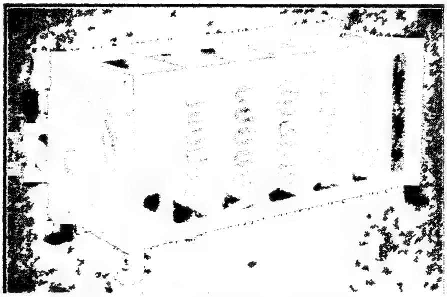


FIG 39—Large field rheostat

4800 joules. If the field circuit is abruptly broken, this energy will have to be dissipated in the arc formed on breaking the circuit; if, for example, the current were made to disappear in ½ sec., the average rate of energy dissipation would be 4800 ÷ ½ = 9600 watts, and the average voltage induced by the collapse

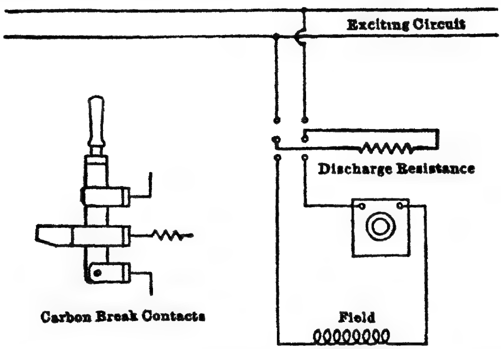


FIG 40—Diagram of connections of field discharge resistance.

of the magnetic field would be $L_s(dI_s/dt) = 600 \times 8 = 4800$ volts. In this case the arc would be destructive, and the high induced voltage would be likely to puncture the insulation of the winding. To obviate this danger the field current must be gradually reduced before breaking the circuit. In large machines the reduction of field current is accomplished by allowing the field windings to discharge through a *field-discharge* resistance, connected in the manner shown in Fig. 40.

30. Polarity of Generators.—In order that a self-exciting generator of any of the types already described may be operative, it is necessary that there be some remanent magnetism in the field system and, further, that the initial flow of current through the exciting winding have such a direction that it will strengthen the remanent field. In other words, the polarity of the machine is determined by that of the remanent magnetism.

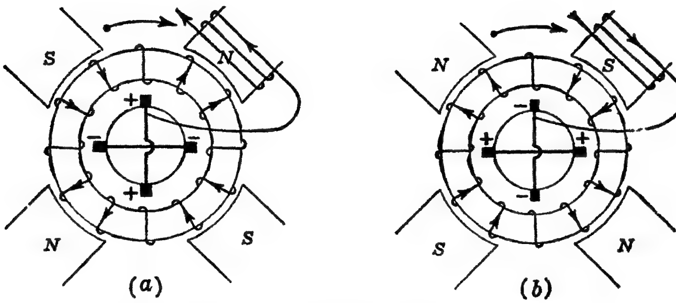


Fig. 41.—Effect of reversal of residual magnetism.

For example, consider the conditions existing in the two shunt-wound generators illustrated in Fig. 41. The machines are identical except that the remanent magnetism of the second is reversed with respect to that of the first. In each case the machine will build up if the direction of rotation is clockwise, but with the polarity of the terminals of the one opposite to the polarity of the other. With the connections shown, counter-clockwise rotation would set up a field current that would wipe out the remanent magnetism, but with counter-clockwise rotation the machines would again become self-exciting if the terminals of the field winding are interchanged.

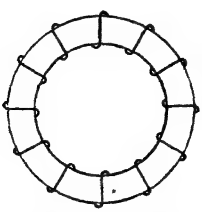


Fig. 42.—Left-handed ring-wound armature.

In both parts of Fig. 41 the armature winding is *right-handed*, that is, it is wound around the core in the manner of a right-handed screw thread. If *left-handed* armature windings had been used (Fig. 42), other conditions remaining the same, annulment of the remanent magnetism would again be the result. Finally, it is clear that the direction of the winding around the poles plays a similar role.

There are therefore four elements that affect the polarity of such a machine: the *sense* of the windings of armature and pole pieces, respectively; the direction of rotation; and the order of connections of the field winding terminals to the armature terminals. With a given remanent magnetism, the machine will operate only when there is a definite relation between them. On the assumption that the conditions for operation are satisfied, a change in any *one* of these four elements will cause the machine to counteract its residual magnetism, but a change in any *two* of them will not affect the operation. Thus, a right-handed armature rotating clockwise in a given field flux will yield the same brush polarity as a left-handed armature rotating counterclockwise in the same field. In general, a change in an *odd* number of the four elements will disturb conditions if they were previously correct, whereas a change in an *even* number of them will not affect the operation.

31. Direction of Rotation of Motors.—The same types of field windings and connections as are used for generators find equal application in the case of motors. Series motors, when supplied with constant terminal voltage, fall off rapidly in speed as the load increases or, to put it in another way, "race" as the load is removed; this characteristic of variable speed at constant terminal voltage is a sort of "mirror" image of the series-generator characteristic, namely, variable voltage at constant speed. The speed characteristic of the series motor adapts this machine to traction and hoisting service. The shunt motor, when supplied with constant terminal voltage, operates at practically constant speed at all loads, just as the shunt generator delivers a nearly constant terminal voltage (within limits of machine capacity) when driven at constant speed. Used as a motor, an over-compounded generator, if sufficiently compounded, will rise in speed with increasing load, if supplied from constant-potential mains (see Chap. XI), the inverse relationship between generators and motors being thus emphasized again, for in an over-compounded generator, operated at constant speed, the voltage rises as the load increases.

Let Fig. 43 represent diagrammatically a series machine used as a generator, the shaded half of the armature circle representing a belt of current flowing into the plane of the paper and the unshaded half representing current of opposite direction. If

this machine is now connected to mains of the polarity indicated in Fig. 44 and is operated as a motor, its direction of rotation will be reversed as may be seen by applying Fleming's left-hand rule. It follows that a series generator supplying a network fed by other generators may reverse its direction of rotation and so buckle the connecting rod of the driving engine. The

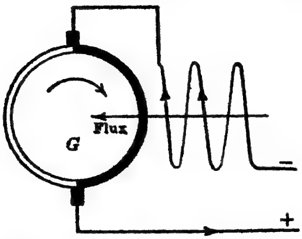


FIG. 43.—Diagrammatic sketch of series generator.

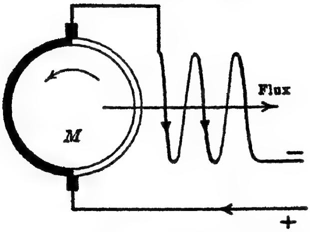


FIG. 44.—Diagrammatic sketch of series motor.

fundamental reason for the reversal of direction of rotation is that both armature current and field current reverse simultaneously, at the same time that the machine changes from generator to motor action. Considering these changes from the point of view of Fleming's right- and left-hand rules, it is seen that there have been *three* changes in all, namely: (1) a change in the direction of the armature current, represented by the middle finger; (2) a change in the direction of the magnetic field, represented by the forefinger; and (3) a change from generator to motor action, involving the use of the left hand instead of the right.

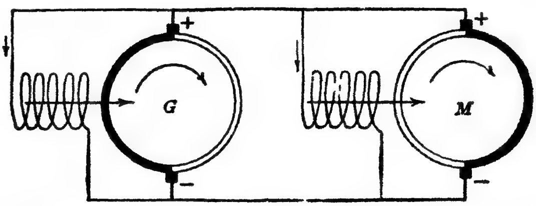


FIG. 45.—Showing direction of rotation of shunt generator and motor

To turn now to the case of two identical shunt machines, one used as a generator, the other as a motor, as in Fig. 45, the direction of rotation is found to be the same in both; consequently, a shunt generator supplying a network fed by other generators is not subject to reversal of its direction of rotation in case its prime mover is disconnected or shut down but will continue to run in the original direction, as a motor. If these

conditions are examined in the light of Fleming's rules, it is clear that there have been *two* changes: (1) reversal of the middle finger, indicating the direction of the armature current; (2) the change from right hand to left hand because of the transition from generator to motor action.

These considerations may be generalized by observing that there are four factors to be considered in applying Fleming's rules: the direction of the armature current, the direction of the magnetic field, the direction of rotation, and the nature of the operation of the machine as generator or as motor. A change in an *odd* number of any of these four factors will change one of the remaining factors, whereas a change in any *even* number of them will not change the remaining ones.

Thus, a reversal of the polarity of the mains supplying a motor, whether it be of the series or shunt type, will change two factors, the direction of the armature current and the direction of the magnetic field; hence, the direction of rotation will not be affected, and it is for this reason that both types will run as a-c motors. If a motor is to have its direction of rotation reversed, only one change must be made, either in the direction of the armature current or in that of the magnetic field, but not in both.

CHAPTER VII

ARMATURE WINDINGS

1. Types of Armature.—Armatures, considered as a whole, may be divided into three classes according to the shape of the core and the disposition of the winding upon it. These three classes are: (1) *ring armatures;* (2) *drum armatures;* (3) *disk armatures.*

The *ring armature* is one in which a ring-shaped core is wound with a number of coils, or elements, each of which winds in and out around the core in helical fashion, in the manner illustrated in Figs. 9, 10, and 11, Chap. VI. In these windings the coils are usually connected successively to each other so as to form a continuous circuit, the end of each element being connected to the beginning of the next adjacent element, but this particular feature is not essential to the definition. The characteristic feature of ring windings is that there are conducting wires inside the ring which do not cut the main flux in the airgap and are, therefore, inactive so far as concerns active generation of e.m.f. The main flux proceeds from pole to pole chiefly through the iron of the armature core, as indicated in Fig. 1. A few lines of force will leak across the inside of the ring core, though the number of such leakage lines is necessarily small because of the high reluctance of the air path; it should be noted, however, that the internal leakage flux, to whatever extent it exists, exerts a deleterious effect upon ring windings, for it produces an e.m.f. in the interior conductors in a direction that opposes the main e.m.f.

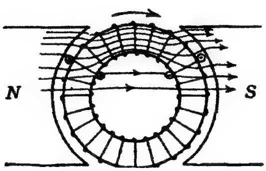


Fig. 1.—Effect of internal leakage flux, ring winding.

The drawings illustrating ring windings indicate smooth-core armatures merely for the sake of simplicity. It is to be understood that toothed cores may be used, if desired. In reality, ring windings are no longer used in commercial designs, and

the chief reason for discussing them is that a study of their properties contributes to a better understanding of the various types of drum-armature windings.

The *drum armature* differs from the ring armature in that no part of the winding threads through the core; the entire winding is external to the core, and the only reason for any opening in the core is to permit ventilation and cooling. The active wires of the winding are those lying on the cylindrical surface of the core, generally in open-type slots, and these are connected at the front and back ends* by means of *end connections* which are

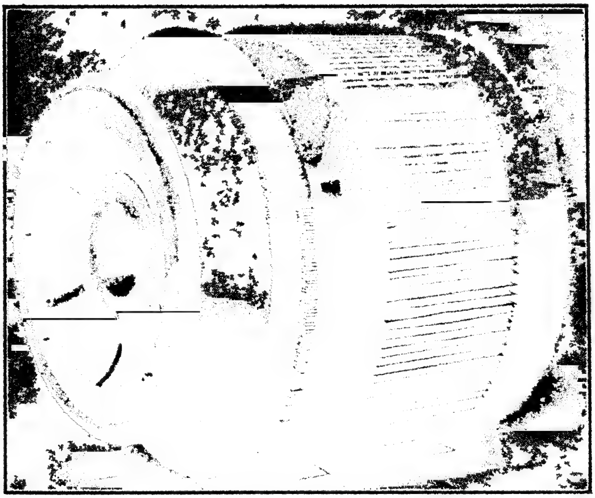


FIG 2.—Partly wound drum armature (lap winding).

supported on cylindrical extensions bolted to the core, as in Fig. 14, Chap. VI. The end connections are not separate and distinct from the active conductors but are merely extended portions thereof. Figure 2 shows a partly wound drum armature and illustrates how the end connections form a two-layer cylindrical arrangement; this particular illustration represents an armature for a multipolar machine, the end connections spanning an arc approximately equal to the pole pitch, in order that the e.m.fs. in the active conductors thus connected may be additive. In bipolar machines, which are used only in very small sizes, the end connections run across the flat ends of the core and join active conductors which are nearly diametrically opposite each

* The front end is the commutator end; the back end is the pulley end.

other. The drum armature may be thought of as evolved from the ring type by moving the inner connections of the ring winding elements to the outer surface and at the same time stretching the coil circumferentially until the spread of the coil is approximately a pole pitch.

The *disk armature* differs from the other two types in that the active conductors, instead of lying on the outer cylindrical surface of the core, are disposed radially on the flat sides of a disk. The disk revolves between a number of pairs of poles of opposite signs, so that the wires on both faces of the disk are active (see Fig. 20, Chap. VI). Disk armatures are obsolete so far as modern practice is concerned.

Of the three types of armature described above, the drum armature is used practically to the exclusion of the others. One of the reasons for its original development was the desire to increase the percentage of active wire in the winding as a whole, the active wire being that part of the winding which cuts the main flux and so contributes to the total generated e.m.f. But the principal advantage of the drum winding is that it eliminates the hand winding required in ring armatures and therefore reduces the cost of manufacture; also, since the coils are wholly outside the core, they may be wound on formers, or winding jigs, and can be thoroughly insulated before being slipped into place.

2. Types of Winding.—All armature windings, for both d-c and a-c machines, belong to one or the other of the two types, *open-coil* and *closed-coil* windings. An open-coil winding is one in which, by starting with any conductor and tracing progressively through the winding, a "dead end" is finally reached; whereas, in a closed-coil winding, the starting point will finally be reached after having passed through all, or some submultiple (one-half, one-third, one-fourth, etc.), of the conductors. The use of open-coil windings is at present confined to a-c machines and need not, therefore, be considered here. Open-coil windings were at one time used to a large extent in d-c series arc-lighting generators, such as the Brush and Thomson-Houston machines.*

3. Closed-coil Ring and Drum Windings.—In designing the armature of a generator or motor, it is necessary to fix the

* See S. P. THOMPSON, "Dynamo Electric Machinery."

number of armature conductors so as to satisfy the following conditions:

1. The numbers of armature conductors Z must be an even integer that satisfies the fundamental equation for the e.m.f., namely,

$$E = \frac{p}{a} \frac{\Phi Z n}{60 \times 10^8}$$

2. The order of connections must be such that the e.m.fs. of the individual conductors will add together to produce the desired total e.m.f.

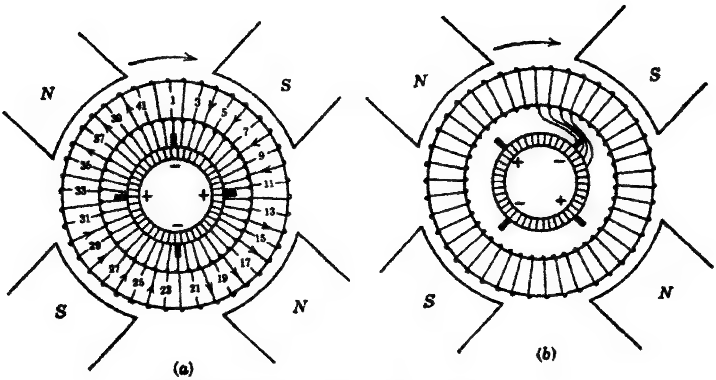


FIG. 3.—Ring winding. $Z = 42$, $p = 4$.

3. The resultant closed winding must be symmetrical, with respect to the brushes, in any position in which it is placed.

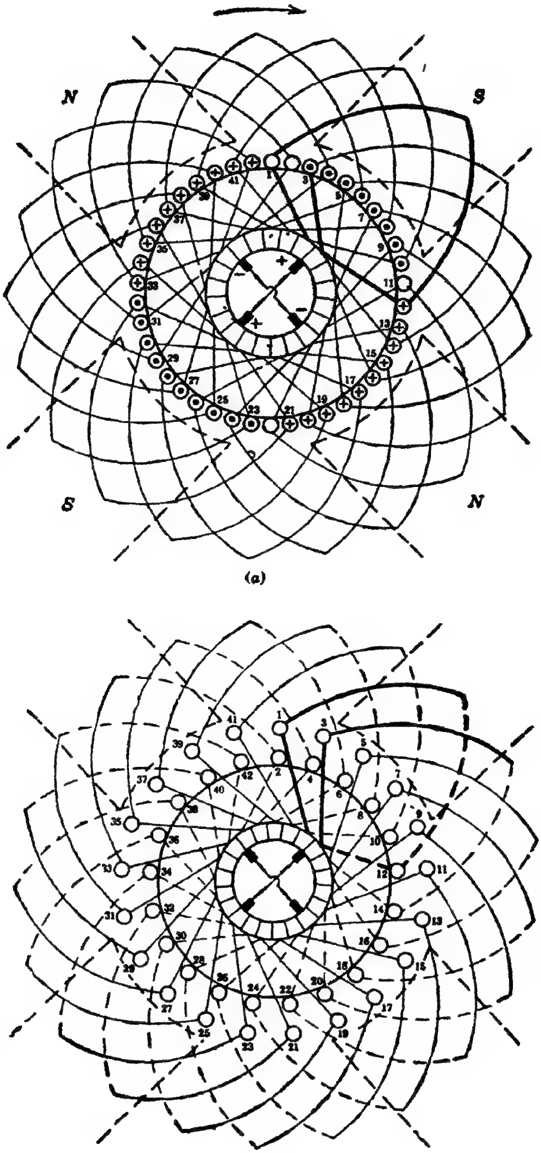
In Figs. 3, 4, and 5 there are shown three distinct types of closed-coil winding for a four-pole machine having 42 active conductors. The two parts of Fig. 3 represent a simple ring winding, and Figs. 4 and 5 represent drum windings. In these drawings the small circles represent the cross-sections of the active conductors lying on the cylindrical surface of the armature core; the end connections which serve to connect the active conductors at the back, or pulley, end of the armature are drawn for convenience outside the bounding surface of the core, although in reality these end connections lie on a cylindrical surface that forms an extension of the iron core and that has a diameter slightly less than that of the core, in the manner shown in Fig. 2; similarly, the front, or commutator, end connections

are shown in Figs. 4 and 5 inside the bounding surface of the core, though these, too, rest on a cylindrical surface. The arrangement of the windings shown in Figs. 4 and 5 is made clearer by the use of the developed diagrams of Figs. 6 and 7, which are derived from Figs. 4 and 5 by rolling out the cylindrical surface of the armature core into a plane.

It will be noted that in the ring winding of Fig. 3*a* the brushes occupy positions midway between the pole tips, whereas in the drum windings of Figs. 4 and 5 the brushes are placed almost directly under the middle of the pole faces. This difference is due solely to the shape and disposition of the connections leading to the commutator segments, as may be seen from Fig. 3*b*; this may be considered to be derived from Fig. 3*a* by turning the commutator and brushes through an angle of 45 deg. and at the same time stretching the connections from the winding to the commutator. In like manner, the brushes of drum windings may be made to occupy any position other than immediately opposite the centers of the pole faces by suitably shaping the end connections, as in Fig. 8, parts (*c*) and (*f*).

Part (*a*) of Figs. 4 and 5 is in each case diagrammatic only, for the sake of comparison with Fig. 3. If all the active conductors were in one layer, as drawn, the crossings of the end connections would introduce insurmountable physical difficulties; accordingly, the active conductors are arranged in two layers, as in Figs. 4*b* and 5*b*. In this way the end connections of each layer lie side by side without interference, and the two layers can be insulated from each other in the manner illustrated in Fig. 2. The transition from the upper to the lower layer is accomplished by forming the individual coils with the bend marked *B* in Fig. 9.

4. Winding Element.—It will be seen that in each case the winding consists of a number of identical *elements* which are shown in heavy lines in Figs. 4 to 7. An element may be defined as that portion of a winding, which, beginning at a commutator segment, ends at the next commutator segment encountered in tracing through the winding. It will be evident at once that an element may consist of more than one turn, i.e., of more than two active conductors; for instance, parts (*b*), (*c*), (*e*), (*f*), of Fig. 8 represent elements of windings similar to those of Figs. 6 and 7, but with three turns each, instead of one.



(a)

(b)

Fig. 4.—Drum winding (lap).

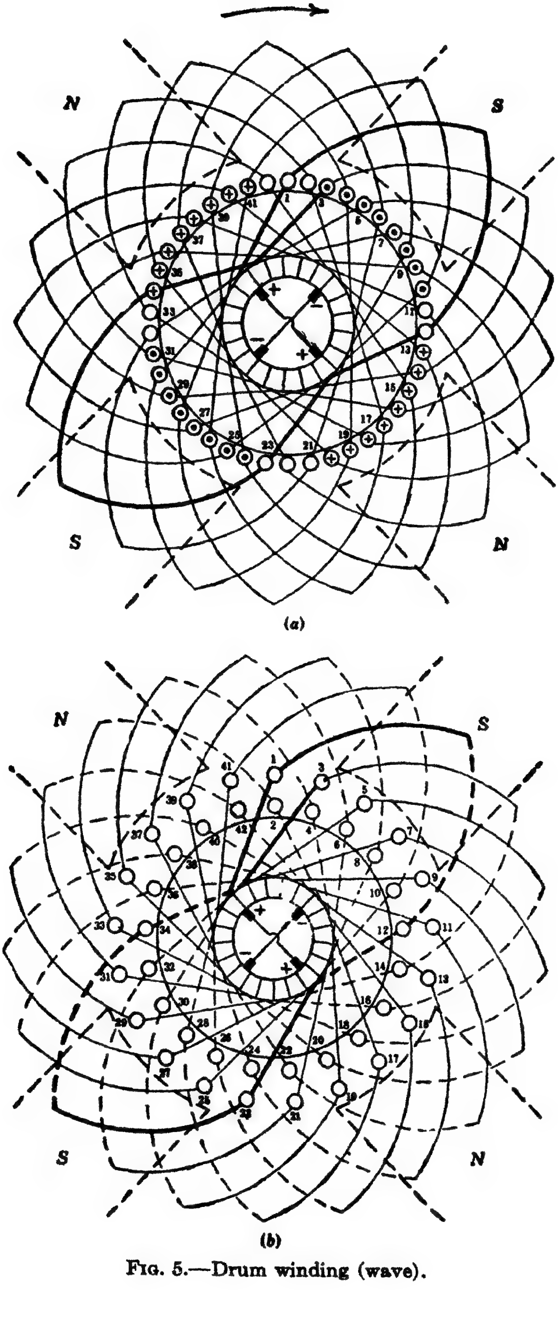


FIG. 5.—Drum winding (wave).

Small machines for relatively high voltage, such as radio generators and railway motors, frequently have numerous turns per element; but in machines of large capacity there is, as a rule, only one turn per element for the purpose of improving commutation. Every time an element passes through the neutral zone of the magnetic field the current that it has been carrying must be reversed in direction; hence, its self-inductance must be kept as small as possible in order that the reversal of the current may not be impeded, and as the coefficient of self-induction is proportional to the square of the number of turns, the number of turns should be a minimum, or unity, for best results.

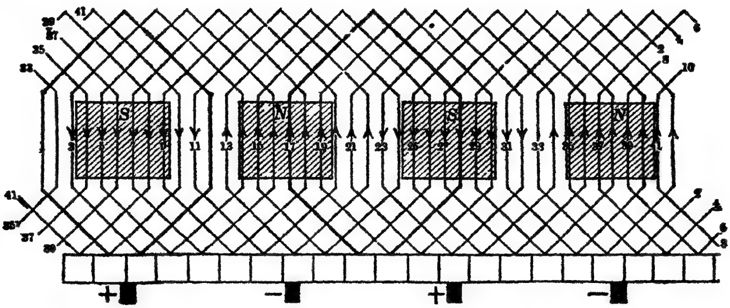


FIG. 6.—Developed lap winding.

5. Lap and Wave Windings.—The windings of Figs. 4 and 5 belong, respectively, to the *lap* and *wave* types of closed-coil winding. The derivation of these terms will be evident from an inspection of Figs. 6 and 7; in the former, the successive elements lap back over each other, whereas in the latter they progress continuously in wave fashion around the periphery of the armature.

In both lap and wave windings the two sides of a coil, or element, are subjected to the influence of adjacent poles of opposite polarity, so that the e.m.fs. generated on the two sides add together. In a simple lap winding, the end of any element, say the xth, connects to the beginning of the $(x + 1)$st element, and the beginning of the $(x + 1)$st element lies under the same pole as the beginning of the xth element; in a wave winding, however, although the end of the xth connects to the beginning of the $(x + 1)$st element, the latter is not under the same pole as the beginning of the xth element but is separated from it by a double pole pitch.

The study of the arrangement of the windings shown in Figs. 6 and 7 is facilitated by preparing winding tables, in the manner illustrated below. Thus, taking the lap-wound armature

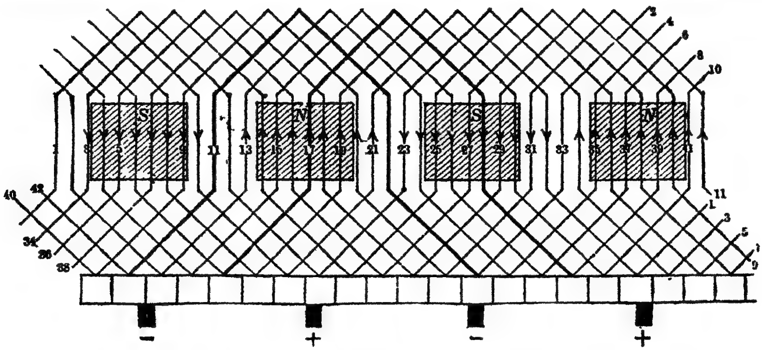


FIG. 7.—Developed wave winding.

first, the order of connections of the conductors is, starting with conductor 1, 1–12–3–14– . . . , or in tabular form,

1	12
3	14
5	16
7	18
9	20
11	22
13	24
15	26
17	28
19	30
21	32
23	34
25	36
27	38
29	40
31	42
33	2
35	4
37	6
39	8
41	10
1 (winding closes)	

The gradual advance, or creep, of the winding around the periphery, plainly evident in Fig. 6, may be emphasized by arranging the numbers of the table in accordance with the fol-

lowing plan, where the letters *S*, *N*, *S*, *N* are spaced apart at a distance representing the pole pitch, the letters themselves being at the center points of the pole faces.

S	*N*	*S*	*N*
1	12		
3	14		
5	16		
7	18		
9	20		
11	22		

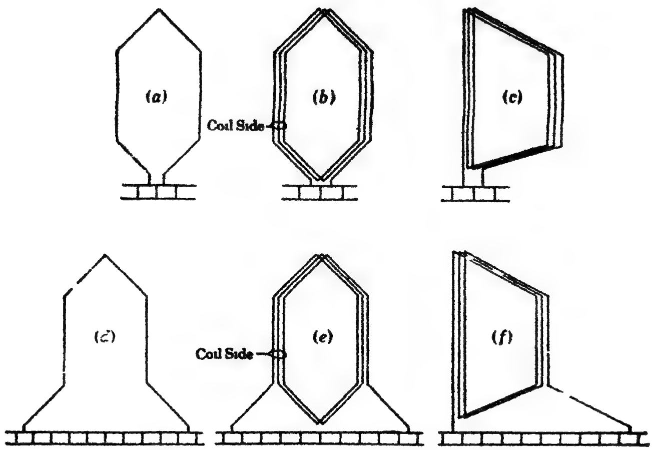


Fig. 8.—Types of winding elements.

Limitations of space prevent the completion of the entire table on the printed page.

Upon proceeding in the same way with the wave winding of Fig. 7, the winding table is

1	12	23	34
3	14	25	36
5	16	27	38
7	18	29	40
9	20	31	42
11	22	33	2
13	24	35	4
15	26	37	6
17	28	39	8
19	30	41	10
21	32	1 (winding closes)	

If this table is arranged like the one immediately preceding, so as to show the creep of the winding, it is in part as follows:

```
        S                  N                      S                   N
1                12                     23                    34
    3                14                     25                    36
       5                 16                     27                    38
          7                  18                     29                    40
            9                    20                     31                    42
               11                    22                     33
     4             13                    24                     35
                       15                    26                     37
        6                  17                    28                     39
                                                                        etc.
```

An examination of the directions of the current flow in Figs. 3, 6, and 7 shows that in the case of the first two diagrams there are four separate and distinct paths for the current through the winding ($a = 4$); each of these paths carries one-fourth of the entire current supplied to the external circuit in the case of generator action, or supplied from the line in the case of motor action. In Fig. 7, on the other hand, though there are four poles as in the other machines, there are only two paths through the winding ($a = 2$). Other things being equal, therefore, the wave winding shown in the diagram will generate twice the e.m.f. of either of the other two in accordance with the fundamental equation

$$E = \frac{p}{a} \frac{\Phi Z n}{60 \times 10^8}$$

or, what amounts to the same thing, the same e.m.f. will be generated in a wave winding with only half the number of conductors required in an equivalent ring or lap winding. Furthermore, the diagrams show that four brushes are required in the cases of the ring and lap windings, whereas two will suffice in the case of the wave winding, though all four may be used. These two facts explain the reason for the use of wave windings in d-c railway motors, where the combination of the cramped space and the moderately high voltage requires a minimum number of conductors; and, no less important, considerations of accessibility for inspection and repairs limit the number of brush sets to two.

Lap and wave windings are often referred to as *parallel* and *series* windings, respectively.

6. Number of Brush Sets Required.—Inasmuch as the current in an element must undergo commutation once for each passage of the element through a neutral zone, it follows that the element may be short-circuited by a brush at each such reversal.

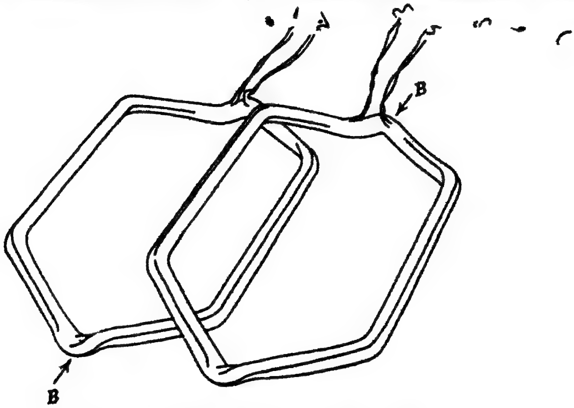


FIG. 9.—Samples of winding elements.

Since the number of neutral zones and consequent reversals is equal to the number of poles, *the number of permissible brush sets may in all cases be the same as the number of poles.* In lap windings and in simple ring windings of the type shown in Fig. 3, the number of brush sets *must* be equal to the number of poles. But in wave windings, though p brushes *may* be used, two

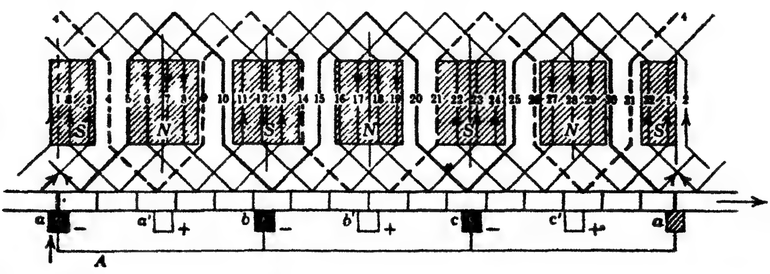


FIG. 10.—Six-pole wave winding, showing elements short-circuited by brush.

brushes will suffice irrespective of the number of poles. Thus, in Fig. 10, which shows a wave winding for a six-pole machine having 32 armature conductors, any two of the three negative brushes a, b, and c may be omitted, as, for example, b and c (provided that a corresponding pair of positive brushes is removed at the same time), in which case the remaining brush a will short-

circuit three elements in series when it is in contact with two adjacent commutator segments. The three elements thus short-circuited by brush a are shown in heavy lines; in the position shown in the figure, brush b', if it alone of the positive brushes were present, would short-circuit the three elements shown in dashed lines. Figure 10 also makes it clear why two brushes, instead of six, will suffice to collect the current, for it will be observed that brushes a, b, and c are connected not only by the external conductor A but also by the short-circuited elements shown in heavy lines; these elements are in the neutral zone, consequently have little or no e.m.f. generated in them, and are, therefore, equivalent to additional dead conductors joining the three brushes; hence, conductor A and any two of the brushes a, b, and c may be omitted. But if brushes b and c are retained,

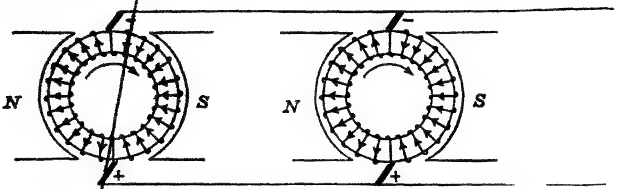


FIG. 11.—Armatures in parallel.

and similarly b' and c', brushes a, b, and c, which are connected to form one terminal of the machine, operate in pairs to short-circuit single elements. This circumstance reduces the e.m.f. of self-induction to one-third of the value that would otherwise have to be handled, commutation conditions being thus improved.

It should be noted, however, that if only two brushes are used in connection with a wave winding, there is no real saving of material since the two brushes must have the same current-carrying capacity as the full number would possess, and this requirement would necessitate an increased length of commutator.

7. Simplex and Multiplex Windings. Degree of Reentrancy. If two identical ring-wound machines are connected in parallel as indicated in Fig. 11, the combined current output will be double that of either machine separately. The same result may be attained, together with economy in the use of material, by placing two independent windings on the same armature core, subjected to the magnetizing action of a single field structure, as indicated in Fig. 12a. Here both the winding elements and the commutator segments of the independent windings are "sand-

wiched," or imbricated. The same result might also be secured by using two independent commutators, one at each end. Windings of the type of Fig. 12 are called *duplex* windings as distinguished from the *simplex* windings of Fig. 11 and those preceding it. Obviously, there is nothing to prevent the multiplication of independent windings so as to form *triplex*, *quadruplex*, . . . , windings, at least so far as the physical arrangement of the constituent windings is concerned.

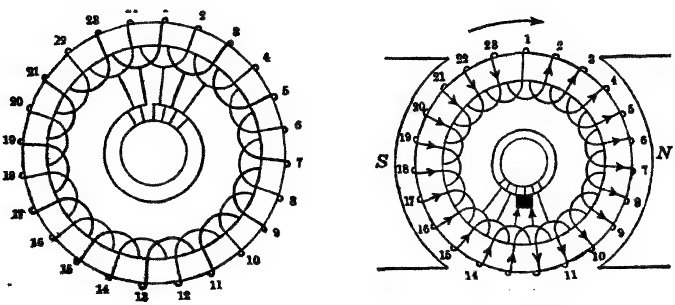
Drum windings, of both the lap and wave varieties, may be treated in the same way as has here been described for the case of ring windings, but the interleaving of the commutator segments of the component windings requires the use of brushes of sufficient width to collect the current from each set of circuits at a neutral point.

A multiplex winding is equivalent to two or more simplex windings in parallel with one another. Thus, a duplex winding is equivalent to two simplex windings in parallel, a triplex winding is equivalent to three simplex windings in parallel, a quadruplex to four, and so on. Multiplex lap and wave windings, except a relatively few duplex windings, have seldom been used, because of complications that are discussed in Art. 21 of this chapter.

It has been pointed out in connection with Fig. 3 that simple ring windings necessarily have as many armature circuits in parallel as there are poles; this feature characterizes simple lap windings of the kind illustrated in Fig. 4, as is easily understood when it is considered that the only difference between ring and lap windings is that in the latter the successive turns lie on the surface of the core instead of looping through it. But in wave windings, of the type shown in Figs. 7 and 10, there are only two paths through the armature irrespective of the number of poles. Because of these facts lap windings are often called *parallel* or *multiple* windings, and wave windings are called *series* or *two-circuit* windings.

From what has been said above, it follows that a duplex lap winding has $2p$ armature circuits in parallel, a triplex lap has $3p$ parallel circuits, and an x-plex winding has xp parallel circuits. Similarly, a duplex wave winding has 4 parallel circuits, a triplex wave has 6, and an x-plex wave has $2x$ parallel circuits, independently of the number of poles.

The arrangement illustrated in Fig. 12*a* shows two independent windings each containing 12 elements, or 24 in all. Suppose, now, that one of the 24 elements is omitted and that the remaining 23 elements, uniformly spaced, are connected alternately, as in Fig. 12*b*. Instead of two independent windings, each closed upon itself, as in Fig. 12*a*, there is now but a single closure;



(*a*) (*b*)

FIG. 12—Duplex armature windings.

but a study of the direction of current flow, indicated by the arrowheads, reveals the interesting fact that there are still four paths through the armature from brush to brush, just as in Fig. 12*a*. In other words, both drawings of Fig. 12 illustrate duplex windings, but the former is *doubly reentrant* whereas the latter is *singly reentrant*. The meaning of these terms will be clear when it is considered that a closed winding may be thought of as reentering upon itself; thus, if, in tracing through a winding, all the conductors are encountered before coming back to the starting point, there is but one closure or reentrancy, and the winding is singly reentrant. But if, on returning to the starting point after tracing through the given connections, it is found that only half the total number of conductors have been encountered, it is necessary to begin to trace through the remaining half before a second closure results, in which case there are two separate closures or reentrancies, and in that case the winding is doubly reentrant. The *degree of reentrancy* of a winding is, therefore, numerically equal to the number of independent, separately closed windings on the armature. Thus, it is possible to design windings as triplex, triply reentrant; triplex, singly reentrant; quintuplex, singly reentrant; etc., though such wind-

ings have no practical application because of the difficulties described in Art. 21.

It should be understood that all these conclusions apply with equal force to lap and wave drum windings, the ring type having been used in the discussion above solely for the sake of simplicity.

8. General Considerations.—The first systematic analysis of the relations to be satisfied in order that a symmetrical closed winding might result was the work of Professor E. Arnold of Karlsruhe, who published the result of his studies in 1891. The following derivation of the fundamental formulas is based upon Arnold's analysis.

Probably the first questions that will present themselves to the student examining diagrams like those of Figs. 6, 7, and 10 are: How does one know in advance the number of conductors to be stepped over in joining the end of one wire to the beginning of the next? Thus, in Fig. 10, the order is 1–6–11–16, . . . : would not some other order of connection do equally well? And what would be the effect of changing the total number of conductors from 32 to some other number? The answer to these and related questions is implicitly involved in a general equation covering all kinds of closed windings; this equation is derived in the succeeding articles.

9. Number of Conductors, Elements, and Commutator Segments.—Without regard to the number of turns per element, ring windings usually have only one active coil side per element, whereas drum windings have as a rule two active coil sides per element; the meaning of the term "coil sides" is shown in Fig. 8. Further, in accordance with the definition of an element, *there must be as many commutator segments S as there are elements.* Consequently, in ring windings the number of commutator segments is equal to the number of active coil sides, whereas in drum windings the number of commutator segments is usually equal to half the number of coil sides. S must, of course, be an integer, but it may be either even or odd; therefore, in ring windings which have one turn per element and in which S is odd, the number of peripheral conductors may also be odd; but the number of elements in ring windings is usually made even, and more particularly in simplex windings the number is a multiple of the number of poles, in order that each branch path of the armature may be at all times identical with all of the others, in which case

the number of conductors will be even. In drum windings, no matter whether S is even or odd and irrespective of the number of turns per element, the number of conductors and the number of coil sides must be even.

Let z represent the number of active conductors per coil side (or the number of turns per element in types like those of Fig. 8); then, in a drum winding, $Z/2z$ must equal the number of elements, or

$$\frac{Z}{2z} = S$$

When z is greater than unity, as in parts (*b*) and (*e*) of Fig. 8, the effect, so far as the connections to the commutator are concerned, is the same as though the multiturn element were replaced by a simple element of the types shown in Figs. 8*a* and 8*d*; it follows, therefore, that the analysis of the rules for armature windings can be confined to a consideration of the number of coil sides, each of which is treated as though it were a single conductor.

10. Winding Pitch, Commutator Pitch, and Slot Pitch.—In Fig. 6 the back, or pulley, end of coil side 1 is connected to the back end of coil side 12, and the front or commutator end of 12 is connected to the front end of 3 The number of coil sides passed over in this way is called the *winding pitch* thus, in Fig. 6, the *back pitch*, which will be designated by y_1, is $+11$, and the *front pitch*, or y_2, is -9. In Fig. 7 both front and back pitches are positive and equal to 11.

Again, in Fig. 6 the beginning and end of each element are connected to adjacent commutator segments, the numbers of which differ by unity. Similarly, in Fig. 7 the terminals of the elements are connected to segments which differ numerically by 11. This numerical difference between the terminal segments of an element is called the *commutator pitch* y.

In slotted armatures the number of slots spanned by a coil or element is called the *slot pitch* (represented by y_s).

Lap windings are *right-handed* or *left-handed*, respectively, depending upon whether y_1 is numerically greater or less than y_2. If one faces the armature at the commutator end, the winding is right-handed if it progresses clockwise from segment to segment of the commutator in tracing through the circuit.

On the other hand, wave windings are right- or left-handed according to whether one arrives at a segment to the right or left, respectively, of the starting point after tracing through $p/2$ elements, where p is the number of poles. Thus, in Fig. 10 the winding is left-handed.

The algebraic sum of the front and back pitches is a measure of the total advance or retreat per element in tracing through the winding. In the case of the simplex lap winding of Fig. 6, the back pitch is 11 coil sides, and the front pitch is -9 coil sides, so there is a net advance of 2 coil sides per element. At the same time the ends of the element are separated by one commutator segment, and hence the net advance in terms of commutator segments is only half as great as the advance in terms of coil sides; this fact is due to the number of commutator segments being only half as great as the number of coil sides. In general, in simplex *lap windings*,

$$\Sigma y = y_1 - y_2 = 2y \qquad (1)$$

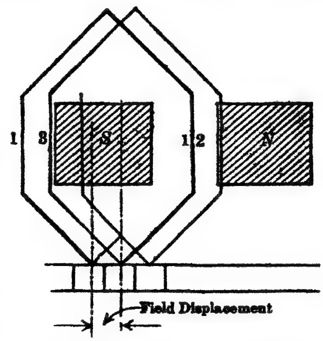


FIG. 13.—Field displacement in lap winding.

In the wave winding of Fig. 7, the front and back pitches are both equal to 11, so that the net advance per element is 22 coil sides; but the advance per element, in terms of commutator segments, is only half as great, or 11, since here again there are only half as many segments as coil sides; hence, in simplex *wave windings*,

$$\Sigma y = y_1 + y_2 = 2y \qquad (2)$$

11. Field Displacement.—Reference to Figs. 6 and 7 shows that the terminals of each element of a winding are connected to commutator segments which do not occupy exactly corresponding positions with respect to the axes of the pole pieces. There is a *field displacement*, or creep of the winding, between them which may be expressed in terms of the number of commutator segments, m, by which they fail to occupy homologous positions. Thus, in Fig. 13, which represents a portion of the lap winding of Fig. 6, the field displacement between the ends of an element is one segment, whence $m = 1 = y$; in the ring windings of Fig. 12, $m = 2 = y$.

In wave windings there is a somewhat similar state of affairs. Thus, Fig. 14 represents a portion of the wave winding of Fig. 7, from which it is seen that although the terminals of an element are separated by an interval approximately equal to a double pole pitch, so that the ends of an element are very nearly similarly placed with respect to poles of the same sign, the actual interval differs from the double pole pitch by an amount which is again a measure of the field displacement, or creep of the winding. It is clear that if this creep did not exist in the case of a wave winding, the winding would close upon itself after traversing a number of coil sides equal only to the number of poles.

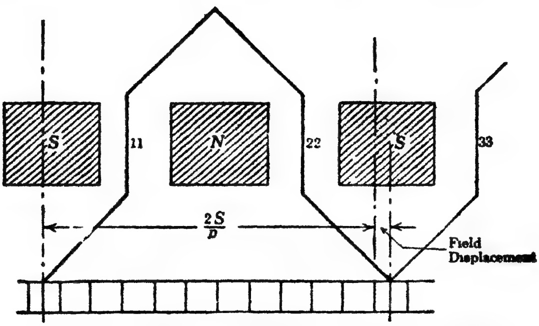


FIG. 14.—Field displacement in wave winding.

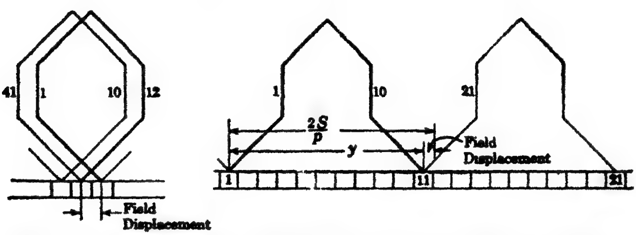


FIG. 15.—Left-handed (retrogressive) lap and wave windings.

In both Figs. 13 and 14 the field displacement is positive in sign; that is, the winding creeps ahead in a right-handed direction. In Fig. 13, if the front and back pitches were +11 and −13, respectively, instead of +11 and −9, the field displacement would be $m = -1$, and the winding would retrogress in the left-hand direction. In the same way, it is possible to connect the 42 coil sides of Fig. 7 to form a wave winding by using a front pitch of +11 and a back pitch of +9, in which case, also, m would be negative and the winding would be left-

handed. These retrogressive (left-handed) windings are illustrated in Fig. 15.

In wave windings of the usual type, as in Fig. 14, it will be seen that the commutator pitch is equal to the double pole pitch (expressed in terms of commutator segments) plus or minus the field displacement (also expressed in commutator segments). Since the number of commutator segments in a double pole pitch is $2S/p$, it follows that

$$y = \frac{2S}{p} \pm m \qquad (3)$$

In lap windings, as Figs. 13 and 15 plainly show,

$$y = \pm m \qquad (4)$$

so that in general

$$y = \frac{fS}{p} \pm m \qquad (5)$$

where $f = 2$ for ordinary wave windings and $f = 0$ for lap windings. The quantity f may be called the *field step* of the elements, the term "field step" meaning the nearest whole number of pole pitches between the ends of an element. Thus, in wave windings of the usual type the ends of an element are separated by nearly two pole pitches, so that $f = 2$, whereas in lap windings the two ends of an element are under the influence of the same pole, so that $f = 0$. In a wave winding of the type illustrated in Fig. 29, $f = 4$.

12. Number of Armature Paths.—In the simple ring winding of Fig. 3, where the ends of each element are separated by one commutator segment, so that $m = 1$, it is easy to see that there are as many circuits, or paths, through the winding as there are poles. But in Fig. 12, both parts of which are ring windings for a bipolar machine, the field displacement is two segments ($m = 2$), and, as has already been pointed out, the number of armature paths is four, or twice as many as there are poles. It may therefore be inferred that there is a definite relation between the field displacement m and the number of armature circuits.

Again referring to Fig. 3, it is easy to see that if we start with an element like number 1, in contact with a *negative* brush, and follow in order through elements 2, 3, 4, etc., the successive field displacements (in this case each equal to one segment) become

cumulative and ultimately amount to S/p segments when the next adjacent *positive* brush has been reached; one complete armature path, or circuit, will then have been traversed.

Precisely the same remarks apply to the lap winding of Fig. 6 and the wave winding of Fig. 7. There is no difficulty in appreciating the truth of this statement in the case of the lap winding; and such difficulty as may exist in the case of the wave winding is easily resolved by actually tracing through such a drawing as Fig. 7 or Fig. 10 and noting carefully the commutator segments as they are encountered in the passage from a brush of one polarity to the next brush of opposite polarity.

In general, therefore, if we trace through a winding beginning, say at a commutator segment in contact with a negative brush, there will have been a field displacement of m segments by the time the next segment, in order, has been reached; if we advance through the second element to another segment, the total field displacement will amount to $2m$; and so on, until the total field displacement amounts to S/p segments, when a complete path will have been traced out. But in this process there will have been encountered a total of, say S' segments; and since the field displacement per element is m segments, the total displacement in tracing through one complete path will be mS' segments. Hence,

$$mS' = \frac{S}{p}$$

or

$$\frac{S}{S'} = mp \tag{6}$$

Since S' segments are encountered per path, the total number of paths must be

$$\frac{S}{S'} = a$$

which is necessarily an integral number. Hence,

$$mp = a$$

or

$$m = \frac{a}{p} \tag{7}$$

Thus, in ordinary ring or lap windings, where the number of paths equals the number of poles ($a = p$), the field displacement

is $m = 1$; in duplex ring or lap windings, which have twice as many paths as poles, $m = a/p = 2$; or, in general, m equals the degree of multiplicity of lap or ring windings. In wave windings, on the other hand, m is generally fractional. Thus, in a simplex wave winding, where a is always 2, $m = \frac{1}{2}$ in four-pole machines (see Fig. 7), $m = \frac{1}{3}$ in six-pole machines (see Fig. 10), $m = \frac{1}{4}$ in eight-pole machines, etc. In duplex wave windings, where $a = 4$, it follows that $m = 4/p$, and in triplex wave windings $m = 6/p$.

13. General Rules.--In the case of ordinary lap windings it has now been shown that

$$y = \frac{y_1 - y_2}{2} = \pm m = \pm \frac{a}{p} \tag{8}$$

whereas in the case of wave windings it is

$$y = \frac{y_1 + y_2}{2} = \frac{2S}{p} \pm m = \frac{2S \pm a}{p} \tag{9}$$

From these equations there may be deduced certain convenient rules for determining the order of connections of the coil sides, the design of the winding elements being thereby fixed.

It has been pointed out in a previous section that the number of coil sides ($2S$) of a drum winding is necessarily even. If, then, the coil sides are numbered, half of them will bear even numbers and the other half odd numbers. Since each coil side proceeding outwardly from a commutator segment must have a return path through another coil side, the numbering may be so arranged that the even numbers will constitute the outgoing group and the odd numbers will comprise all of the return group. It follows that even-numbered coil sides must be connected to odd-numbered coil sides at both ends, and therefore, *front and back pitches must be odd.* This rule is a general one for all drum windings provided that the numbering is carried out in accordance with the system indicated in Fig. 16.

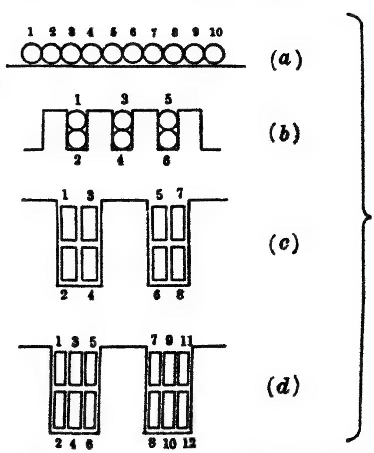


FIG. 16—Standard numbering of coil sides.

1. *Lap or Parallel Windings.*—An examination of the formula $y = \pm(a/p)$ shows that there are no restrictions upon the number of elements, which may, accordingly, be even or odd. In most windings there are only two coil sides per element, so that

$$y_1 - y_2 = 2y = \pm 2\frac{a}{p} = \pm 2m$$

from which it follows that *the pitches must differ by twice the degree of multiplicity in addition to the requirement that they must both be odd.* There remains the further condition that both y_1 *and* y_2 *must not differ too greatly from the pole pitch* $2S/p$, since otherwise the e.m.fs. of the connected sides will not be effectively additive. It is not essential that the average pitch approximate $2S/p$ so far as mere closure is concerned, and in certain so-called *chord* windings or *fractional-pitch* windings the average pitch is purposely made smaller than $2S/p$.

As an example of these rules, it is seen that in Fig. 6

$$Z = 42 \qquad S = 21 \qquad p = 4 \qquad a = 4$$

$$y = m = \frac{a}{p} = 1 \qquad y_1 - y_2 = 2y = 2 \qquad y_1 = 11 \qquad y_2 = 9$$

Had the pitches been made 9 and 7, respectively, or 7 and 5, the winding would close, but it would be an exaggerated form of chord winding.

Since $m = a/p = y$, it follows that in an *m-plex lap winding the commutator pitch equals the degree of multiplicity.* Thus, in a simplex lap winding the ends of an element are connected to adjacent segments; in a duplex winding they are separated by one segment; etc.

2. *Wave or Series Windings.*—The general formula

$$y = \frac{fS \pm a}{p}$$

reduces to $y = (y_1 + y_2)/2 = (2S \pm a)/p$ for most windings of this type. It is clear that the choice of S, and therefore of the number of coil sides and of active conductors, is not unlimited as in lap windings. In Fig. 10, for instance, which represents a simplex wave winding for a six-pole machine, $a = 2$, $p = 6$, $2S = 32$, and hence $y = (32 \pm 2)/6 = 5$ or $5\frac{2}{3}$. The latter value of y being impossible, we must take $y = 5$. Since the pitches must approximate $2S/p = 5\frac{1}{3}$, select $y_1 = y_2 = 5$.

though values of 7 and 3 would result in closure, but with an exaggerated amount of chording. In this connection it is hardly necessary to point out that the winding of Fig. 10 has been used for illustrative purposes only; it is not a practical design, for the number of elements is too small in comparison with the number of poles.

The restriction upon the number of elements in wave windings frequently causes the use of "dummy coils." Suppose, for example, it is necessary to design a simplex four-pole wave winding to be placed on an armature core having 65 slots, each slot being of sufficient size to accommodate four coil sides, in the manner of Fig. 16c. This arrangement implies that $Z = 260$, each coil side being assumed to consist of a single bar conductor, and this value of Z must check with the fundamental equation (7), Chap. II. To summarize: $2S = 260$, $a = 2$, $p = 4$; whence

$$y = \frac{260 \pm 2}{4} = 64\tfrac{1}{2} \text{ or } 65\tfrac{1}{2}$$

But since y must be an integer, it is plain that Z cannot be equal to 260; the value of $2S$ nearest to 260 that will satisfy the equation is 258 ($2S = 262$ is impracticable because the maximum number of coil sides that can be placed in the slots is 260). If we take $2S = 258$, it follows that there must be one element, consisting of two conductors, that is not a part of the winding; it is put in simply to fill up the space in the two slots which contain only three active conductors each. Therefore, $y = (258 \pm 2)/4 = 64$ or 65. Since y_1 and y_2 *must be odd* and, further, $(y_1 + y_2)/2 = y$, the following pairs of pitch values are apparently possible:

$$\begin{cases} y_1 = 65 \\ y_2 = 65 \end{cases} \begin{cases} y_1 = 63 \\ y_2 = 67 \end{cases} \begin{cases} y_1 = 67 \\ y_2 = 63 \end{cases} \begin{cases} y_1 = 65 \\ y_2 = 63 \end{cases} \begin{cases} y_1 = 63 \\ y_2 = 65, \text{ etc.} \end{cases}$$

Practical considerations in this particular case dictate the use of $y_1 = 65$. For if the coil sides are numbered in accordance

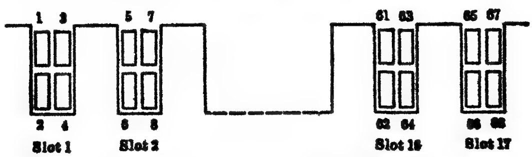


FIG. 17.—Numbering of coil sides.

with Fig. 16, it will be seen from Fig. 17 that the back ends of coil sides 1 and 3 may then be joined to coil sides 66 and 68,

respectively, and the elements may therefore be taped together in pairs, whereby their insertion in the slots is facilitated. For further discussion of this case, see Art. 17.

The formula for the commutator pitch of a simplex wave winding

$$y = \frac{2S \pm 2}{p}$$

may be written in the form

$$y = \frac{z_s N_s \pm 2}{p} = \frac{z_s N_s}{p} \pm \frac{2}{p}$$

where z_s = the number of coil sides per slot.

N_s = the total number of slots.

It follows, therefore, that if dummy coils are to be avoided for all values of p greater than 2, the product $z_s N_s$ must not be a multiple of p.

In wave windings the field displacement is given by $m = a/p$, so that after tracing through $p/2$ elements, corresponding to one circuit of the periphery, the total displacement is $p/2 \times a/p = a/2$ commutator segments. Therefore, in simplex windings ($a = 2$) the end of the $p/2$th element connects to a segment adjacent to the starting segment; in duplex windings it connects to the next but one; etc.

3. *Series-parallel Windings.*—The ordinary wave winding results in but two paths ($a = 2$) through the armature irrespective of the number of poles. But it is possible to secure any multiple of this number of paths by a suitable choice of S in the general formula. Wave windings having more than two paths are called *series-parallel* windings. Thus, if an eight-pole armature has 188 conductors wound to form 94 elements, it may be arranged as a four-circuit (duplex) wave winding; on substituting in the equation $y = (fS \pm a)/p$ the data $f = 2$, $S = 94$, $a = 4$, $p = 6$, there results $y = 23$ or 24, from which $y_1 = y_2 = 23$; or $y_1 = 25$, $y_2 = 23$.

14. General Rule for the Degree of Reentrancy.—*If, in the* general formula

$$y = \frac{fS \pm a}{p}$$

the two sides of the equation have a common factor q, we have

$$\frac{y}{q} = \frac{f\frac{S}{q} \pm \frac{a}{q}}{p}, \qquad \text{or} \qquad y' = \frac{fS' \pm a'}{p} \tag{10}$$

which means that the original winding is really made up of q independent windings, each of which has $S' = S/q$ elements and a commutator pitch of y', the latter counted with respect to the S' segments. That is, the winding will be multiplex and multiply reentrant of the qth degree in the event that y, S, and a have a common factor q; it will be singly reentrant if there is no such common factor.

In ordinary duplex wave windings ($f = 2$),

$$y = \frac{2S \pm 4}{p} = \frac{2(S \pm 2)}{p}$$

from which it follows that if y is even, that is, contains 2 as a factor, S must also be even because $(S \pm 2)/p$ must be an integer and p is always an even number. This leads to the simple rule that *a duplex wave winding is doubly reentrant if y is even* and to the corollary that it is singly reentrant if y is odd.

In triplex wave windings in which $f = 2$,

$$y = \frac{2S \pm 6}{p} = \frac{2(S \pm 3)}{p} \tag{11}$$

Suppose now that y contains 3 as a factor, in which case $y = 3x$, where x is an integer; then, from Eq. (11),

$$3x \cdot \frac{p}{2} = S \pm 3$$

$$\frac{p}{2}x = \frac{S}{3} \pm 1$$

Therefore, since $(p/2) \cdot x$ is integral, S must be a multiple of 3, and hence the winding is triply reentrant. It follows that a *triplex wave winding will be triply reentrant if y is a multiple of* 3, and it will be singly reentrant if y is not a multiple of 3.

In the case of quadruplex wave windings, however, such simplifications of the general rule are not possible. Such windings may be singly, doubly, or quadruply reentrant. Thus, if $f = 2$,

$a = 8$, and $p = 6$, $S = 79$ leads to a singly reentrant winding in which $y = 25$; $S = 82$ results in a doubly reentrant winding, $y = 26$; and $S = 80$ gives quadruple reentrancy, $y = 28$. Quadruplex windings are of theoretical interest only and need not be considered further.

15. Recapitulation of Winding Rules.—The conditions that must be satisfied by the various windings discussed in the preceding articles may be summarized as follows:

1. *Lap Windings.*

a. The number of elements, S, may be any number, even or odd (though preferably a multiple of the number of poles), consistent with the condition that it must satisfy the relation

$$S = \frac{Z}{2 \times \text{number of turns per element}} = \frac{Z}{2z}$$

where the number of turns per element is commonly one, though it may be two or more, depending upon commutation requirements; and the number of peripheral conductors, Z, must satisfy the equation

$$E = \frac{p}{a} \frac{\Phi Z n}{60 \times 10^8}$$

b. The winding pitches y_1 and y_2, expressed in terms of the number of coil sides spanned, must be approximately equal to $2S/p$, and both must be odd.

c. The numerical difference between the winding pitches must be 2 in simplex windings, 4 in duplex windings, 6 in triplex windings. In general, the difference between the pitches must be twice the degree of multiplicity, or $2m$.

d. The commutator pitch, expressed in terms of the number of commutator segments between the ends of an element, must be 1 in simplex windings, 2 in duplex windings, 3 in triplex windings. In general it must be equal to the degree of multiplicity.

e. The number of armature circuits in parallel is equal to the number of poles times the degree of multiplicity, or $a = pm$.

f. The degree of reentrancy is necessarily single in simplex windings; in duplex windings it will be double if S is even, single if S is odd; in triplex windings it will be triple if S is a multiple of 3, single if S is not a multiple of 3.

2. *Wave Windings.*

a. The number of elements, S, must satisfy the condition that

$$S = \frac{Z}{2 \times \text{number of turns per element}} = \frac{Z}{2z}$$

where Z in turn satisfies the equation

$$E = \frac{p}{a} \frac{\Phi Z n}{60 \times 10^8}$$

In addition, S must satisfy the relation

$$y = \frac{2S \pm a}{p}$$

where y must be an integer.

b. The winding pitches y_1 and y_2 must be approximately equal to $2S/p$, they must both be odd, and their average must be equal to the commutator pitch y.

c. The number of armature circuits, a, to be substituted in the formula

$$y = \frac{2S \pm a}{p}$$

must be twice the desired degree of multiplicity.

d. The degree of reentrancy is necessarily single in simplex windings; in duplex windings it will be double if y is even, single if y is odd; in triplex windings it will be triple if y is a multiple of 3, single if y is not a multiple of 3.

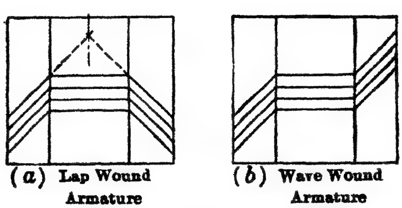


FIG. 18.—Showing direction of end connections in lap and wave windings

16. Construction of Winding Elements.—It is easy to recognize an armature as lap- or wave-wound, when the conductors are made of bars or strips of copper, by observing the relative directions of the top end connections at the two ends of the armature. Thus if the top end connections, when produced, meet at or near the center of the core, as in Fig. 18*a*, the winding is a lap winding;

whereas if the top end connections are parallel, as in Fig. 18*b*, the winding is a wave winding. If, however, the winding elements consist of two or more turns of wire, the end connections,

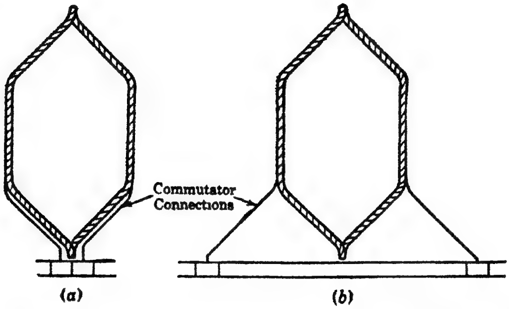


FIG. 19.—Wire-wound elements. (*a*) Lap. (*b*) Wave.

after they have been taped, have the same appearance in the case of both lap and wave windings, as may be seen from Fig. 19 The free ends of such winding elements connect to the commutator segments in the usual manner; but since they are covered

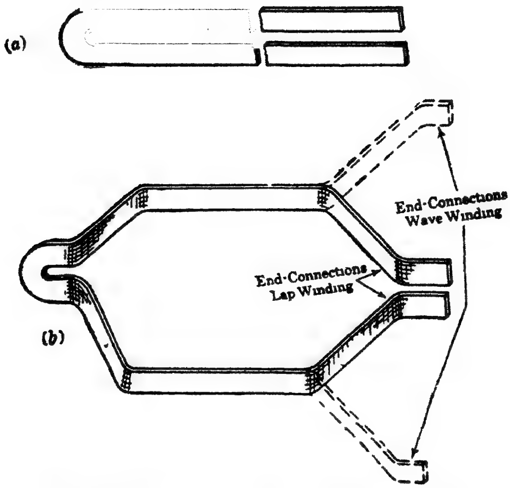


FIG. 20.—Formation of bar-wound element.

more or less by strips of insulating cloth and by banding wires, their directions cannot readily be traced.

The formation of elements for bar- or strip-wound armatures is illustrated in Fig. 20. Such elements can have only one turn. or two active conductors. The original piece of strip copper,

cut to the proper length, is bent edgewise at the middle, as in part (*a*), by means of a special bending machine which may be hand-operated in the case of small or moderate sizes of conductor but which must be power-driven when the cross-section of the strip is large. The hairpin-shaped loop is then opened out in the manner indicated in part (*b*) of the drawing. The end connections for a lap winding are formed by bending the ends inward, as shown by the full lines, whereas those for a wave winding are bent outward, as shown by the dashed lines.

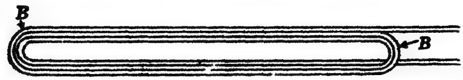


FIG. 21.—Initial form of wire-wound element.

Wire-wound elements having two or more turns are usually made by winding the wire on forming jigs in the manner shown in Fig. 21, the turns being held in position after removal from the jig by tying pieces of string at intervals. The elongated loop thus formed is placed in a forming machine which clamps the loop along those portions of the sides which are to occupy the slots; the clamps are mounted on pivoted arms which turn about a common axis, the radius of each arm corresponding to the

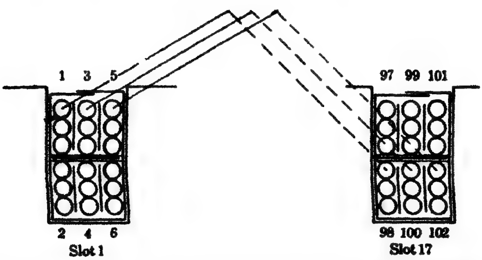


FIG. 22—Three-turn elements, six coil sides per slot.

radius of the cylindrical surface on which the active conductors are to be placed. On rotating the clamps through an angle that corresponds to the slot pitch of the element and at the same time clamping the ends of the loops to preserve the bend, as at *B* (Figs. 9 and 21), the element takes the form shown in Fig. 9.

17. Examples of Simplex Lap and Wave Windings.—The disposition of wire-wound elements in the slots is illustrated in Fig. 22 for the case of elements having three turns each, with six coil sides per slot. If, for example, the armature has 65 slots

and is to be used in a four-pole frame, then, without regard to whether the winding is to be of the lap or wave type, the slot pitch must approximate 65 ÷ 4, or say 16. Since there are 3 elements per slot, or 195 in all, the winding pitch must be approximately $2S/p = 390/4 = 97\frac{1}{2}$. By selecting a back pitch equal to 97, the three coil sides occupying the top of, say slot 1, will then connect, in order, to the three coil sides occupying the bot-

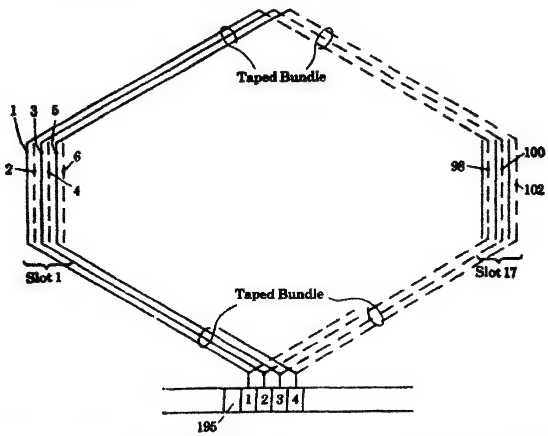


FIG. 23.—Lap winding. $S = 195$, $p = 4$, $y_1 = +97$, $y_2 = -95$, $y = +1$; 65 slots. Right-handed.

tom of slot 17; it follows that the end connections of all the individual elements will then be identical in shape and dimensions, and in the case here discussed the elements can be taped together in sets of three and inserted in the slots as units. The taped bundle of elements will in this case have three pairs of terminal wires, instead of the single pair indicated in Fig. 19, and in order to facilitate the identification of the two wires of a pair they should be marked in some distinctive manner; one way of doing this is to provide both wires of a pair with cotton sleeving having a color that differs from the colors of the other pairs.

The winding discussed in the preceding paragraph may be connected to form either a simplex lap or a simplex wave winding. Upon using a back pitch of 97 for the reasons given, a simplex lap winding will result if the front pitch is made either −95 or −99, being right-handed in the former case, left-handed in the latter. Portions of the developed winding diagrams for these two cases are shown in Figs. 23 and 24.

To convert this winding into a simplex wave, the commutator pitch must satisfy the condition

$$y = \frac{2S \pm a}{p} = \frac{390 \pm 2}{4} = 97 \text{ or } 98$$

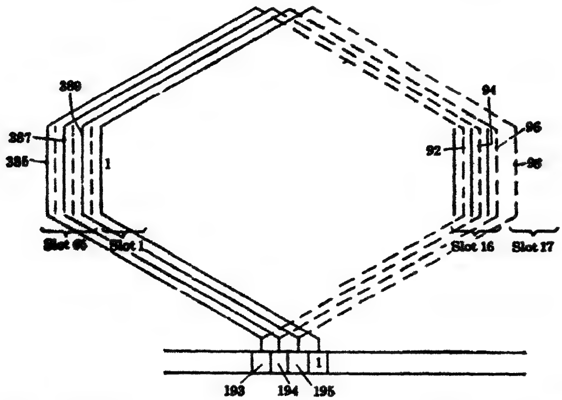


FIG. 24.—Lap winding. $S = 195$, $p = 4$, $y = +97$, $y_2 = -99$, $y = 1$; 65 slots. Left-handed.

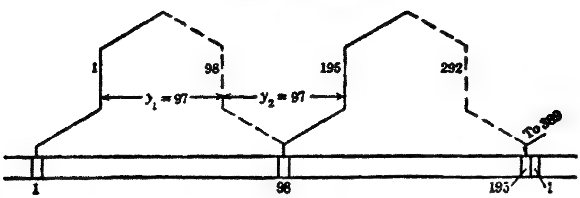


FIG. 25.—Wave winding. $S = 195$, $p = 4$, $y_1 = 97$, $y_2 = 97$, $y = 97$; 65 slots. Left-handed.

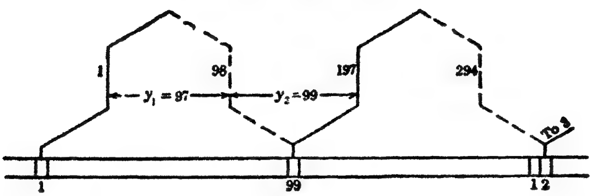


FIG. 26.—Wave winding. $S = 195$, $p = 4$, $y_1 = 97$, $y_2 = 99$, $y = 98$; 65 slots. Right-handed.

Since the back pitch has already been fixed at 97, the front pitch must be 97 if $y = 97$, or 99 if $y = 98$. These two cases are illustrated in Figs. 25 and 26, the former being a left-handed, the latter a right-handed winding. It should be noted that in this

instance all the elements are active; that is, there is no need for a dummy coil.

Effect of Dummy Coil on Numbering of Coil Sides.—In the example cited in Art. 13 (page 339) and illustrated in Fig. 17, the

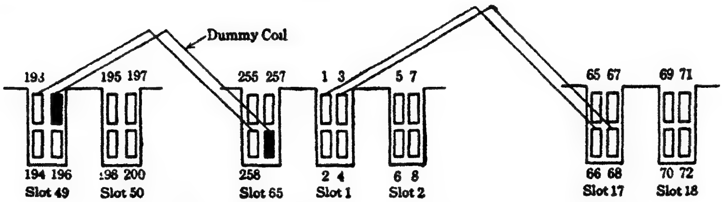


FIG. 27.—Effect of dummy coil on numbering of coil sides.

65 slots accommodate 260 coil sides, but 2 of these are merely fillers. The presence of the dummy coil introduces a slight complication in the numbering of the coil sides, as may be seen from Fig. 27, which is similar to Fig. 17 but shows more of the slots.

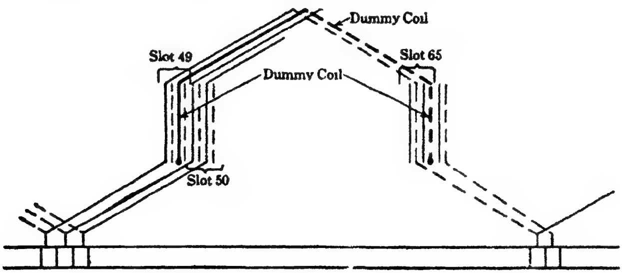


FIG. 28.—Development of wave winding having dummy coil.

The dummy coil is here shown by the solid black cross-section. It may be located anywhere, but in the illustration is shown in slots 49 and 65; obviously, it must not be numbered, with the result that the numbers of the coil sides in slots 49 to 65 are not consecutive numbers, as in all the remaining slots. In the actual process of making connections to the commutator segments there is, however, no complication, since all that is necessary is to cut off the front end connections of the dummy coil, in the manner indicated in Fig. 28, and then to insert the remaining front connections in the slots of the commutator risers in the usual way.

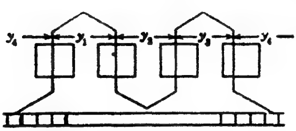


FIG. 29.—Element having four active coil sides.

Windings Having Reduced Number of Commutator Segments.—Although the armature windings of generators and motors are

almost always designed with two active coil sides per element, cases may arise in which it becomes desirable to reduce the number of commutator segments to one-half the number required with elements of the usual type. This result may be attained in the case of wave windings by using elements having four active sides, as in Fig. 29. In this case the field step f is equal to 4, instead of the usual 2, so that

$$y = \frac{4S \pm a}{p} \tag{12}$$

and

$$y = \frac{y_1 + y_2 + y_3 + y_4}{4} \tag{13}$$

Equation (12) indicates that an element of the form of Fig. 29 cannot be used in a four-pole simplex wave winding, for in that case y would become $(S \pm \frac{1}{2})$, which, being a mixed number because S is necessarily integral, represents an impossible condition.

As an example of this type of winding, consider the following data:

Number of poles = 6.
Number of slots = 74.
Number of coil sides per slot = 4.
Total number of coil sides = $4 \times 74 = 296$.
Number of winding elements = $296 \div 4 = 74$.

Upon applying Eq. (12), it is found that the sign of a must be taken as negative, whence

$$y = \frac{4 \times 74 - 2}{6} = 49$$

If $y_1 = y_2 = y_3 = y_4 = 49$, Eq. (13) being thereby satisfied, each of the complete elements like that of Fig. 29 can be made up

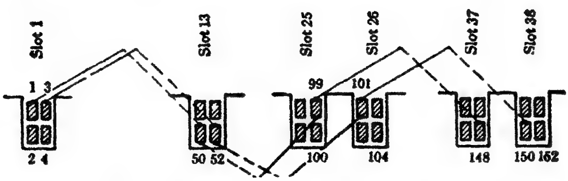


FIG. 30.—Slot arrangement of special winding.

of two identical halves, and the two coil sides lying side by side in a slot can be taped together to form a pair, as shown in Fig.

30. A portion of the developed winding diagram is shown in Fig. 31. It is seen that this winding will require only 74 commutator segments instead of the 148 segments needed if each element has only two active sides; it follows that the reactance voltage in the double element, when it is undergoing commutation, will be nearly double that developed in the two-sided element, and this feature must be carefully checked to ensure satisfactory commutation.

It is obvious that a wave winding can be worked out for the data of the preceding example by winding each element with

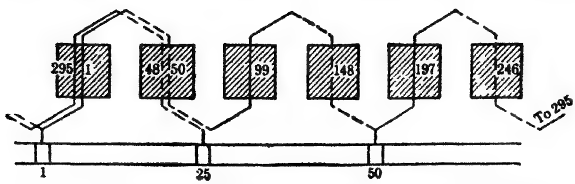


FIG. 31.—Commutator connections of special winding.

two turns in the manner indicated in Fig. 8*e*. There would again be 74 elements and 74 commutator segments, but the commutator pitch would be

$$y = \frac{2 \times 74 + 2}{6} = 25$$

The difference between this winding and the one first considered is due chiefly to the reactance voltage induced in the coils undergoing commutation. If each of the two halves of the element of Fig. 29 has a reactance x, then the reactance of the whole element will be approximately $2x$; but if the two partial coils are combined into one two-turn coil, the reactance becomes $4x$, since the reactance of a coil varies as the square of the number of turns.

18. Examples of Multiplex Windings.—If the total armature current is represented by I_a, the current per path in the armature winding itself is $I_a/a = I_a/p$ in the case of a simplex lap winding or $I_a/2$ in the case of a simplex wave winding. The magnitude of the current per path determines the cross-section of the conductor, since the current density must not exceed a definite value in order to keep the temperature of the winding within allowable limits. If the cross-section thus determined calls

for conductors that are inconveniently large, the winding may theoretically be subdivided into the equivalent of two or more simplex windings in parallel by arranging it as a multiplex winding.

Examples of duplex lap and wave windings* are represented by the diagrams of Figs. 32 to 35. The lap windings of Figs. 32 and 33, having 62 and 64 coil sides, respectively, must have winding pitches that average approximately $62/4 = 15\frac{1}{2}$ in the former and $64/4 = 16$ in the latter. Since the front and

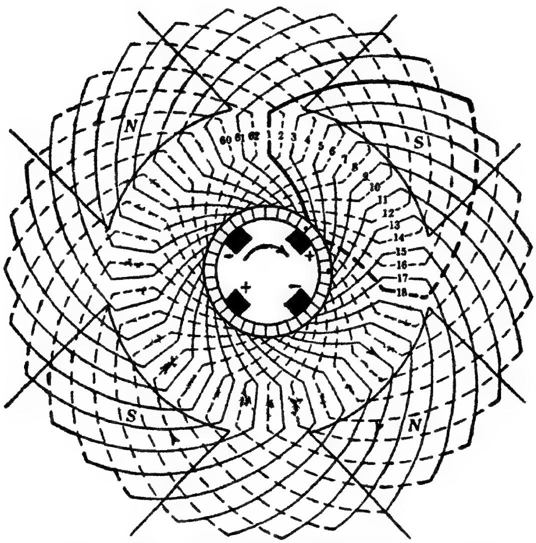


FIG. 32.—Duplex lap winding, singly reentrant. $Z = 62$, $S = 31$, $y = 2$, $y_1 = +17$, $y_2 = -13$.

back pitches must both be odd and must in these cases diffe by 4 ($=2m$), the pitches have been made $y_1 = +17$ and $y_2 =$ -13 in both windings. It will be seen that in both diagrams the winding progresses right-handedly, in accord with the fact that $y = m = +2$, and that the ends of an element, instead of connecting to adjacent commutator segments, are separated by the width of two segments. Figure 32 is singly reentrant, since $S = 31$ (an odd number); Fig. 33 is doubly reentrant (S is even).

* These examples are inserted for illustrative purposes only, and must not be regarded as representative of actual practice. (See Art. 21.)

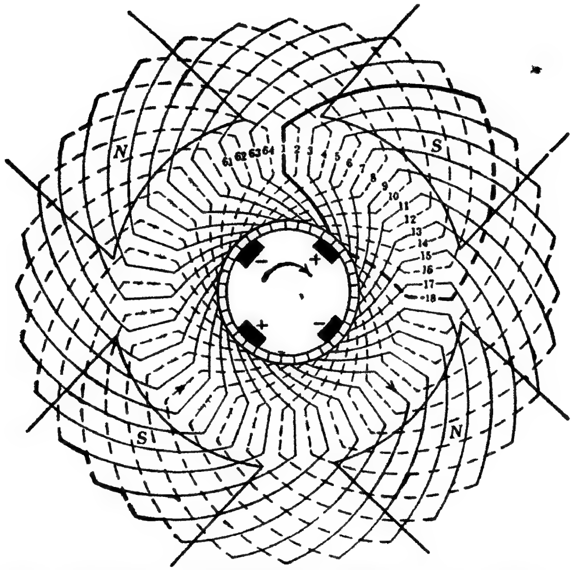


FIG. 33.—Duplex lap winding, doubly reentrant. $Z = 64$, $S = 32$, $y = 2$, $y_1 = +17$, $y_2 = -13$.

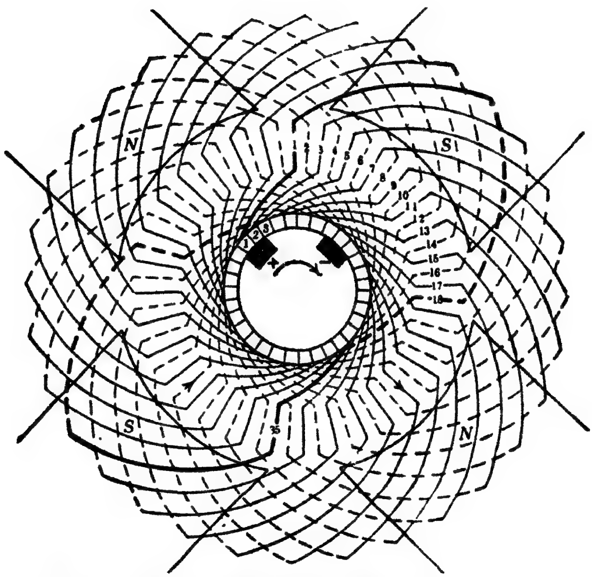


FIG. 34.—Duplex wave winding, singly reentrant. $Z = 64$, $S = 32$, $y = y_1 = y_2 = 17$.

In the duplex wave winding of Fig. 34, the winding pitches must satisfy the condition

$$y = \frac{y_1 + y_2}{2} = \frac{2S \pm a}{p} = \frac{2 \times 32 \pm 4}{4} = 17 \text{ or } 15$$

The diagram as constructed uses $y_1 = y_2 = 17$, corresponding to $y = 17$, but an equivalent chorded winding would result if the pitches had been made $y_1 = y_2 = y = 15$. The winding is singly reentrant since y is odd.

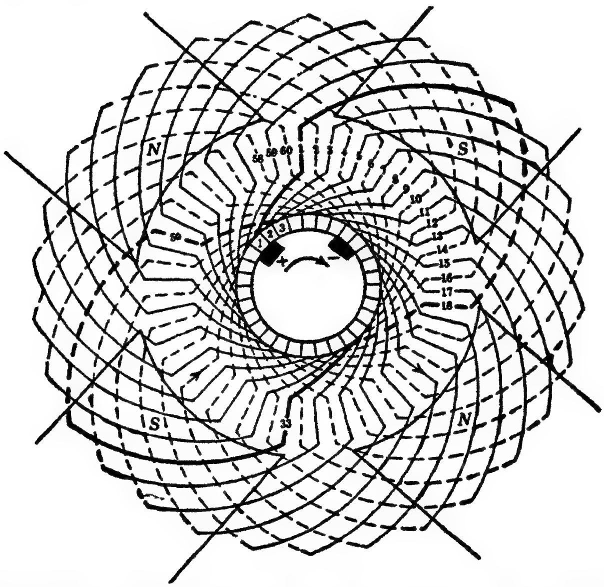


FIG. 35.—Duplex wave winding, doubly reentrant. $Z = 60$, $S = 30$, $y = 16$, $y_1 = 17$, $y_2 = 15$.

Similarly, in the duplex wave winding of Fig. 35, the winding pitches must satisfy the condition

$$y = \frac{y_1 + y_2}{2} = \frac{2 \times 30 \pm 4}{4} = 16 \text{ or } 14$$

The diagram is constructed with $y = 16$, $y_1 = 17$, $y_2 = 15$; an equivalent chorded winding would result from using $y = 14$, $y_1 = 15$, $y_2 = 13$. With $y = 16$ or 14, the winding is doubly reentrant. It is to be noted in both Figs. 34 and 35 that after tracing through $p/2$ $(=2)$ elements the commutator segments

at the beginning and end are separated by the width of two segments.

19. Double Commutator Windings.—It is clear that an armature may be wound with two completely independent windings, occupying the same slots but entirely insulated from each other, each with its own commutator, one at either end of the armature core. Each of the two constituent windings may be simplex or multiplex, as demanded by circumstances, though ordinarily simplex, or at most duplex, windings would be used. The two independent commutators have the advantage over the interleaved segments of the commutator of a multiplex winding in that

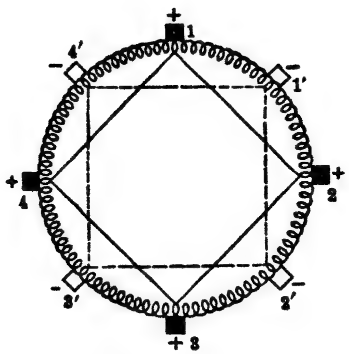


FIG. 36.—Parallel-wound armature with equalizer connections.

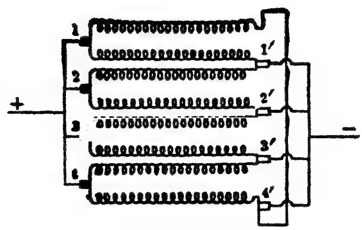


FIG. 37.—Diagrammatic scheme of connections of armature of Fig. 36.

it is more readily possible to keep the current density at the brushes within reasonable limits without making the segments unduly long. A still greater advantage follows from the fact that with two independent commutators the constituent windings, if identical, may be connected either in parallel or in series; if each winding is designed for V volts, the machine is adapted for use in systems that have voltages of either V or $2V$ volts.

20. Equipotential Connections.—Consider a parallel-wound armature like Fig. 36, which represents diagrammatically a winding for an eight-pole machine. The eight parallel paths through the armature from terminal to terminal are shown somewhat more clearly in their relations to one another in the diagram of Fig. 37. It will at once appear that if each path is to carry its proportionate share of the total armature current, each path must at all times generate the same e.m.f. and have the same

resistance as all the other paths. In any case, the current will divide between the eight paths in accordance with Kirchhoff's laws. If for any reason the e.m.f. generated in one path is greater than in another, for instance if that of circuit 3–2′ is greater than that of 3–3′, the brushes 2′ and 3′ would tend to have different potentials; but since they are metallically connected by a conductor of low resistance, the difference of potential will cause an equalizing current to flow in the lead joining brushes 2′ and 3′. Even very small differences of potential may give rise to internal equalizing currents of large magnitude, owing to the low resistance of the circuits, so that excessive heating of the winding and sparking at the brushes may result if preventive measures are not employed.

The possible causes of unequal e.m.fs. in the various paths are as follows:

1. The armature may not be exactly centered with respect to the pole shoes, owing to irregularities in construction or to wear of the bearings. The airgap is consequently not uniform, and some of the poles therefore carry more flux than others, more e.m.f. being thus generated in the coils subject to their influence than is generated in coils under the weaker poles. This cause is of importance in lap windings, where each armature circuit is at a given moment under the influence of one pair of poles only; in wave windings each path is simultaneously acted upon by all the poles, and hence this type of winding is free from the disturbing effect of nonuniform polar flux.

2. The poles may not all be identical in construction, so that their fluxes may differ even if the airgap is uniform. Thus, the joints between the poles and the yoke, or between the pole cores and the shoes, may not be equally good, or the magnetizing effect of the field windings may differ, especially in cases where the field coils are connected in parallel instead of in series.

3. The armature circuits may be unsymmetrical, owing to the use of a number of elements that is not an exact multiple of the number of paths.

The equalizing currents, though a source of loss because of the extra heating that they produce, are mainly objectionable because of the sparking at the commutator that results if they are allowed to complete their paths by way of the brushes. This sparking at the commutator is caused by the excessive loading of

some of the brushes at the expense of the others. To avoid this difficulty it is necessary first of all to strive for the greatest possible degree of magnetic and electrical symmetry of design; any remaining irregularities of construction may then be prevented from affecting commutation by providing *equalizing* or *equipotential connections.*

The principle to be applied in equipping an armature with "equalizer" connections is that those points in the winding which under ideal conditions have no difference of potential between them may be permanently connected by conductors of low resistance. Thus, in the ring winding of Fig. 38, points like *a*, *b*, *c*, *d*, which occupy identical positions with respect to poles of like polarity, may be connected in the manner indicated. Should there be accidental structural irregularities, which of themselves would tend to cause the points so connected to have different electrical potentials, equalizing currents will flow through the several closed circuits; one such closed circuit is seen to include conductors 2, 3. . . . , 12, 13, the path being completed by the connection from *b* to *a*. Similarly, other sets of points, like *a'*, *b'*, *c'*, *d'*, which are separated from one another by a double pole pitch, may also be connected. It is clear from Fig. 38 that the total number of conductors must be a multiple of the number of pairs of poles in order that equalizer connections may be used.

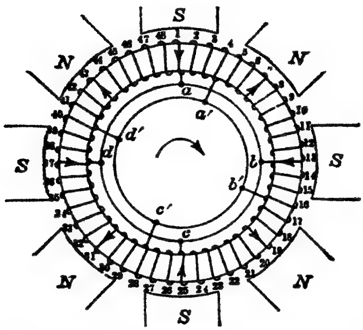


FIG. 38.—Equalizer connections in parallel winding.

It is important to note that the equalizing current which will flow in the closed circuits provided by the equalizer connections is an *alternating current;* the frequency of its alternation is determined by the consideration that there will be one complete cycle per pair of poles per revolution. This alternating current is caused by the very irregularities of flux distribution whose effects upon commutation it is desired to suppress; but by Lenz's law an induced current opposes the action which produces it, and hence it follows that the magnetizing effect of the equalizing currents will react upon the main field and more or less

completely annul the irregularities of flux distribution which, without this neutralizing action, would give rise to trouble.*

In a simplex lap winding, equalizer connections may be provided in the manner illustrated in Fig. 39, subject to the condition that the *number of slots is a multiple of the number of pairs of poles;* for if this condition were not satisfied, there would be no sets of points in the winding that are at all times similarly situated with respect to poles of like sign. In the case of large armatures in which the sid of the core opposite the commutator

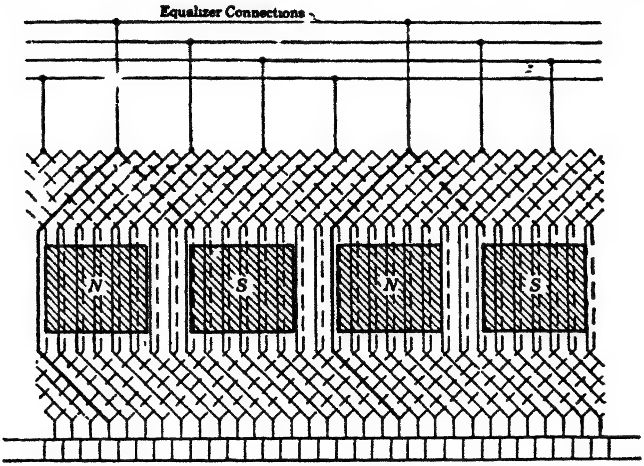


FIG. 39.—Equalizer connections in lap winding.

is generally exposed, the equalizer connections may be made in the form of concentric rings, as in Fig. 40, which are connected to appropriate points in the winding by means of radially disposed bars. The number of equalizer rings to be used is largely a matter of good judgment; it is theoretically possible to use as many as there are elements in a double pole pitch, but this construction would result in crowding of the rings, as well as in increased cost, without corresponding gain in effectiveness; accordingly, the windings are tapped at intervals of three or four elements.

Instead of using equalizer rings as in Fig. 40, an involute type, made of copper ribbon, may be made in the manner sketched in Fig. 41. These strips resemble the end connections of the ele-

* A. D. MOORE, Theory of the Action of Equalizer Connections in Lap Windings, *Elec. Jour.*, **23**, 624, 1926.

ments and nest together on the exposed flank of the core in a two-layer arrangement.

It is obvious that the equalizer connections may be tapped into the winding at the commutator end as well as on the pulley side. In small machines, where there is not sufficient room on the pulley

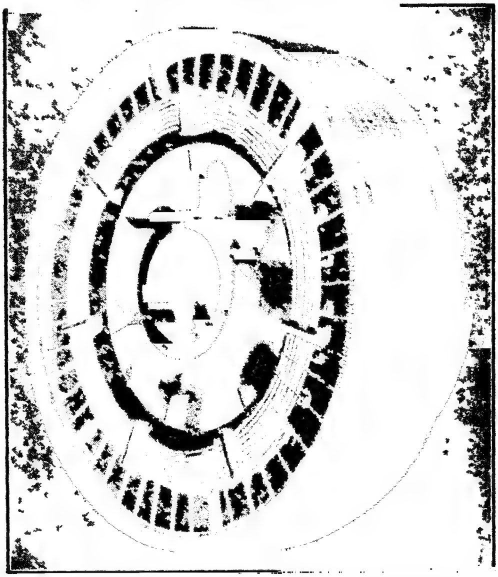


FIG. 40—Equalizing rings of large lap-wound armature.

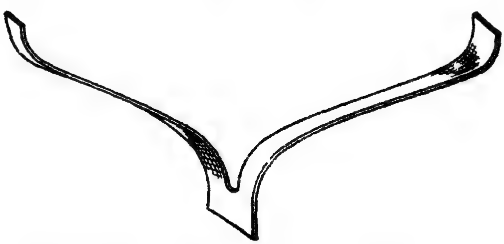


FIG. 41.—Involute type of equalizer connection

flank of the core to insert the rings they are frequently arranged behind the commutator in the manner indicated in Fig. 42; connections are made from the rings directly to the appropriate commutator segments, the number of which must be a multiple of the number of pairs of poles. A diagrammatic sketch of this type of equalizer connection is shown in Fig. 43 for the case of a four-pole lap winding having 24 elements. The equalizers in

this case connect opposite segments, though only a few of the connections are shown in order to avoid confusing the diagram. When such equipotential connections were first introduced by

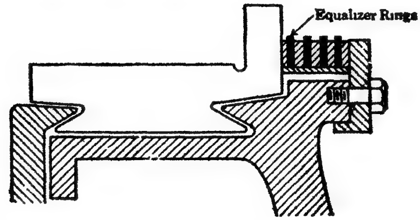


FIG. 42.—Equalizer rings at commutator end.

Mordey, they were intended to reduce the usual number of brush sets; for it is clear from Fig. 43 that two brushes, instead of four, might be expected to collect the entire current, since the segments that would contact with brushes of like polarity are already

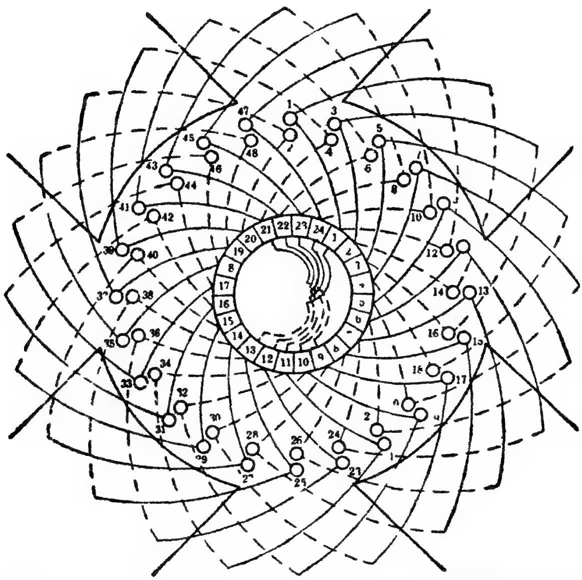


FIG. 43.—Equalizer connections at commutator. (Complete set not shown.)

permanently connected by way of the equalizer connections. In actual practice this apparently attractive scheme is not feasible, for the reason that if only two brushes are used the several parallel armature circuits are not all identically situated with respect to the line terminals, some being more remote than

others. The extra resistance of the equalizing connections between the remote armature paths and the line, though small, is still a large percentage of the resistance of the circuits themselves, so that an electrical unbalancing results which causes an unequal distribution of current through the armature winding, unequal loading of the brushes, and consequent sparking at the commutator. For these reasons the full number of brush sets is always used even when a complete set of equalizer connections is present.

Simplex wave windings do not require the use of equalizers since the conductors of each of the two parallel paths are subjected equally to the inductive action of all of the poles of the machine. As a matter of fact, there are no points in a simplex wave winding that are at the same potential, except at those points where contact must be made with brushes of like sign.

21. Equalizer Connections in Multiplex Windings.*—If the constituent parts of a multiplex lap winding are completely independent of one another, each reentrant upon itself and occupying its own set of slots, it is clear that each of these component simplex windings may have its own set of equalizers independently of the others, provided that the several windings are so designed that each includes points of equipotential when the magnetic field is symmetrically distributed. Although these independent sets of equalizers will ensure that the current in any one winding will be uniformly distributed through the $a = p$ paths of that winding, there can be no assurance that the total current of the entire machine will be divided equally between all of the constituent windings unless the several sets of equalizers are themselves thoroughly interconnected. If a multiplex lap winding is singly reentrant, independent sets of equalizers are of course impossible; but in that case equipotential connections should nevertheless be provided in order to ensure proper distribution of current among the various circuits.

In the case of a multiplex, multiply-reentrant wave winding, none of the component reentrant windings can have equalizer connections of its own, but it should be possible to interconnect points of the several constituent simplex windings to facilitate the uniform distribution of the entire current. Similar interconnec-

* Carl C. Nelson, Multiplex Windings for Direct-current Machines *Trans. A.I.E.E.*, **45**, 976, 1926.

tion should be possible in a multiplex wave winding that is singly reentrant.

The general principles that must be observed in making provision for equalizer connections in any winding, including the above-discussed multiplex windings, may be summarized as follows:

1. There must be symmetrically situated points in the winding that are angularly spaced at intervals equal to the double pole pitch (or exact multiples thereof).

2. The active conductors that are to be connected by the equalizers must be similarly situated in their respective slots.

3. Since the only windings of practical importance are those in which each element has two active sides, there must also be symmetrically spaced commutator segments that are similarly situated with respect to brushes of like polarity.

If consideration is limited to those cases in which the points to be connected by equalizers are separated by a double pole pitch, the number of commutator segments per pair of poles must be an integer, that is,

$$\frac{S}{p/2} = \frac{2S}{p} = \text{an integer} \tag{14}$$

4. For any winding whatever, in which

N_s = numbers of slots.

z_s = numbers of coil sides per slot (= an even number).

y_s = slot pitch.

y_1 = back pitch of winding.

$$N_s z_s = \text{total number of coil sides} = 2S \tag{15}$$

and

$$y_s z_s + 1 = y_1 \tag{16}$$

The relation expressed by Eq. (16) may be understood by referring to Fig. 22; for, in order that the coils in the top of slot 1 may connect, in order, to those in the bottom of slot $(y_s + 1)$, it is plain that the coil side numbered $(z_s - 1)$ in the upper right-hand corner of slot 1 must be joined to the coil side numbered $z_s(y_s + 1)$ in the lower right-hand corner of slot $(y_s + 1)$, and the difference between these two numbers must equal y_1. Upon assembling these relations, it follows that, in general,

$$\frac{2S}{p} = \frac{N_s z_s}{p} = \frac{N_s}{p}\,\frac{y_1 - 1}{y_s} = \text{an integer} \tag{17}$$

subject to the expressed limitation, namely, that each equalizer must connect points (segments) separated by a double pole pitch.

In order to examine these conditions somewhat more in detail, first with respect to lap windings and second with respect to wave windings, Fig. 44 has been constructed to illustrate diagrammatically the armatures of machines having 4, 6, 8, . . . , 18 poles, the inscribed polygons indicating in each case the minimum number of points that under appropriate conditions may be connected by equalizers. This minimum is multivalued in the 8-, 12-, 16-, and 18-pole machines, but in general the number

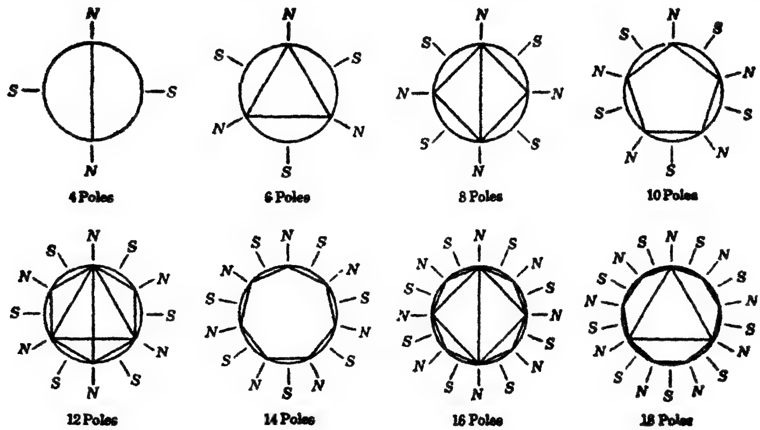


FIG. 44.—Possible locations of equipotential points.

of such points is equal to either $p/2$ or any integral factor of $p/2$ that is greater than unity.

1. *Duplex Lap Windings.*— If the winding is doubly reentrant (S even), the successive taps to an equalizer ring (see Fig. 44) must connect first to one of the component windings, then to the other, and there must be an *even* number of such taps if the arrangement is to be symmetrical. This consideration at once rules out the possibility of doubly reentrant duplex lap windings for all machines in which $p/2$ is an odd number ($p = 6, 10, 14, 18$); hence it follows that in machines in which $p/2$ *is odd, only singly reentrant duplex lap windings can be used;* that is, in these cases S must be an odd number. But if both S and $p/2$ are odd numbers, their quotient [which by Eq. (14) must be an integer] will also be an odd number; and this requirement, by

Eq. (17), imposes a restriction upon N_s and z_s. In particular, if $z_s = 2$, $N_s = S$; that is, the number of slots must be odd. Again, if $z_s = 4$, machines having 6, 10, 14, 18, . . . , poles become impossible, but if $z_s = 6$, N_s again becomes odd for a six-pole machine, and 10- and 14-pole machines are possible only provided that $2S/p$, in addition to being odd, is also a multiple of 3.

In the case of machines having an *even* number of pairs of poles ($p = 4, 8, 12, 16, . . .$) the successive taps at intervals of $2S/p$ commutator segments can produce a symmetrical arrangement only when the winding is doubly reentrant, that is, *when S is an even number.* But since the commutator pitch must equal 2 in a duplex lap winding, it follows that all the odd-numbered segments will belong to one of the two component windings and all the even-numbered segments will belong to the other; hence, since the interval $2S/p$ must serve to interconnect the two component windings by spanning from an odd- to an even-numbered segment, it follows that $2S/p$ must again be an odd number; that is, *$2S/p$ must be an odd integer no matter whether $p/2$ is odd or even*

When $p/2$ is even, Eq (17) again imposes restrictions upon the number of slots, depending upon the value of z_s. Thus, when $z_s = 2$, $2N_s/p =$ odd integer; hence, when $p = 4, 8, 12, 16, . . .$, N_s must always be even. But if $z_s = 4$, N_s must be odd when $p = 4$, even when $p = 8$, odd when $p = 12$, etc.; that is, when $z_s = 4$, N_s must be odd if p is an odd multiple of 4, and N_s must be even if p is an even multiple of 4. If $z_s = 6$, it will be seen from Eq. (17) that $2S/p$ must be a multiple of 3 to permit equalizer connections, except when p is a multiple of 6; and in all cases in which $z_s = 6$, N_s will be even.

2. *Duplex Wave Windings.*—Here the conditions to be met are circumscribed, to an even greater degree than in duplex lap windings, by the requirement that the commutator pitch

$$y = \frac{2S \pm 4}{p} = \frac{2S}{p} \pm \frac{4}{p} \tag{18}$$

must be an integer. The conditions that arise in the case of various numbers of poles are as follows:

a. Four Poles.—In this case, Eq. (18) shows that $2S/p$ must be an integer, and a duplex wave winding will therefore permit the

use of equalizers; but since such a winding has the same number of armature paths as a simplex lap, it presents no advantages over the latter and need not be considered.

b. Six Poles.—This is clearly an impossible case, as may be seen from Fig. 44, which shows that a balanced interconnection of the two component windings is out of the question; moreover, $2S/p = S/3$, and by Eq. (18) $S/3$ must be a mixed number, whereas Eq. (17) demands that it be an integer.

c. Eight Poles.—By Eq. (18), $2S/p = S/4$ must be a whole number *plus* ½. Equalizer taps at intervals of a double pole pitch are therefore impossible, but they can be used if they span twice this interval, that is, if they join diametrically opposite points, as in Fig. 44. If we write

$$\frac{S}{4} = x + \frac{1}{2}$$

where x may be either an even or an odd integer, it is seen that S must be an even number $(4x + 2)$; but by Eq. (18) y may be either even or odd, corresponding, respectively, to double or single reentrancy. The number of coil sides per slot, z_s, which must be an even number (such as 2, 4, 6, . . .) then requires the following conditions:

(1) $z_s = 2$. By Eq. (15), $N_s = S$, so that the number of slots must be even.

(2) $z_s = 4$. By Eq. (15), $N_s = S/2$; and since S is itself even, it would at first appear that N_s may be even or odd. But N_s cannot be even; for in that case $S/2$ would be divisible by 2, and $S/4$ would be an integer which is contrary to the requirement that $S/4$ must be $(x + \frac{1}{2})$. Consequently, N_s must be odd.

(3) $z_s = 6$. By Eq. (15), $N_s = S/3$; and from Eq. (18), $y = S/4 \pm \frac{1}{2}$, whence $N_s = (4y \pm 2)/3$, where both N_s and y are necessarily integers. The number of slots is therefore limited to the series of even numbers that follow the progression 2, 6, 10, 14, 18, . . . ; that is, the number of slots must be double an odd number.

Upon applying the same analysis to larger numbers of poles, it is easily seen that equalizers are impossible in all cases where $p/2$ is an odd number; in other words, duplex wave windings are not practicable when $p = 6, 10, 14, 18$, etc. They are, however,

possible in the sense that equalizers may be used when $p = 8, 12, 16$, etc., that is, when $p/2$ is an even number, but subject to restrictions imposed by Eqs. (15) and (18).

3. *Triplex Windings.*—The principles explained in the preceding discussion of duplex windings may readily be extended to cover triplex, and higher orders of multiplex, windings, but this analysis is left to the initiative of the student. Very few windings of higher order than duplex have ever been attempted on a commercial basis, probably because of discouraging performance of experimental windings that were not properly designed to permit full interconnection of the various armature circuits. An interesting exception is the so-called "frog-leg" winding* of the Allis-Chalmers Manufacturing Company, which consists of a multiplex wave winding in the same slots with a separate simplex (or multiplex) lap winding, both having the same number of paths and both connected to the same commutator. Thus, if a winding of this type is to be designed for an eight-pole machine, the lap winding, if simplex, will have eight paths, and the wave winding would then have to be quadruplex; in tracing through such a wave winding, the terminals of any consecutive $p/2$ $(=4)$ elements will connect to commutator segments between which there are three other segments; and between these terminals of the $p/2 = 4$ elements of the wave winding there will be four elements of the simplex lap winding. Since these two sets, of four elements each, are in parallel with each other, they must each develop the same e.m.f., a condition that can be attained only if the eight coil sides of the one set are situated with respect to the poles in identically the same way as the eight coil sides of the other set. This requirement means that the front and back pitches of the two windings must be respectively the same. Such a winding for an eight-pole machine will have 16 paths, and each of the two separate windings will serve to cross-connect the other without using external equalizers.

* See paper by W. H. Powell and G. M. Albrecht, *Iron and Steel Engineer*, September, 1925, p. 345.

CHAPTER VIII

THE MAGNETIZING EFFECTS OF THE FIELD AND ARMATURE WINDINGS

1. Magnetization and Saturation Curves.—Every dynamo, whether used as a generator or as a motor, consists of one or more electrical circuits interlinked with a magnetic circuit. The armature winding is that part of the electrical circuit which, in the case of a generator, serves to produce the active e.m.f. by virtue of the rotation of the winding through the magnetic flux; in the case of a motor, the armature winding serves to produce the torque because of the reaction of the armature current upon the magnetic flux. In either case, therefore, the performance characteristics of the machine are closely related to those magnetic characteristics which determine the magnitude of the main flux. It has been shown in Chap. III that the flux produced in a magnetic circuit is determined by the relation

$$\text{Flux} = \frac{\text{m.m.f.}}{\text{reluctance}}$$

In other words, the flux is directly proportional to the excitation and inversely proportional to the reluctance. If the reluctance were constant, as would be the case if the permeability of the iron were constant, the relation between flux and excitation would be represented by a straight line; but since the reluctance of a magnetic circuit increases rapidly after saturation sets in, the actual relation between flux and excitation is represented by a curve (Fig. 1) that departs more and more from a simple rectilinear relationship as the excitation becomes greater and greater.

Since the e.m.f. generated in an armature winding is given by

$$E = \frac{p}{a}\frac{\Phi Z n}{60 \times 10^8} = \frac{p}{a}\frac{Zn}{60 \times 10^8} \cdot \Phi \tag{1}$$

where Φ is the flux per pole, expressed in maxwells, it follows that

$$E = \frac{p}{a}\frac{Zn}{60 \times 10^8} \cdot \frac{\text{m.m.f.}}{\text{reluctance}} \tag{2}$$

the m.m.f. being the excitation which is responsible for the flux Φ, and the reluctance being that of the magnetic circuit through which the flux passes.

When the frame of a given machine has been completely specified as to dimensions and materials, it is clear that without regard to the kind and connections of its windings there is one, and only one, relation between the excitation applied to it (measured either in gilberts, or in ampere-turns per pole or per pair of poles) and the flux Φ that will result.* That is to say, for a given frame, regardless of its winding, there is a definite characteristic curve, of the form of Fig. 1, which shows in what manner the flux per pole, Φ, will vary as the excitation applied to the magnetic

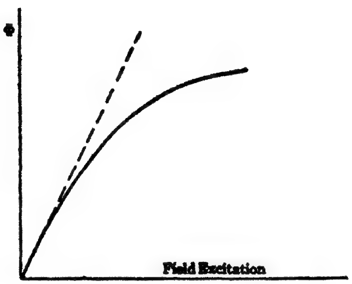


FIG. 1.—Magnetization curve.

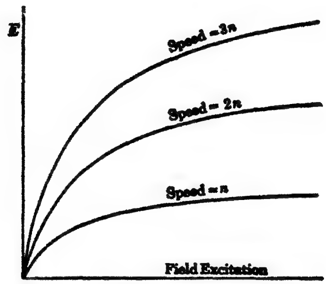


FIG. 2.—Saturation curves.

circuit is altered. Such a curve is the *magnetic characteristic* of the frame and may be referred to as the *magnetization curve.*

Inspection of Eqs. (1) and (2) shows that the e.m.f. generated in an armature winding is directly proportional to the flux Φ, that the proportionality constant $\frac{p}{a}\frac{Zn}{60 \times 10^8}$ is dependent upon the number and arrangement of the armature inductors (as fixed by Z and a) and upon the speed n, and that for a given armature winding and a given value of the flux Φ the e.m.f. E is directly proportional to the speed of rotation, n. If we start, therefore, with such a magnetization curve as is represented in Fig. 1, the relation between E and the field excitation, for a series of assigned values of n, will have the form shown in Fig. 2, in which each curve corresponds to a particular value of n. Such curves are called *saturation curves.* It will be seen that if E, n, and the field excitation are plotted along three mutually perpendicular axes of reference, as in Fig. 3, the several

* Provided that the effect of hysteresis is ignored.

saturation curves corresponding to successively different values of n are sections of a curved surface.

2. Main Magnetic Circuit.—The dashed lines in Figs. 4 and 5 represent the mean paths of the flux in typical forms of bipolar and multipolar machines. These lines are so drawn that they pass through the centroids of the sections of the tubes of induction. It will be observed that a complete path or magnetic circuit, such as C', Fig. 5, includes the armature core, two sets of teeth, two airgaps, two pole shoes, two poles cores, and the

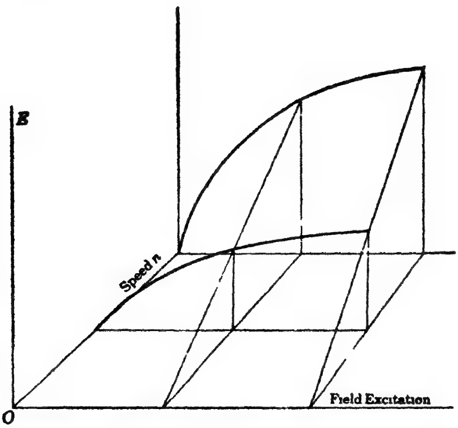


FIG. 3.—Three-dimensional diagram showing dependence of saturation curves upon speed of rotation.

connecting yoke. A magnetizing winding P on one pole will set up the same flux in each of the paths C and C' (perfect symmetry of construction being assumed) since these paths are in parallel. A similar winding on every *alternate* pole would then magnetize all the poles equally; hence, the excitation required to drive the flux through a complete magnetic circuit such as C or C' is the excitation *per pair of poles.* The excitation per pole is, therefore, that required to maintain the flux in a magnetic circuit such as *abcd* (Fig. 5) consisting of a single airgap, one set of teeth, one pole shoe and core, and half of the connecting circuit through the armature core and the yoke. Field excitation is generally expressed in terms of the number of ampere-turns per pole or in terms of ampere-turns per pair of poles.

The magnetization curve is of great importance. Whether the machine is to be used as a generator or as a motor, the form of the

magnetization curve will largely determine its operating characteristics. Conversely, a given set of performance specifications will fix the form of the magnetization curve and therefore the design of the magnetic circuit (that is, the frame of the machine). It is, therefore, apparent that the determination of this curve is of fundamental importance. In the case of a machine that exists only as a paper design, it is possible to compute the coordinates of points on the curve by methods that are described in Chap. IX; in the case of a completed machine, the magnetization curve can be found by an experimental determination of the form of the saturation curve.

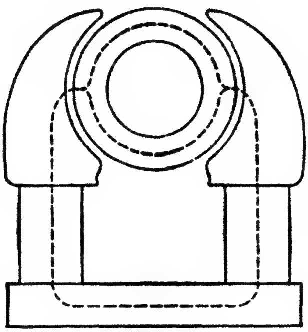


FIG. 4.—Magnetic circuit of bipolar machine.

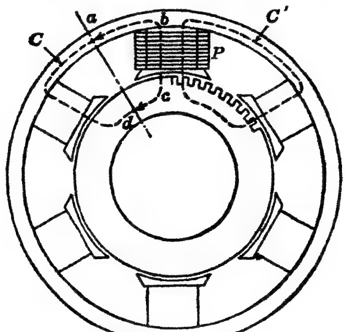


FIG. 5.—Magnetic circuits of multipolar machine.

3. Experimental Determination of Saturation Curve.—Since

$$E = \frac{p}{a}\,\frac{Zn}{60 \times 10^8} \cdot \frac{\text{m.m.f.}}{\text{reluctance}} = kn \times \text{function of field ampere-turns} \qquad (3)$$

it follows that all that is necessary is to run the machine at a constant speed n (driving it with a motor or other suitable prime mover) and to observe a series of simultaneous pairs of values of E and exciting ampere-turns. In Eq. (3), E is the e.m.f. generated in the armature by rotation through the flux produced by the field current. Therefore, to measure E directly, the armature must be without current; that is, it must be on open circuit. The machine must therefore be separately excited during this test, as in Fig. 6, current being supplied to the field winding from a suitable external source, controlled by a variable resistance R and measured by an ammeter A. The procedure consists of varying

the current A by means of R and taking a reading of voltmeter V for each setting of A, the speed being kept constant throughout.

Inspection of Eq. (3) indicates that with a fixed value of field excitation the generated e.m.f. E should be directly proportional to the speed, but in reality it is not exactly so; for it is possible that current may flow in those elements of the armature winding which are short-circuited by the brushes, and these short-circuit currents may easily attain sufficient magnitude to react upon the flux and so affect the generated e.m.f. To reduce the disturbing effect of the short-circuit currents to a minimum, it is necessary to cut down the e.m.f. that gives rise to them; and, with a given field excitation, this result can be accomplished by reducing the speed. It is best, therefore, to determine the saturation curve at a speed considerably below the rated speed and then to multiply the observed voltage by the ratio of rated speed to the speed actually used. The effect of the current in the short-circuited coils of the armature is discussed in more detail in Art. 14, Chap. XII.

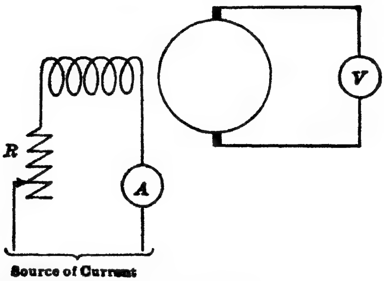


FIG. 6.—Experimental determination of saturation curve.

The form of the saturation curve obtained experimentally is not the same if the exciting current in the field winding is first gradually increased from zero to a maximum and then gradually reduced from this maximum back again to zero. The observed readings when plotted take the form of Fig. 7. The difference between the two curves is due to the hysteresis of the iron part of the magnetic circuit.

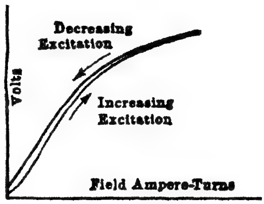


FIG. 7.—Effect of hysteresis on saturation curve.

It is obvious that the magnetization curve can be computed from the saturation curve by dividing the ordinates of the latter by the factor $\frac{p}{a}\frac{Zn}{60 \times 10^8}$.

The magnetization curve of a machine being a function of the B-H curves of the materials of which it is constructed, it is

impossible to devise a theoretically correct equation for it in the absence of any known relation between B and H. There is, however, an empirical formula, known as Froelich's equation, that expresses with a fair degree of accuracy the relation between flux (or the corresponding generated e.m.f.) and the excitation; it has the form

$$\text{Flux} = \frac{\text{constant} \times \text{excitation}}{\text{constant} + \text{excitation}}$$

This equation represents a hyperbola passing through the origin (as in Fig. 1) and having a horizontal asymptote located at a distance above the origin equal to the constant in the numerator. There is another (vertical) asymptote to the left of the origin at a distance equal to the constant in the denominator. Froelich's equation and the use that may be made of it are discussed at greater length in Art. 15, Chap. X; it may be noted, however, that the equation in the form given above fails to take into account the residual magnetism, since if the excitation is placed equal to zero the flux likewise reduces to zero value. But the equation may be modified to include the effect of residual magnetism by the simple expedient of transferring the origin to a point to the right of the one determined by the equation as given above. Thus, if we call the excitation X and let a and b represent the two constants, the original equation may be written

$$\Phi = \frac{aX}{b + X} \tag{4}$$

If the origin is transferred to the right by an amount X_0, the equation becomes

$$\Phi = \frac{a(X' + X_0)}{b + (X' + X_0)}$$

where X' is the excitation measured from the new origin, and this may be written

$$\Phi = \frac{aX' + m}{b' + X'} \tag{5}$$

where $m = aX_0 =$ constant and $b' = b + X_0 =$ constant, so that the revised equation has the form

$$\Phi = \frac{\text{constant} \times \text{excitation} + \text{constant}}{\text{constant} + \text{excitation}}$$

In other words, Froelich's equation may be made to take residual

magnetism into account by the introduction of an additional constant term in the numerator of the right-hand expression.

4. Predetermination of Magnetization and Saturation Curves. The computation of the coordinates of points on the magnetization curve is based upon the method previously outlined in Art. 24, Chap. III. Thus, on the assumption that it is desired to find the excitation corresponding to any assigned value of the flux per pole, Φ, the procedure is substantially as follows: From the assumed value of Φ determine the flux density in each part of the magnetic circuit by dividing the flux in that part of the circuit by the cross-section. From these values of the flux density it is then possible to find from the *B-H* curve of the corresponding material the necessary number of ampere-turns per unit length; and by multiplying the latter by the known length of the portion of the circuit in question, the product will be the number of ampere-turns required for that portion. Upon adding together all these partial products, the sum will represent the total number of ampere-turns for the entire circuit.

If this process is repeated for other assumed values of the flux, any desired number of points on the magnetization curve can be determined. The corresponding points on the saturation curve for any speed n can be found by multiplying each value of Φ by the factor $\frac{p}{a}\frac{Zn}{60 \times 10^8}$. Five such points will generally be sufficient to construct the curve, corresponding, say to quarter, half, three-quarters, and full voltage and to a value 10 to 20 per cent greater than full voltage.

ASSUMED VALUE OF FLUX $\Phi = \cdot \cdot$; $E = \frac{p}{a}\frac{\Phi Zn}{60 \times 10^8} = \cdot \cdot \cdot$

Part of circuit	Cross-section	Length of path	Flux density	Amp.-turns per unit length	Total amp.-turns
Armature core......	A_a	l_a	B_a	at_a	AT_a
Teeth, one set.......	A_t	l_t	B_t	at_t	AT_t
Air-gap, single......	A_g	δ	B_g	at_g	AT_g
Pole shoe, single.....	A_s	l_s	B_s	at_s	AT_s
Pole core, single.....	A_c	l_c	B_c	at_c	AT_c
Yoke..............	A_y	l_y	B_y	at_y	AT_y

The calculations will be facilitated and systematized by tabulating the results in the manner indicated in the table on p. 372, which also serves to show the symbols referred to in subsequent articles where details of the calculations are explained.

If the number of ampere-turns for each part of the complete magnetic circuit is known, the total number of ampere-turns per pair of poles is

$$AT = AT_a + 2AT_t + 2AT_g + 2AT_s + 2AT_c + AT_y \quad (6)$$

in accordance with the discussion in Art. 2, from which it follows that

$$AT = at_a l_a + 2at_t l_t + 2at_g \delta + 2at_s l_s + 2at_c l_c + at_y l_y \quad (7)$$

The magnetization (L-H) curves of magnetic materials are usually plotted in terms of B and at (the latter being equal to

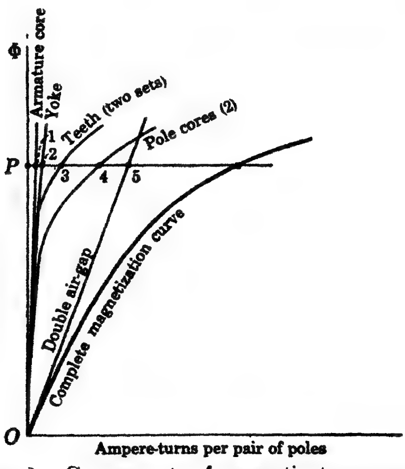


FIG. 8.—Components of magnetization curve.

$0.8H$), so that when the values of B for the various parts of the circuit have been determined, the corresponding values of at may be read from the curve and substituted in the expression for AT. Figure 37, Chap. III, shows curves for the usual commercial materials, the coordinates being plotted in c.g.s. and also in English units. If English units are used, the expression for AT becomes

$$AT = at_a'' l_a'' + 2at_t'' l_t'' + 2at_g'' \delta'' + 2at_s'' l_s'' + 2at_c'' l_c'' + at_y'' l_y'' \quad (8)$$

By making computations of this kind for several different values of Φ, the results may be plotted as in Fig. 8 to obtain the

complete magnetization curve. There is shown in this diagram the magnetization curve for each separate part of the magnetic circuit, that is to say, curves whose abscissas show the ampere-turns required for each of the several parts of the complete circuit and whose ordinates show the corresponding value of Φ. It is readily seen, in accordance with Eq. (7) or (8), that for any value of Φ, as OP, the total excitation (in ampere-turns

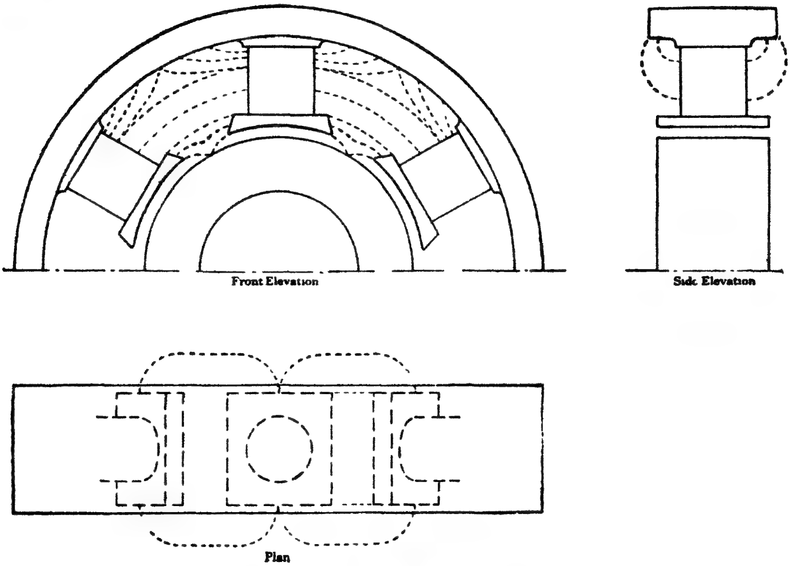


FIG. 9.—Leakage flux.

per pair of poles) is the sum of the abscissas $P1$, $P2$, . . . , each of which represents the excitation for a particular part of the complete magnetic circuit.

5. Magnetic Leakage.—The flux per pole, Φ, may be designated the useful flux, since it is this flux that is involved in the production of the generated e.m.f. But the entire flux produced by the magnetizing action of the field winding does not penetrate the armature, an appreciable part of it "leaking" across from pole to pole in the manner indicated in Fig. 9; in general, leakage flux will exist between all points that have between them a difference of magnetic potential. This leakage flux φ increases the total flux in the pole cores from Φ to

$$\Phi_t = \Phi + \varphi$$

The ratio

$$\frac{\Phi_t}{\Phi} = 1 + \frac{\varphi}{\Phi} = \nu \tag{9}$$

is called the *leakage coefficient* or, preferably, the *coefficient of dispersion.* It is always greater than unity, and in machines of the usual radial multipolar type ranges from about 1.1 to 1.25, the larger values corresponding to small machines. Since the leakage flux must traverse the poles and yokes, the cross-section of these parts must be sufficiently large to carry it as well as the useful flux, and hence the necessity of keeping down leakage as much as possible. The conditions to be satisfied to attain this end are, accordingly, minimum reluctance of the main magnetic circuit and maximum reluctance of leakage paths; these conditions call for a compact magnetic circuit made up of short poles, the interpolar spaces being wide and of small section.

The coefficient of dispersion is not constant for a given machine under all conditions. The leakage flux φ, being mainly in air, is very nearly proportional to the m.m.f., whereas Φ is less and less proportional to the m.m.f. as the saturation of the iron is increased. In general, therefore, $\nu = 1 + (\varphi/\Phi)$ increases more or less with increasing excitation.

Methods for the calculation of the coefficient ν are explained in Chap. IX; for the present it will suffice to state that ν is a function of the dimensions of the machine. This fact introduces a difficulty in designing a new machine; for the flux densities cannot be determined until the dimensions have been decided upon, and the dimensions are themselves dependent upon Φ_t and, consequently, also upon ν. It is therefore necessary in such a case to assume a value of ν in accordance with previous experience and to proceed then to the calculation of Φ_t and the dimensions. The true value of ν can be calculated later and the tentative computations modified in case the discrepancy is sufficiently large to warrant a readjustment.

6. Magnetizing Action of Armature.—In the foregoing discussion of the behavior of a dynamo, it was tacitly assumed that the armature was not carrying any current. Under this no-load condition the magnitude and distribution of the magnetic flux are dependent only upon the excitation due to the field winding and upon the shape and materials of the frame. But under load

conditions the current in the armature conductors gives rise to an independent excitation which alters both the magnitude and distribution of the flux produced by the field winding alone. This magnetizing action of the armature is called *armature reaction.*

For the sake of simplicity, let us examine first the conditions in a bipolar machine. If the armature is without current, the flux due to the field excitation will be symmetrically distributed in the manner illustrated in Fig. 10. The line *ab*, drawn through the center of the shaft at right angles to the polar axis, is the *geometrical neutral* axis; it is an axis of symmetry of the flux

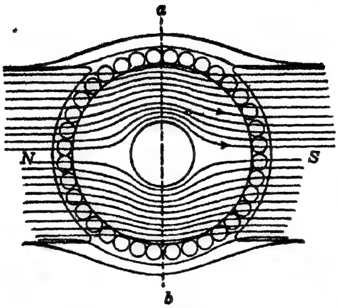


Fig. 10.—Distribution of magnetic field, armature currentless.

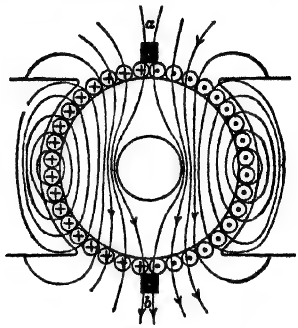


Fig. 11.—Magnetic field due to armature current, field magnets not excited.

under no-load conditions. Armature conductors on opposite sides of the geometrical neutral axis will then be the seat of oppositely directed e.m.fs. If, under no-load conditions, the brush axis coincides with the geometrical neutral, the winding elements lying in the neutral axis will be short-circuited by the brushes during a very brief interval in which only a small e.m.f. is generated in them; hence, the short circuit is harmless.

If the field excitation is now removed and the armature is supplied with current from some external source, there will result a magnetic field the distribution of which is approximately as shown in Fig. 11. Magnetic poles will be developed in the line of the axis of commutation. Most of the flux will be concentrated in the region covered by the pole shoes, since the reluctance there is much less than in the interpolar gap.

Under load conditions, the armature current and the field excitation exist simultaneously, and the resultant flux can be thought of as compounded of the two fields shown separately in Figs. 10 and 11, at least as a first approximation.* The form of the resultant field is shown in Fig. 12, which serves equally well for generator and motor action. Figure 12 shows that in the case of the generator *the field is strengthened at the trailing tips A and A′ and weakened at the leading tips B and B′*, whereas in the case of the motor the exact reverse is true. Moreover, the neutral axis (that in which the winding elements are not cutting lines of induction) has been shifted to the position *a′b′*; the effect is the same as though the flux had been twisted, or skewed, in the direction of rotation in the case of the generator and in the opposite direction in the case of the motor.

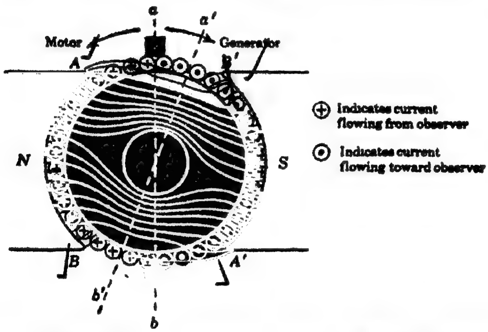


FIG. 12.—Distribution of magnetic field under load conditions.

As a result of the shift of the neutral axis, the brushes (assumed to be in the axis *ab*) short-circuit elements which are cutting lines of induction and in which an active e.m.f. is being generated. Currents of large magnitude may therefore flow in such elements because of the low resistance of the circuit, which includes only

* It is not exactly true that the resultant field is made up of the separate fields of Figs. 10 and 11 as components. What actually happens is that the windings of the field structure and of the armature each produce a definite m.m.f., and that these m.m.fs. then combine to form a resultant m.m.f. which in turn produces the resultant flux. The composition of the separate fields would give correct results only if the flux were at all points proportional to the m.m.f.; and this condition is, of course, not satisfied in the presence of iron cores, especially if the iron is worked at a flux density at or near the knee of the magnetization curve (see Art. 14, page 388).

the short-circuited winding elements and the brush contacts; and the rupture of this current, as the commutator segments pass from under the brush, may cause sparking and perhaps blistering of the commutator. Furthermore, the machine will not develop its full e.m.f.; for of the $Z/2$ conductors in series, say on the left-hand side of the armature of Fig. 12, those between b and b' will generate an e.m.f. opposite in sign to that of the e.m.f. due to conductors between b' and a. Both these effects are objectionable, the former because it reduces the life of the commutator and lowers the efficiency, the latter because it unnecessarily reduces the available output of the machine. Obviously, the remedy for both troubles is to make the distortion, or skewing, so small that the neutral axis is not appreciably shifted; or if this effect cannot be accomplished, to shift the brushes until they are in (or near) the neutral axis; but when the brushes are shifted, the armature field (Fig. 11) moves with them in such a way that the resultant polarization of the armature coincides with the axis of commutation.

The net effect is that the resultant field tends to skew more and more as the brushes are moved toward the neutral axis. Fortunately, however, the piling up of the flux in the pole tips A and A' results in their saturation, so that further skewing becomes insignificant and the brush axis may even pass the neutral axis.

brush

7. Commutation.—It is desirable at this point to examine the phenomena occurring during commutation somewhat more in detail than has yet been done, in order to settle in a general way the conditions that must be satisfied by the brush position. Figure 13 represents three elements a, b, and c of a ring winding operating as a *generator*. It is evident that element a will occupy successively the positions of b and c and that during the b position its current must change from the value existing in conductors to the left of the brush to the equal and opposite value existing in conductors on the right. This change cannot occur instantaneously; in the ideal case the current should change *uniformly* from the initial to the final value in exactly the time required for the element to pass under the brush, during which time the element is short-circuited in the manner of coil b. This variation of current is represented diagrammatically in Fig. 14, where $+I$ and $-I$ are the initial and final values of the current in the

element and T is the duration of the short circuit, or the *period of commutation.*

Now, the self-inductance of the element undergoing commutation tends to keep the current at its original value and in the original direction, and in order to counteract this tendency the e.m.f. of self-induction must be balanced by an opposing e.m.f. If the brushes were exactly in the neutral axis, no e.m.f. would, on the average, be generated in the short-circuited coil, the effect of self-induction would not be opposed, and the reversal of the current could not be satisfactorily completed in the time T. Since the direction of the e.m.f. required to neutralize the e.m.f. of self-induction must be the same as the final direction of the current (see coil c, Fig. 13), it follows that the *short-circuited*

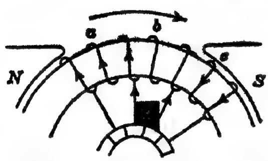


FIG. 13.—Reversal of armature current during commutation (generator).

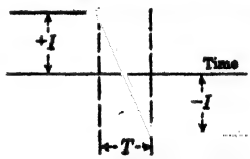


FIG. 14.—Ideal variation of current in commutated coil.

coil must be under the influence of the pole in advance of the neutral axis, that is, in the direction of rotation in the case of a generator.

In the case of a motor similar considerations show that the brushes must likewise be displaced from the neutral axis, but in the backward direction with respect to the direction of rotation. The angular displacement of the brush from the neutral axis will be nearly the same whether the machine is used as a generator or as a motor.

When the angular displacement of the brushes is in the direction of rotation, as in a generator, the angle is called the *angle of brush lead;* when in the direction opposite to the rotation, as in a motor, it is called the *angle of backward lead.*

8. Components of Armature Reaction.—Imagine the brushes of the armature of Fig. 11 supplied with constant current from some external source, the field being unexcited. If the brushes are rocked forward or backward, the armature m.m.f. will follow, and it will remain constant in magnitude. It may, therefore, be represented by a line of constant length, OA, Fig. 15, in line with the brushes. If the fields are now excited, their m.m.f.

may be represented by a line OF (the direction of current flow in armature and field windings being taken the same as in Fig. 12). Upon resolving OA into the components OC and CA, it is seen that the armature magnetizing action is equivalent to a *cross-magnetization* due to OC (so called because it acts across the main m.m.f. OF) and a *demagnetization* due to CA, which directly opposes the main excitation OF. The demagnetizing action of the armature is a direct consequence of the rocking of the brushes to the position most favorable for commutation.

It will be clear from Fig. 15 that, if the brushes of a generator have a backward lead, the armature will assist in magnetizing the field, that is, the axial component CA becomes magnetizing.

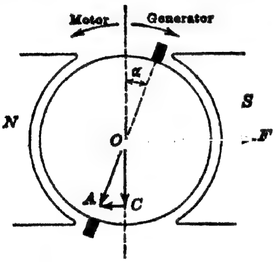


FIG. 15.—Components of armature m m.f.

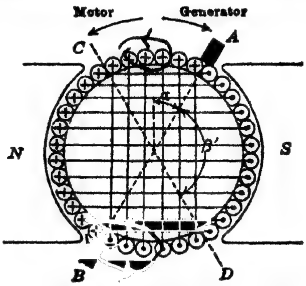


FIG. 16.—Cross-magnetizing and demagnetizing belts of conductors.

A similar result follows if the brushes of a motor are given a forward lead; but in both cases the brushes will spark viciously.

Were it not for the fact that a negative brush lead affects commutation unfavorably, the armature reaction might be purposely exaggerated to such an extent as to self-excite the fields. This feature is taken advantage of in the Rosenberg type of generator for train lighting (see Chap. XIV), but in general, special auxiliary devices are required to take care of the commutation difficulties.

9. Cross-magnetizing and Demagnetizing Ampere-turns.—The resolution of the armature m.m.f. OA in Fig. 15 into components is not a strictly accurate proceeding; it is qualitative rather than quantitative. But it leads directly to the conclusion that the entire armature winding of Fig. 16 may be considered to be made up of two distinct "belts" of conductors, namely, those between AD and CB, and those between CA and BD. The

former conductors, when grouped in pairs in the manner indicated by the horizontal lines, constitute a coil whose magnetizing effect is directly across that of the main exciting winding; they are called the *cross-magnetizing turns*. The remaining conductors, grouped in vertical pairs, constitute the *demagnetizing turns*, since their effect is in direct opposition to the main exciting winding. It follows, therefore, that in a *bipolar* machine the demagnetizing turns per pair of poles are equal to the number of armature conductors within the double angle of lead, 2α; and the *demagnetizing* (or back) *ampere-turns* per pair of poles, AT_d, equal this number multiplied by the current per conductor.

$$\therefore AT_d = \frac{2\alpha Z}{360} \cdot \frac{I_a}{2} = \frac{\alpha Z I_a}{360} \tag{10}$$

Similarly, the *cross-magnetizing ampere-turns* are given by

$$AT_c = \frac{\beta' Z}{360} \cdot \frac{I_a}{2} = \frac{\beta' Z I_a}{720} \tag{11}$$

where

$$\beta' = 180 - 2\alpha$$

It is important to realize that the demagnetizing turns are a consequence of the cross-magnetizing action of the armature; for if the brushes are originally in the geometrical neutral axis, the entire magnetizing action of the armature is across the main field, and the resultant field is thereby distorted. The shifting of the brushes to a position near the resultant neutral axis, for the purpose of improving commutation, then brings the demagnetizing turns into existence.

10. Cross-magnetizing and Demagnetizing Effect in Multipolar Machines.—In the foregoing discussion of the case of bipolar machines, a ring-wound armature was assumed. Accordingly, the brush axis and the axis of commutation coincided, and no distinction was made between them. It must be remembered, however, that the end connections of lap and wave windings are generally so shaped that the brushes are opposite the middle of the poles when the sides of the coil undergoing commutation are in the geometrical neutral.

An extension of the principles developed for the case of the bipolar machine leads to the generalization that all the conductors lying within the double angle of lead have a demagnetizing

effect upon the field, whereas the remaining conductors produce a cross-magnetization. This conclusion holds accurately for all windings of the ring type and for lap and wave windings of full pitch. But in short-chord windings it will be found that the

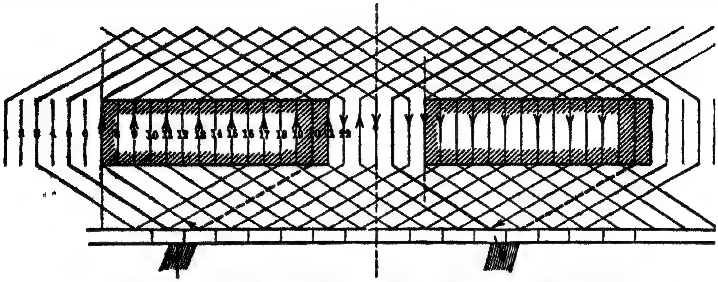


FIG. 17.—Reduction of demagnetizing action caused by fractional pitch of winding.

conductors occupying the space between pole tips carry currents that are partly in one direction and partly in the other, the demagnetizing effect being thus partly neutralized. For example, assume a four-pole simplex lap winding having 80 conductors;

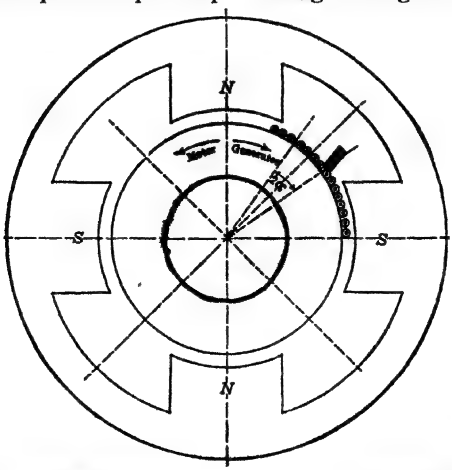


FIG. 18.—Demagnetizing belt of conductors in multipolar machine.

that is, $p = 4$, $a = 4$, $Z = 80$, $m = 1$. Take $y_1 = 15$ and $y_2 = -13$. On tracing through the winding diagram, a portion of which is shown in Fig. 17, it will be found that the current in the interpolar region is alternately in opposite directions.

If we omit from consideration the special case of short-chord windings, the number of demagnetizing ampere-turns per pair of poles can be determined as follows (see Fig. 18):

The total number of conductors lying within the demagnetizing belts is

$$\frac{Z}{360} \cdot 2\alpha \cdot p$$

and, therefore, the number of *demagnetizing ampere-turns per pair of poles* is

$$AT_d = \frac{1}{2} \cdot \frac{Z}{360} \cdot \frac{2\alpha p}{p/2} \cdot \frac{I_a}{a} = \frac{\alpha Z I_a}{180a} \qquad (12)$$

11. Corrected Expression for Demagnetizing Effect of Back Ampere-turns. 1. *Lap Windings.*—Figure 19 is a development

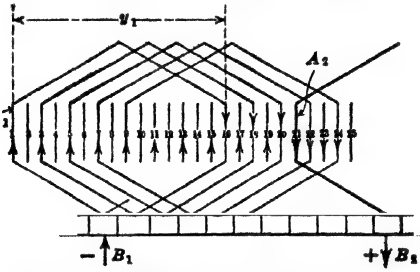


FIG. 19.—Fractional-pitch lap winding.

of that portion of a "chorded" (fractional-pitch) lap winding embraced in a span slightly greater than the pole pitch. It is required to determine the reduction in the value of $AT_d = \alpha Z I_a/180a$ due to the fact that the coils in the neutral zone carry currents which are not all in the same direction.

In the first place, it will be evident that if the winding were of full pitch, all the $2S/p$ coil sides lying to the left of A_2 would carry current in the same direction (vertically upward in the figure). In the second place, however short the chording may be, the current in the coil sides immediately to the right* of, and including, A_2 will all be in the same direction (downward in the figure). It follows, therefore, that the reversed currents all lie in

* This is a consequence of the fact that the winding sketched in Fig. 19 is right-handed, i.e., $y_1 > y_2$. If the winding were left-handed, $y_1 < y_2$, the words "right" and "left" would have to be interchanged.

a zone to the left of A_2, the extent of this zone depending upon the difference between y_1 and the pole pitch. Similarly, there will be zones of reversed currents to the left of all the coil sides, which, like A_1 and A_2, are connected to commutator segments touched by the brushes.

If coil side A_1 is numbered 1, the first coil side carrying reversed current is $(y_1 + 1)$, the second is $(y_1 + 3)$, etc. The number corresponding to the last one in the group will evidently be $\frac{2S}{p} = y_1 + \left(\frac{2S}{p} - y_1\right)$.

Summarizing these results,

$$\begin{array}{ll} y_1 + 1 & \text{corresponds to the 1st.} \\ y_1 + 3 & \text{corresponds to the 2d.} \\ y_1 + (2k - 1) & \text{corresponds to the } k\text{th.} \\ y_1 + \left(\frac{2S}{p} - y_1\right) & \text{corresponds to the } n\text{th.} \end{array}$$

Thus, there are n of such reversed bundles, where

$$2n - 1 = \frac{2S}{p} - y_1$$

or

$$n = \frac{1}{2}\left(\frac{2S}{p} - y_1 + 1\right) \tag{13}$$

Since the current in each of these n coil sides balances the demagnetizing effect of the current in n bundles the direction of which is normal, the total reduction will be that due to $2n$ bundles. Since each element contains $Z/2S$ turns, AT_d will be less than the computed value by

$$\frac{Z}{2S}\frac{I_a}{a}\left(\frac{2S}{p} - y_1 + 1\right) \quad \text{ampere-turns} \tag{14}$$

It should be noted that extreme chording may cause some of the n reversed coil sides to fall outside the double angle of lead and, therefore, vitiate the correction above. But such extreme chording would not be used in practice, and hence the correction may be safely used. It should be noted, further, that neither the formula for AT_d nor that for the correction due to chording takes account of the number of coil sides in the

neutral zone which are short-circuited by the brushes during commutation.

2. *Wave Windings* (Fig. 20).—If y_1 is the back pitch of the winding and y_2 is the front pitch,

$$y = \frac{y_1 + y_2}{2} = \frac{2S + a}{p}$$

the positive sign of a indicating that the winding is right-handed. The extent to which y_1 falls short of the pole pitch is then a measure of the chording; obviously, then, $y_2 > y_1$. Full-pitch winding would result in uniform opposition of direction of current on either side of coil sides like A_1, A_2 (Fig. 20) that are in contact with the brushes.

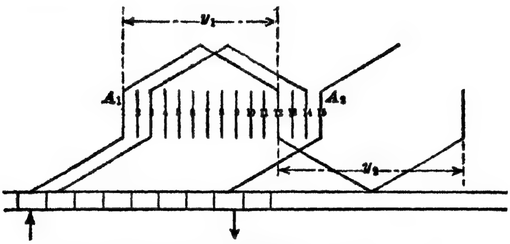


FIG. 20.—Fractional-pitch wave winding.

Owing to chording, however, the first coil side that carries reversed current is $(y_1 + 1)$, the second is $(y_1 + 3)$, etc. If there are n such coil sides, the number of the nth coil side will be $y_1 + (2n - 1)$. All these coil sides bear even numbers, since y_1 is necessarily odd. Now, the number of coil sides per pole pitch is $2S/p = y - (a/p)$, which is a mixed number but which may be taken equal to y. If y is odd, the last even number in the group is $y - 1$, whence

$$y - 1 = y_1 + (2n - 1)$$

or

$$2n = \frac{y_2 - y_1}{2} \tag{15}$$

If y is even,

$$y = y_1 + (2n - 1)$$

$$2n = \frac{y_2 - y_1}{2} + 1 \tag{16}$$

Hence, the number of ampere-turns to be deducted from AT_d is

$$\left.\begin{array}{ll} \dfrac{Z}{2S}\dfrac{I_a}{a}\left(\dfrac{y_2 - y_1}{2}\right) & \text{if } y \text{ is odd} \\ \dfrac{Z}{2S}\dfrac{I_a}{a}\left(\dfrac{y_2 - y_1}{2} + 1\right) & \text{if } y \text{ is even} \end{array}\right\} \qquad (17)$$

12. Shape of Magnetic Field Produced by Armature Current.—The current in the armature conductors lying to one side of the commutated coil has a direction opposite to that of the current

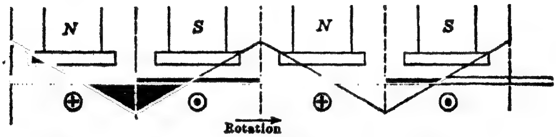


FIG. 21.—Peripheral distribution of armature m.m.f.

in the conductors on the other side; or the current is distributed around the periphery in a series of alternately directed bands or belts, equal in number to the number of poles. This fact is indicated in Fig. 21, which represents a development of a four-pole generator. The peripheral distribution of m.m.f. due to the armature current is represented by the ordinates of the broken line. The m.m.f. is a maximum at the points where the current reverses, and it is zero at the points midway between them.

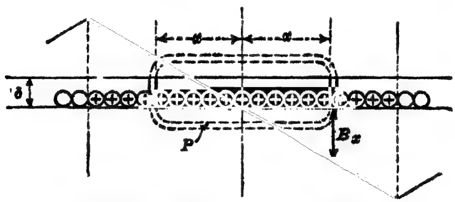


FIG. 22.—Calculation of armature flux.

If the pole shoes completely surrounded the armature and the surface of the armature were smooth, the armature flux would at all points be directly proportional to the m.m.f., since in such case the reluctance would be constant all around the gap (the reluctance of the flux paths in the iron being neglected in comparison with the reluctance of the air paths).

The number of conductors surrounded by the elementary tube of flux P, Fig 22, is $(Z/\pi d) \cdot 2x$ where d is the diameter of the

armature. Since each conductor carries I_a/a amp., I_a being the total armature current, the m.m.f. acting on tube P is $\frac{4\pi}{10}\frac{Z}{\pi d}\frac{I_a}{a}\cdot 2x = \frac{4\pi}{10}q\cdot 2x$, where $q = \frac{Z}{\pi d}\cdot\frac{I_a}{a}$ is the number of *ampere-conductors per unit length* of periphery. The flux density at a point distant x from the center of the pole is

$$B_x = \frac{(4\pi/10)q\cdot 2x}{2\delta}\mu_0 = \frac{xq\mu_0}{0.8\delta} \tag{18}$$

In the usual case of machines having pole shoes separated from each other by an intervening air space, the flux distribution curve is not similar in form to the curve of m.m.f. Under the pole shoe it will closely follow the m.m.f. curve because of the practically

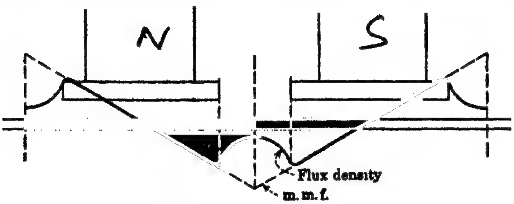


FIG. 23.—Distribution of armature flux.

uniform reluctance; but between the pole tips the reluctance increases at a much greater rate than the m.m.f., and hence the armature flux density will be small at the point midway between them, as shown in Fig. 23.

13. Approximate Distribution of the Resultant Field.—Parts a, b, and c of Fig. 24 represent the effect of the armature field in modifying the magnitude and distribution of the resultant magnetic field for three positions of the brushes. In each diagram, curve F shows the flux distribution due to the field excitation alone, curve A is the flux curve due to the armature, and curve R is the resultant of F and A. The diagrams are drawn for the cases of commutation:

a. Midway between pole tips.

b. Between the pole tips but nearer the leading tip.

c. Under the middle of the poles.

In case a the distortion of the magnetic field is clearly shown. The symmetry of curve A with respect to F means that the flux added to the trailing tip (generator action being assumed) is exactly equal to the flux removed from the leading tip, and hence

the flux per pole remains unaltered. In case *b* there is distortion and also a resultant demagnetization, as it is clear that the flux *A* under a pole is more subtractive than additive. In case *c* there is no distortion, but only demagnetization, as might be expected from the fact that the brushes have been shifted to such an extent as to eliminate all the cross ampere-turns and to raise the back ampere-turns to a maximum value.

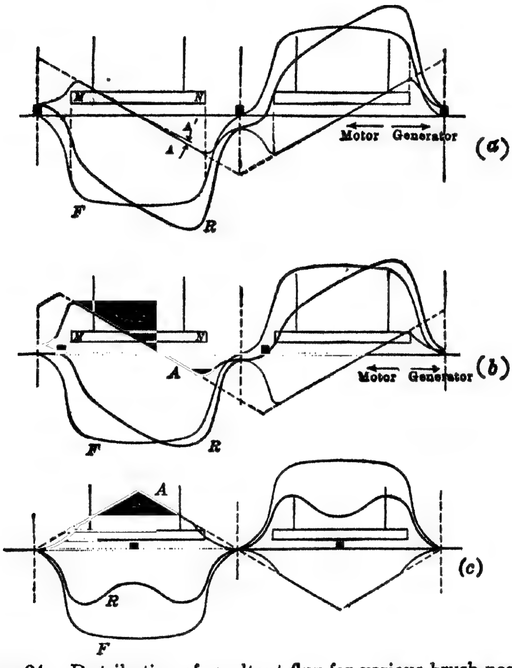


FIG. 24.—Distribution of resultant flux for various brush positions.

14. Demagnetizing Component of Cross-magnetization.—In the preceding article the shape of the resultant field *R* was determined on the theory that it may be considered to be made up of two components: one, a field produced by the m.m.f. of the field winding, acting alone; the other, a field produced by the armature m.m.f., acting alone. As a matter of fact, this theory is not strictly correct, as the following analogy will show:

Imagine a rod of cast iron acted upon simultaneously by compressive and tensile stresses, and suppose that these stresses are

equal. If we assume that the stresses act independently in deforming the rod, the elongation due to the tension would considerably exceed the shortening due to the compression, provided that the tensile stress is beyond the elastic limit; on this basis there would be a resultant elongation. But it is clear from the assumed equality of the stresses that the resultant stress and, therefore, the resultant deformation are both zero, and hence the absurdity of the first method.

In the case of the magnetic circuit, m.m.f. is analogous to stress, and flux to deformation. Hence, we must conclude that the only correct procedure is first to combine the several m.m.fs. to form a single resultant and from the latter determine the distribution and magnitude of the resultant flux.

It follows from the considerations above, taken in connection with diagrams *a* and *b* of Fig. 24, that the increased m.m.f. on the side *N* of the pole shoe cannot raise the resultant flux on that side to the same extent as the diminished m.m.f. at *M* lowers the flux on that side, because of the fact that the permeability of the iron of the pole shoe and armature teeth decreases with increasing magnetizing force. Therefore, even when commutation takes place midway between the poles, corresponding to zero brush lead and an entire absence of back ampere-turns, there is still a resultant demagnetizing action due to the cross ampere-turns; although this effect is usually not very pronounced, it is nevertheless appreciable. The general nature of the effect is indicated in Fig. 24*a*, where the broken line marked *A′* indicates that the increased flux due to *A* on the side of the pole marked *N* is less than proportional to *A*. A more detailed analysis of the magnitude of this demagnetizing component of the cross-magnetizing action is given in the following article.

16. Excitation Required under Load Conditions.—Let curve *OM* of Fig. 25 represent the saturation curve of a generator, and let *OV* be the terminal voltage *V* at rated load. The excitation required to generate this e.m.f. at no load is $(AT)_0$ ampere-turns per pair of poles. When the armature delivers current to the load, the excitation required to maintain constant terminal voltage must be greater than $(AT)_0$ in order to compensate:

1. The ohmic drop, or drop in potential, caused by the flow of the current through the resistance of the armature.

2. The demagnetizing effect of the armature back ampere-turns.

3. The demagnetizing component of the armature cross ampere-turns.

1. If the terminal voltage under load conditions is to remain the same as at no load, the total *generated* e.m.f. E_1 must be greater than V by the drop I_aR_a, where R_a is the resistance of the armature. With reference to Fig. 25, the excitation required to generate E_1 volts is $(AT)_1$.

2. If the armature demagnetizing ampere-turns per pair of poles (corrected, if necessary, to take care of chording) amount to $(AT)_d$, the field excitation must be increased over and above

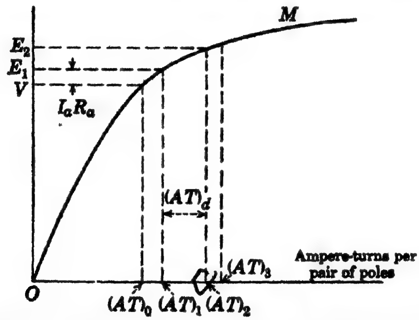


FIG. 25.—Excitation required under load conditions.

$(AT)_1$ by $(AT)_d$, that is, to the value $(AT)_2$ corresponding to an open-circuit e.m.f. of E_2 volts.

3. Because of the fact that saturation of the pole tips may reduce the flux at one pole tip more than it is increased at the other, it may be necessary to increase the field excitation still further, as from $(AT)_2$ to $(AT)_3$, Fig. 25, in order that the terminal voltage under load conditions may be maintained at its no-load value. In order to determine this increase, the following facts may be noted:

The cross field which is responsible for the demagnetization of one pole tip and the magnetization of the other is due to the conductors lying under the pole face, and the path of this cross flux is indicated by the line marked C, Fig. 26. The number of cross ampere-turns acting around this path is $(\beta Z/360) \cdot (I_a/a)$, half of which oppose the main excitation at pole tip A, the other half reinforcing it at B. The main field excitation acts on the

circuit marked F. Circuits C and F have in common the reluctance made up of the *double airgap, two sets of teeth, the armature core* (in part), *and the pole shoes*, and these parts constitute a large percentage of the reluctance of the entire magnetic circuit. If the difference between the reluctances of circuits C and F were negligible, it would follow that a given m.m.f. acting around such a circuit as C would produce a flux equal to that produced by the same m.m.f. acting around circuit F, in which case the relation between flux and m.m.f. would be represented by the magnetization curve of the machine. Actually, however, the relation between flux and m.m.f. in such a circuit as C, Fig. 26, will be given by the curve marked C, Fig. 27, which lies to the left of the magnetization curve M; ordinates of this curve represent flux, and abscissas represent the ampere-turns required to maintain this flux through the double airgap, two sets of teeth, and the small reluctance of the pole shoes and armature core comprised in circuit C, Fig. 26. Abscissas of curve C, Fig. 27, are less than those of curve M for a given flux because of the greater reluctance of the entire magnetic circuit. In Fig. 8, the summation of the abscissas of the curves corresponding to the double airgap, the two sets of teeth, the armature core, and the two pole shoes will give the curve C of Fig. 27, in which the abscissa $(AT)_1$ represents the ampere-turns per pair of poles required to develop the e.m.f. E_1, as in Fig. 25; abscissa OP then represents the number of ampere-turns required to produce a corresponding flux in the circuit C, Fig. 26. These relations are shown separately in Fig. 28. When the armature is delivering a current

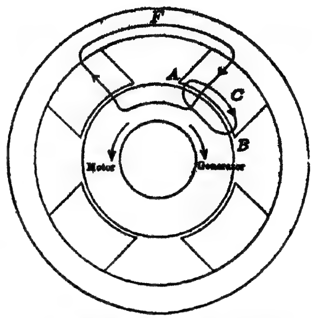


Fig. 26.—Comparison of paths of main and cross field.

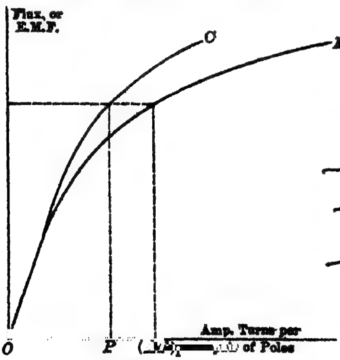


Fig. 27.—Magnetization curves.

I_a, the brushes being assumed to be in the neutral axis, the excitation at the middle of the pole face MN is not affected by the cross-magnetizing action; but at the tip M the magnetizing action is decreased by bd ampere-turns, and at the tip N it is increased by be ampere-turns, where $bd = be = (\beta Z/360) \cdot (I_a/a)$. The flux density in the airgap at M is therefore proportional to the ordinate of the point a of curve C, that at N is proportional to the ordinate of point c, and at intermediate points the flux density is propor-

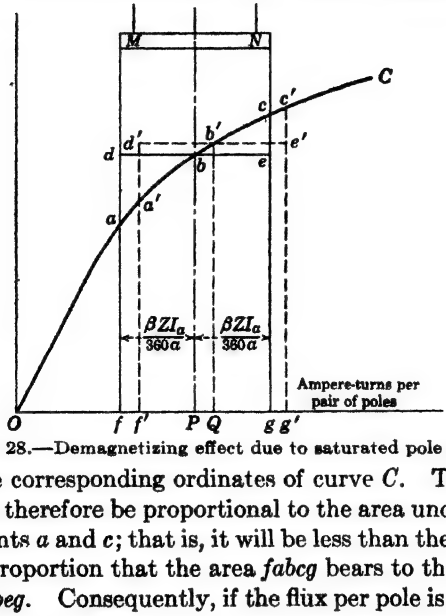


FIG. 28.—Demagnetizing effect due to saturated pole tips.

tional to the corresponding ordinates of curve C. The total flux per pole will therefore be proportional to the area under the curve between points a and c; that is, it will be less than the undistorted flux in the proportion that the area $fabcg$ bears to the area of the rectangle $fdbeg$. Consequently, if the flux per pole is not to suffer a reduction, the excitation will have to be increased by an amount PQ, such that the area under the curve between points a' and c' shall be the same as that of the rectangle $fdeg$. The excitation PQ is the amount that must be added to $(AT)_2$, Fig. 25, to give the required total excitation, $(AT)_3$ in that figure.

It is possible to compute the magnitude of the additional excitation PQ if the analytical expression for the curve C of Fig. 28 is known. A close approximation to such an expression, within the limits of the curve that enter into this problem, is given by Froelich's equation

$$y = \frac{ax}{b + x}$$

The diagram of Fig. 28 has been reproduced in somewhat different form in Fig. 29, abscissas (ampere-turns per pair of poles) being represented by the symbol x and the quantity $\beta ZI_a/360a$ by c. The area of the rectangle *fdeg* of Fig. 28 is

$$2cy_0 = \frac{2acx_0}{b + x_0} = \text{area of rectangle} \tag{19}$$

and the area under the curve between abscissas x_1' and x_2' is

$$\int_{x_1'}^{x_2'} y\,dx = a\int_{x_1'}^{x_2'} \frac{x\,dx}{b + x} = a\left[x_2' - x_1' - b \log_e \frac{b + x_2'}{b + x_1'}\right] \tag{20}$$

Upon equating the expressions (19) and (20) and substituting the identity $x_2' - x_1' = 2c$, it follows that

$$\log_e \frac{b + x_0' + c}{b + x_0' - c} = \frac{2c}{b + x_0} \tag{21}$$

The increased excitation PQ, Fig. 28, is equal to $x_0' - x_0$, where x_0 is the excitation $(AT)_2$ of Fig. 25. Since $c = \beta ZI_a/360a$ and b is known when Froelich's equation has been determined, Eq. (21) readily serves to compute x_0' and therefore $(x_0' - x_0)$, also.

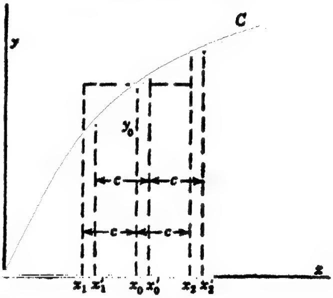


FIG. 29.

16. Limitation upon Armature M.M.F.—The distortion and possible weakening of the main magnetic field produced under load conditions by the cross-magnetizing action of the armature implies a limit within which the armature m.m.f. must remain unless special features for compensating armature reaction are incorporated in the design. In particular, it is essential that at the weakened pole tip—the leading tip in a generator and the trailing tip in a motor—the flux density B_c, conceived to be produced by the armature acting independently, shall remain smaller than the flux density B_g set up by the main excitation acting alone; for if B_c were greater than B_g, the field necessary to ensure reversal of current in the coil undergoing commutation would have the wrong direction.

The cross-magnetizing m.m.f. acting upon a path C, Fig. 30, amounts to $(\beta Z/360)(I_a/a)$ amp.-turns; and if the armature

core were smooth, half of it would be consumed in the airgap at each pole tip, on the assumption that the reluctances of pole face and armature core are negligible. Under these conditions the flux density B_c would be

$$B_c = \frac{4\pi}{10} \frac{\beta Z I_a}{360a} \cdot \frac{\mu_0}{2\delta} \tag{22}$$

where δ cm. is the length of airgap at the pole tip. But in the case of a slotted armature (Fig. 30), the teeth at tip T_2 become highly saturated, so that the reluctance at that point is much greater than at tip T_1. The effect is the same as though in a smooth-core armature the airgap δ_1 at T_1 were smaller than the gap at T_2; incidentally, the main flux Φ would in that case not divide equally between the two halves of the pole face, somewhat more than half of it passing to the left in Fig. 30, and less than half to the right. The actual distribution could be found only by a trial-and-error method. In general, however, it may be said that the value of B_c at tip T_1 is less than that given by Eq. (22), so that

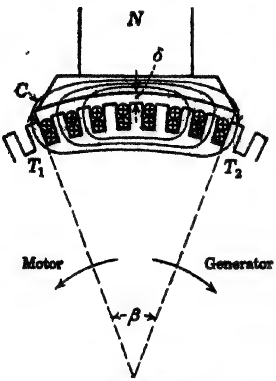


FIG. 30.—Cross field in multipolar machine.

$$B_c = \frac{4\pi}{10} \frac{\beta Z I_a}{360a} \cdot \frac{\mu_0}{2k\delta} \tag{23}$$

where $k > 1$ is a factor that compensates for these disturbing influences.

In general, in order that there may be a commutating field of sufficient strength, B_g must be greater than B_c by about 2000 to 3000 lines per sq. cm.; and since B_g is usually between 6000 and 10,000, it follows that

$$B_g = (1.25 \text{ to } 2.0)B_c \tag{24}$$

The excitation required to maintain the flux density B_g being given by

$$B_g = \frac{4\pi}{10} AT_g \frac{\mu_0}{\delta} \tag{25}$$

substitution of the relations (23) and (24) yields

$$AT_g = (1.25 \text{ to } 2.0)\frac{\beta Z I_a}{360a} \cdot \frac{1}{2k} \tag{26}$$

Now, $\beta = 360\psi/p$, where ψ, the ratio of pole arc to pole pitch, is usually about 0.7 in d-c machines; $(Z/2) \cdot (I_a/a)$ is the total number of *armature ampere-turns;* and AT_g is ordinarily 0.7 to 0.9 of the total *field ampere-turns* per pole. Upon inserting these relations in Eq. (26), it is found that

$$\text{Armature amp.-turns per pole} = (0.5k \text{ to } 1.0k) \text{ field amp.-turns per pole} \quad (27)$$

Ordinarily, the armature ampere-turns per pole amount to 80 or 90 per cent of the field ampere-turns per pole; but if auxiliary devices such as commutating poles or compensating windings are used, as described later, the armature may be magnetically much more powerful.

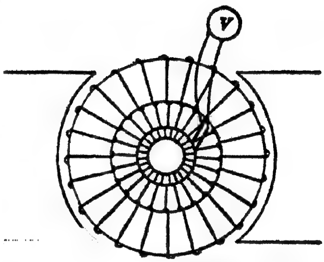


FIG. 31.—Double pilot brush.

17. Experimental Determination of Flux Distribution.—Since the instantaneous e.m.f. generated in an armature conductor is proportional to the radial component of the flux density at the point occupied by the conductor at the moment in question (see Art. 2, Chap. VI), the measurement of this e.m.f. will provide data for the calculation of the flux distribution.

Consider the case of a simplex ring-wound armature (Fig. 31) provided with such a large number of commutator segments that the turns of each element may be assumed to be concentrated in a radial plane, or that all the turns of the element (if there be more than one turn per element) are simultaneously in a field of the same intensity. Take a narrow strip of tough paper (sheet fiber or pressboard) the length of which is equal to the periphery of the commutator, and along its axis drill a series of small holes the spacing of which is the same as that of the commutator bars. Wrap the strip around the commutator, and fasten it to the brush studs so that the commutator may rotate within it without binding. The free ends of a pair of leads connected to a low-reading voltmeter are then to be provided with contact points made of moderately hard lead pencils. When the contact points are inserted into adjacent holes in the strip, the reading of the voltmeter will be equal to the e.m.f. generated in the

element minus the ohmic drop due to the current flowing through the resistance of the element, on the assumption that the experiment is made when the machine (generator) is running under load conditions.

Instead of the perforated strip described above, there may be employed a "pilot" brush made of two thin pieces of sheet brass screwed on opposite sides of a strip of wood or ebonite, the thickness of which is such that the metal strips are separated by the distance from center to center of adjacent commutator segments.

A similar arrangement will suffice in the case of a simplex lap or a simplex wave winding, provided that the elements are of full pitch; moreover, in the case of the simplex wave winding, the voltmeter reading will be due to $p/2$ elements in series, instead of only one element. If the winding is duplex, the spacing of the contact points must be two segments.

As stated above, the observed readings of the voltmeter are less than the true values of generated e.m.f. by an amount equal to the ohmic drop in the element (or elements) due to load current, if the machine, used as a generator, is supplying an external circuit. In order to eliminate this correction of the observed readings, an auxiliary "search" coil, of full pitch, may be wound on the armature, one of its terminals being grounded on the shaft and the other connected to an insulated stud on the end of the shaft. Connect one terminal of a voltmeter of the D'Arsonval type to the frame of the machine (or, better, to a metal brush rubbing on the shaft) and the other terminal to a brush that makes contact once per revolution with the insulated stud. If the moving coil of the voltmeter has sufficient inertia and is well damped, it will give a steady reading proportional to the e.m.f. generated in the search coil at the instant when the contact is established between the brush and the rotating stud. If the brush is made capable of adjustment around the arc of a circle concentric with the shaft, the contact can be made to occur when the search coil is in any desired position under the poles.

18. Potential Curve.—In Fig. 32 the ordinates $e_1, e_2, e_3, \ldots,$ represent the e.m.fs. generated in individual coils. It is evident, therefore, that the expression

$$e_1 + e_2 + e_3 + \cdots + e_n$$

will be equal to the reading of a voltmeter one of whose terminals is connected to the main brush B and the other to a single pilot brush separated from B by an angle equivalent to the spread of n coils. If the pilot brush (P, Fig. 33) is moved around the

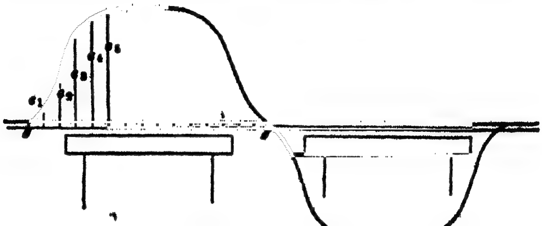


FIG. 32.—Variation of voltage per element.

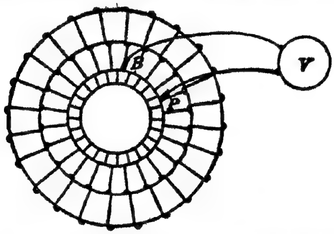


FIG. 33.—Determination of potential curve.

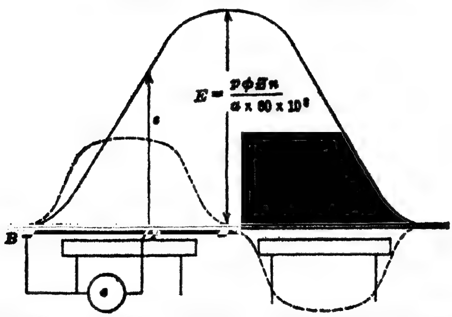


FIG. 34.—Relation between voltage per element and potential curve.

periphery of the commutator and voltmeter readings (corrected for drop of potential if current is flowing) are taken at various points, a *potential curve* of the kind shown in full line in Fig. 34 will result.

Since any ordinate e of this curve is the sum of the ordinates of the dotted curve (which is the same as that of Fig. 32) lying

to the left of e, it follows that the first derivative of the function which represents the potential curve will represent the curve of flux distribution, provided that the winding is divided into a large number of elements; or the slope of the potential curve at any point is proportional to the e.m.f. generated in the coil corresponding to that point.

19. Predetermination of Flux Distribution in the Airgap.—The change in the distribution of the airgap flux due to the

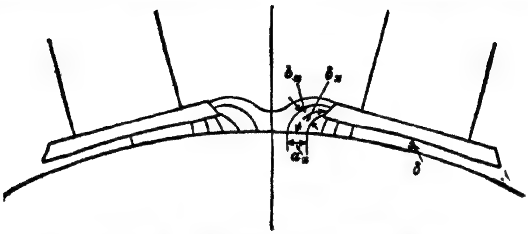


FIG. 35.—Distribution of main field.

magnetic reaction of the armature current is very important because of its effect upon the commutating characteristics of the machine, as explained in a preliminary manner in Art. 7, page 378, and in greater detail in Chap. XII. It is therefore occasionally desirable, in designing a new machine, to be able to predetermine the curve of flux distribution due to the field excitation alone (curve F, Fig. 24) and also the curve of flux distribution due to the armature m.m.f. (A, Fig. 24). Several methods* for determining these curves have been developed; but all of them, except that of Carter, are approximate, and Carter's method, though mathematically correct, is derived by assuming a simple shape of pole core and pole face that is not ordinarily used in practice.

For determining the curve of field-flux distribution, the method recommended by Arnold involves mapping the paths of the lines of force in the airgap and in the interpolar spaces in the manner illustrated in Fig. 35. This leaves much to one's judg-

* W. E. GOLDSBOROUGH, *Trans. A.I.E.E.*, **15**, 515; **16**, 461; **17**, 679. S. P. THOMPSON, "Dynamo Electric Machinery," 7th ed., Vol. II, p. 206. F. W. CARTER, *Elec. World*, **38**, 884, 1901. E. ARNOLD, "Die Gleichstrommaschine," 2d ed., Vol. I, p. 320. T. LEHMANN, *Elektrotech. Z.*, **30**, 996, 1019, 1909. B. G. LAMME, *Trans. A.I.E.E.*, **30**, Part III, 2362, 1911. C. R. MOORE, *Trans.* A.I.E.E., **31**, Part I, 509, 1912. F. W. CARTER, *Jour. Inst. Elec. Eng.*, **64**, 1115, 1926.

ment, but some guidance is afforded by the consideration that the lines are substantially perpendicular to the surfaces of the pole faces and of the armature where they enter or leave the iron; it is also true that the flux will distribute itself in such a manner

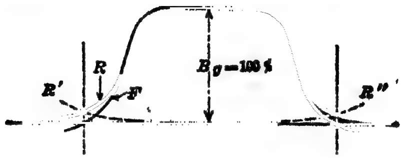


FIG. 36.—Curve of flux distribution.

that the total reluctance is a minimum, or, what amounts to the same thing, for a given m.m.f. between the pole face and armature surface, the total flux will be a maximum. If several trial solutions are carried out, that which yields the largest total flux will be most nearly correct.

Under the central part of the pole, where the airgap δ is uniform, the flux density B_g will also be uniform and inversely

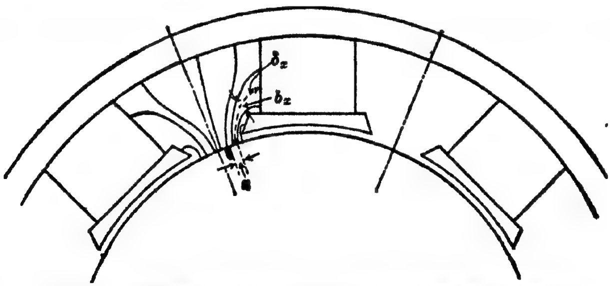


FIG. 37.—Magnetic lines of force due to armature current.

proportional to δ; at any other point, a tube of force of length δ_x and mean section b_x (a unit length along the core being taken) will have a permeance $\mu_0 b_x/\delta_x$, and hence the flux density at the armature surface will be

$$B_x = \frac{\text{m.m.f.}}{a_x} \cdot \frac{\mu_0 b_x}{\delta_x} = B_g \frac{\delta}{\delta_x} \cdot \frac{b_x}{a_x} \tag{28}$$

If B_g is taken as 100 per cent, values of B_x can then be found from the scaled values of δ_x, b_x, and a_x, and the computed results when plotted along the developed armature surface will determine a curve like R, Fig. 36. Curves R' and R'' represent portions of

similar curves for adjoining poles (of opposite polarity), so that the resultant of R, R', and R'' gives the desired curve F. The area of one loop of curve F, multiplied by l' (the corrected length of armature core), must then equal Φ.

The determination of the curves of flux distribution due to the armature m.m.f. (curve A, Fig. 24) is more difficult than in the case of the field flux, but the same general method is applicable.

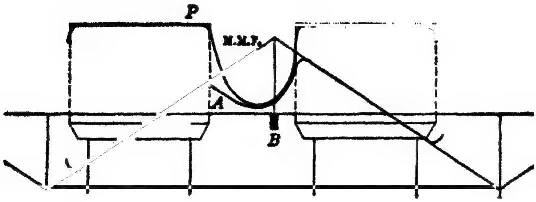


FIG. 38.—Calculation of flux distribution due to armature m.m.f.

Thus, in Fig. 37 are indicated the paths of the lines of force issuing from the armature. At any distance x from the brush the permeance of the tube of force of unit depth parallel to the shaft is equal to $\mu_0 b_x/\delta_x$; and if the peripheral distribution of permeance is plotted as curve P of Fig. 38 and the ordinates of curve P are multiplied by the corresponding ordinates of the curve of armature m.m.f. (M.M.F., Fig. 38), the resultant products will give the curve of armature flux A. The field due to the armature m.m.f. has greatest influence in the axis of commutation, shown at B in the figure; in the paper by Lamme (*loc. cit.*) it is recommended that the mean path of the flux issuing from the axis of commutation be taken as the arc of a circle extending to the middle point of the pole core and intersecting the surfaces of armature and pole core at right angles.

CHAPTER IX

PREDETERMINATION OF MAGNETIZATION CURVE AND MAGNETIC LEAKAGE

1. General.—The important part played by the magnetization curve in determining the performance characteristics of a machine, whether used as a generator or as a motor, has been emphasized in Chap. VIII, but the full significance of its effect cannot be appreciated until it has been shown, as in Chaps. X and XI, how the generator and motor characteristics can be predicted from the shape of the magnetization curve and the principal winding data. The designer who must proportion all parts to meet given operating specifications is naturally obliged to be thoroughly familiar with the methods of computing the magnetization curve, but even operating engineers should know something of the design considerations which underlie performance specifications, if for no other reason than that those specifications may not call for impossible or conflicting conditions.

2. Ampere-turns Required for the Airgap.—The great permeability of iron as compared with air is responsible for the fact that the reluctance of the airgap often constitutes 70 to 90 per cent of the entire reluctance of the magnetic circuit. The accurate determination of the excitation consumed in the airgap is, therefore, of predominant importance.

Two different cases arise in practice: (1) smooth-core armatures, and (2) slotted armatures.

1. *Smooth-core Armatures.*—In a machine having p poles, the angle subtended by the pole pitch is $2\pi/p$. The angle β subtended by the pole shoe is usually between 0.55 and 0.7 of $2\pi/p$; the quantity $(\beta \times 100)/(2\pi/p)$ is called the percentage of polar embrace. If the flux crossed the airgap along radial lines, the determination of B_g and AT_g would be very simple; actually, however, the flux spreads out beyond the pole tips, and there is a further spreading at the flanks, as illustrated in Figs. 1 and 2.

The flux always distributes itself in such a way that the total reluctance is a minimum. The spreading of the flux is equivalent to

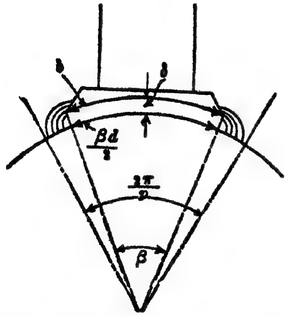


FIG. 1.—Fringing flux at pole tips.

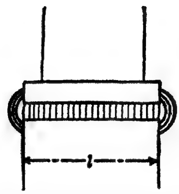


FIG. 2.—Spread of flux at flanks of pole shoes.

increasing b to b' and l to l'. The mean flux density in the gap is, accordingly, $B_g = \Phi/b'l'$; therefore,

$$AT_g = \frac{0.8\ B_g\delta}{\mu_0} \tag{1}$$

all dimensions being in centimeters. In inch units,

$$AT_g = \frac{0.8}{\mu_0}\frac{B_g''}{(2.54)^2}(\delta'' \times 2.54) = \frac{0.3133\ B_g''\delta''}{\mu_0} \tag{2}$$

For practical purposes it is sufficiently accurate to take b' as the average of the polar arc b and of the arc on the armature subtended by the angle β and increased by 2δ on each side; that is,

$$b' = \frac{1}{2}\left[b + \left(\frac{\beta d}{2} + 4\delta\right)\right] = \frac{\beta}{2}(d + \delta) + 2\delta \tag{3}$$

Similarly, l' may be taken as

$$l' = l + 2\delta \tag{4}$$

in case the axial lengths of pole shoes and armature core (between heads) are the same. If these lengths are not equal, let them be represented by l_s and l, respectively; then $b'l'$ in the preceding equation for B_g should be replaced by an area A_g such that

$$A_g = \frac{A_g' + A_g''}{2} \tag{5}$$

where A'_g is the area of the pole shoe and A''_g is the area on the armature core threaded by the flux. Obviously,

$$A'_g = bl_s \tag{6}$$

and

$$A''_g = \left(\frac{\beta d}{2} + 4\delta\right)l \tag{7}$$

2. *Slotted Armatures.*—In this case the calculation of B_g is complicated by the fact that the flux tends to tuft at the tips of the teeth and that more or less of it enters the teeth by way of the slots, as indicated in Fig. 3. It is clear that a given difference of magnetic potential between the pole face and the armature core will produce less flux when slots are present than when the armature surface is smooth, the clearance δ being the same in both cases. In other words, the slots increase the gap reluctance, and this effect may be allowed for either by assuming δ to have been increased to a larger value or by assuming a contraction of the pole arc b to a smaller value b'.

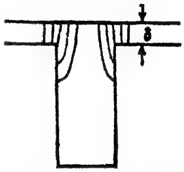


FIG. 3.—Fringing flux at tooth tip.

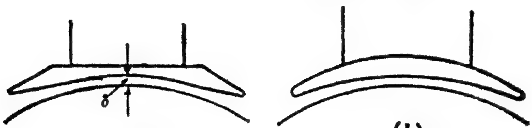


FIG. 4.—Chamfered and eccentric pole shoes.

The problem is further complicated by the fact that the air-gap is frequently not of uniform length over the entire pole face. To improve commutation it is common to chamfer the pole tips (Fig. 4*a*) or to make the cylindrical surfaces of the armature and pole face eccentric (Fig. 4*b*). The effect of the increased gap length at the pole tips is to produce a fringing flux in the interpolar space, as shown by the flux distribution curve of Fig. 5. Ordinates of this curve represent the radial component of flux density at corresponding points on the armature periphery. The ripples at the crest of the curve are caused by the slots and teeth.

Similarly, there is a fringing field at the ends of the core, as shown in Fig. 6; and if ventilating ducts are provided, there will be dips in the curve of axial flux distribution corresponding

to the depressions opposite the ducts, just as in Fig. 5. The extra reluctance due to the ventilating ducts is equivalent to a reduction in the axial length l, and the fringing at the flanks is equivalent to an increase in l, so that the two effects tend to neutralize each other.

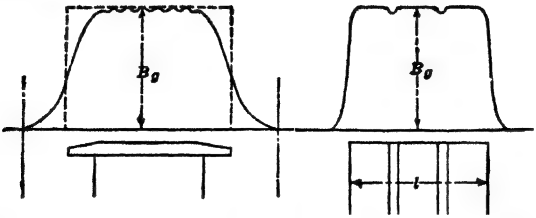


FIG. 5. FIG. 6.
FIGS. 5 and 6.—Peripheral and axial distribution of field intensity.

3. Correction to Pole Arc.—It has been shown by F. W. Carter* that the presence of slots increases the effective length of the airgap from δ to δ', where

$$\delta' = \delta \frac{t}{t - \sigma b_s} \tag{8}$$

and where t = tooth pitch, and b_s = width of slot opening and

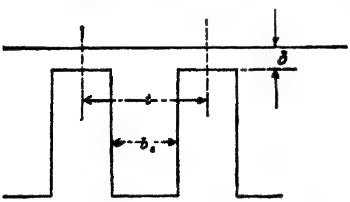


FIG. 7.—Dimensions of teeth and slots.

$$\sigma = \frac{2}{\pi}\left[\text{arc tan}\,\frac{b_s}{2\delta} - \frac{\delta}{b_s}\ln\left(1 + \frac{b_s^2}{4\delta^2}\right)\right] \tag{9}$$

If, however, the effect of the slots is taken into account by reducing the pole arc instead of lengthening the gap, it follows that

$$b' = b\frac{t - \sigma b_s}{t}$$

Values of the factor σ are plotted in Fig. 8 in terms of the ratio b_s/δ = slot opening/clearance.

The fringing field at the pole tips is equivalent to an increase in the value of b, but this effect is generally offset by the increased gap space at the tips.

* Air-gap Induction, *Elec. World*, **38**, 884, 1901.

4. Corrected Axial Length.—The reluctance due to the ventilating ducts may be considered as reducing the axial length l to

$$l'_1 = l\frac{t_v - \sigma b_v}{t_v} \tag{10}$$

where t_v is the distance between centers of the ventilating ducts and b_v is the width of the duct (Fig. 9), and where σ is to be

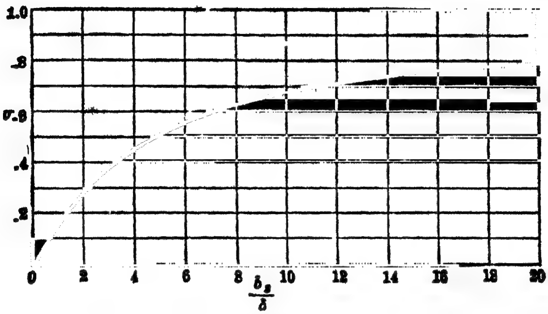


FIG. 8.—Correction factor, σ.

found from Fig. 8, b_v/δ being used as argument. The length l'_1 can be further corrected to take account of the flux that enters the sides of the core from the flanks of the pole shoes, as indicated

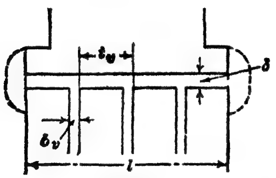


FIG. 9.

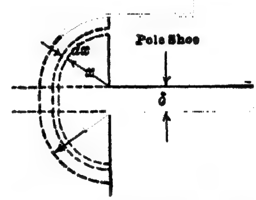


FIG. 10.

FIGS. 9 and 10.—Correction of axial length due to fringing of flux.

by the dotted lines in Fig. 9; the correction takes the form of an additional length l'_2, so that the equivalent axial length is

$$l' = l'_1 + l'_2 \tag{11}$$

The value of l'_2 may be estimated as follows: Assume that the lines of force of the fringing flux are made up of quadrants of circles and of straight lines, as in Fig. 10. The permeance of

an elementary tube of width dx and breadth b' is

$$dP = \frac{\mu_0 b' \, dx}{\delta + \pi x}$$

and the entire permeance of all the tubes between the limits $x = 0$ and $x = r$, on both sides of the core, is

$$P = 2\mu_0 \int_0^r \frac{b' \, dx}{\delta + \pi x} = \frac{2}{\pi}\mu_0 b' \ln \frac{\delta + \pi r}{\delta}$$

But the permeance is to be made equivalent to that of a tube of force of length δ and cross-section $b'l_2'$, hence,

$$\frac{b'l_2'}{\delta} = \frac{2}{\pi} b' \ln \left(1 + \frac{\pi r}{\delta}\right)$$

and

$$l_2' = 1\ 5\delta \log_{10} \left(1 + \frac{\pi r}{\delta}\right) \tag{12}$$

For values of r from 1 to 5 times δ, l_2' varies from 0.9δ to 1.8δ. Generally it is sufficiently accurate to take $l_2' = 1.5\delta$.

When b' and l' are found, the corrected value of flux density in the airgap is

$$B_g = \frac{\Phi}{b'l'}$$

and therefore

$$AT_g = \frac{0.8 B_g \delta}{\mu_0}$$

if c.g.s. units are used (flux density in lines per square centimeter and airgap in centimeters); or

$$AT_g = \frac{0.3133 B_g'' \delta''}{\mu_0}$$

if flux density is given in lines per square inch (B_g'') and airgap in inches (δ'').

5. Ampere-turns Required for the Teeth.—The same difference of magnetic potential that maintains the flux through the teeth also produces a certain amount of flux through the slots, since the two paths are in parallel. When the teeth are not highly saturated, their permeance is so considerable that the flux passing down the slots is relatively insignificant and may

be neglected; but in many machines the iron of the teeth is purposely worked at high flux density in order to limit the effect of armature reaction, and in such cases the permeance of the teeth is decreased to such an extent that the slot permeance becomes comparable with it. If, then, it were assumed that the entire flux per pole passed through the teeth immediately under the pole (with an allowance for the spread of the flux at the pole tips), the resultant tooth density would be higher than it is in reality, and the ampere-turns per unit length corresponding to this apparent density might be greatly in excess of the true value because of the flatness of the magnetization curve at high saturation. The actual tooth density B_t must therefore be distinguished from the apparent density B'_t.

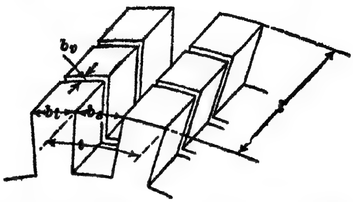


FIG. 11.—Dimensions of teeth and slots.

The conditions that determine the relation of the actual to the apparent density are: (1) that the total flux per pole is equal to the sum of the flux in the iron of the teeth and in the air of the slots, ventilating ducts, and insulation space between laminae; and (2) that the magnitudes of the flux in the iron and in the air are proportional to the permeances of the respective paths. Therefore,

$$\Phi = \Phi_{iron} + \Phi_{air} \tag{13}$$

$$\frac{\Phi_{iron}}{\Phi_{air}} = \frac{\mu \times \text{cross-section of iron}}{\text{cross-section of air}} = \mu K \tag{14}$$

where μ is the relative permeability of the iron corresponding to the *actual* tooth density B_t at the tip of the tooth.

Referring to Fig. 11,

$$\text{Cross-section of iron} = b_t(l - n_v b_v)k \tag{15}$$

where k is the lamination factor (usually about 0.9), and

$$\text{Cross-section of air} = b_s l + b_t \cdot n_v b_v + b_t(l - n_v b_v)(1 - k) \tag{16}$$

From Eqs. (15) and (16) K may be determined for a given set of dimensions. It follows that

$$\frac{\Phi}{\Phi_{iron}} = \frac{\Phi_{iron} + \Phi_{air}}{\Phi_{iron}} = \frac{1 + \mu K}{\mu K} = \frac{B'_t}{B_t} \tag{17}$$

For a given value of K it is possible to compute from this equation a series of simultaneous values of B_t and B'_t by assuming values for B_t, finding the corresponding values of μ from the magnetization curve of the core material, and substituting in Eq. (17). Thus, in Fig. 12, curve M shows the relation between B_t and at_t, as determined by test, for commercial sheet steel. The remaining curves give B'_t for various values of K.

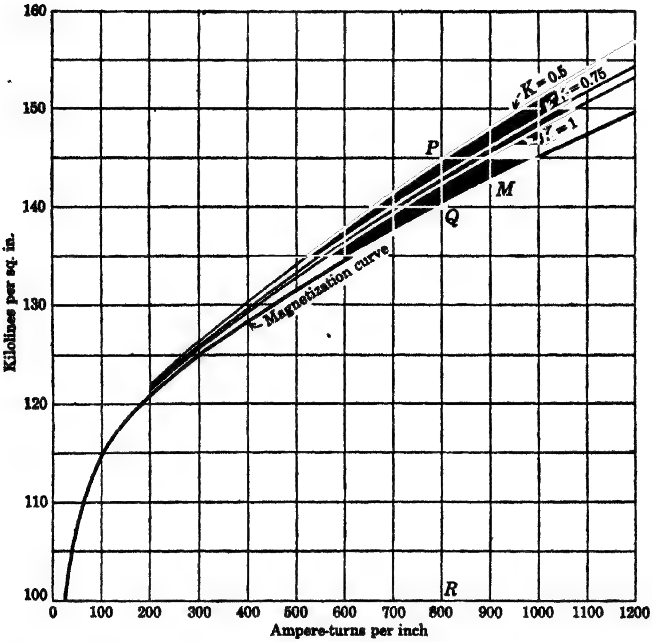


FIG. 12.

The preceding method of determining B_t (at the tip of the tooth) when B'_t is known has the disadvantage that K may differ from any of the values for which curves have been prepared. It is, however, possible to find B_t from B'_t, for any value of K, directly from the magnetization curve (M, Fig. 12), as follows:

From the relation

$$\frac{B'_t}{B_t} = \frac{1 + \mu K}{\mu K}$$

we have

$$B'_t = B_t + \frac{1}{K}\frac{B_t}{\mu} = B_t + \frac{\mu_0 H}{K} \tag{18}$$

provided that B is expressed in lines per square centimeter.

In Fig. 13, let C represent the magnetization curve plotted in terms of B and H, and assume for the present that B and H are plotted to the same scale. To the left of the origin lay off any convenient scale to represent values of K, and lay off ON equal to $\mu_0 = 1$ to the scale of K. Then, if OM is any value of K, $ON/OM = \mu_0/K = \tan\alpha$; and if OR is drawn parallel to MN the intercept QR will equal $\mu_0 H/K$, corresponding to $H = OQ$. Therefore, $PR = OS = B'_t$ since $PQ = B_t$. That is, if the actual tooth density is to be found for a given value of apparent tooth density, all that is necessary is to lay off OS on the axis of ordinates equal to this given value B'_t and to draw through S a line parallel to MN until it intersects curve C in a point P. Ordinate PQ is then the actual tooth density B_t and OQ is the corresponding value of H.

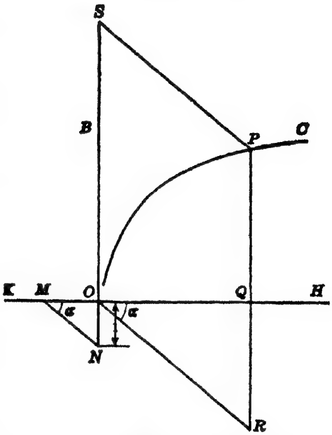


FIG. 13.—Graphical relation between apparent and actual tooth induction.

Since B and H are never plotted to the same scale and since magnetization curves are usually drawn in terms of B and *at*, suitable modifications must be made in the construction. If B is plotted to represent lines per square centimeter and *at* in ampere-turns per centimeter, the length ON must be made equal to $4\pi A_0/10B_0$ to the scale of K, where

A_0 = number of ampere-turns per centimeter per unit length of horizontal axis.

B_0 = number of gausses per unit length of vertical axis.

If B is in lines per square inch and *at* in ampere-turns per inch, ON must be made equal to $2.54 \times (4\pi A'_0/10B'_0) = 3.19(A'_0/B'_0)$ to the scale of K where

A'_0 = number of ampere-turns per inch per unit length of horizontal axis.

B'_0 = number of lines per square inch per unit length of vertical axis.

Effect of Taper of Tooth.—It does not immediately follow that $AT_t = at_t \cdot l_t$; for the tapering of the teeth results in an increasing density from the tip to the root, and consequently at_t changes in value from point to point along the length of the tooth. Nor does it follow that the value of at to be used is that corresponding to the flux density at the middle of the tooth. If values of B_t are computed for a number of points along the length of the tooth and the corresponding values at at_t, found from the appropriate *B-H* curve, as Fig. 12, are plotted, a curve like Fig. 14 will in general result. Evidently the true value of at_t is the mean ordinate of the curve; on the assumption that the curve is parabolic, the mean ordinate, by Simpson's rule, is

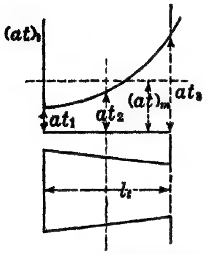


FIG. 14.—Variation of excitation along tooth axis.

$$(at_t)_{mean} = \frac{at_1 + 4at_2 + at_3}{6} \tag{19}$$

whence

$$AT_t = (at_t)_{mean} \cdot l_t \tag{20}$$

6. Ampere-turns Required for the Armature Core.—It is clear from Figs. 4 and 5, Chap. VIII, that the iron of the core below the roots of the teeth carries half the useful flux per pole. Therefore,

$$B_a = \frac{\Phi}{2A_a} \tag{21}$$

If the radial depth of the iron under the teeth is h,

$$A_a = kh(l - n_v b_v) \tag{22}$$

To the value of B_a thus determined there corresponds a value of at_a amp.-turns per unit length, whence

$$AT_a = at_a \cdot l_a \tag{23}$$

7. Ampere-turns Required for Pole Cores and Pole Shoes.—The flux carried by the pole cores and pole shoes varies from

section to section, but it may be assumed without sensible error that the flux is uniform and equal to $\nu\Phi$. We have, therefore,

$$B_c = \frac{\nu\Phi}{A_c} \qquad \text{and} \qquad B_s = \frac{\nu\Phi}{A_s} \tag{24}$$

to which values of flux density there correspond unit excitations of at_c and at_s, respectively; hence,

$$AT_c = at_c \cdot l_c \qquad \text{and} \qquad AT_s = at_s \cdot l_s \tag{25}$$

8. Ampere-turns Required for the Yoke.—The flux carried by the yoke is equal to either $\nu\Phi$ or $\frac{1}{2}\nu\Phi$, depending upon the type of machine, illustrated by Figs. 4 and 5, Chap. VIII, respectively; Fig. 5 is representative of most modern machines. Usually, therefore,

$$B_y = \frac{\frac{1}{2}\nu\Phi}{A_y} \tag{26}$$

to which value there corresponds at_y amp.-turns per unit length, and

$$AT_y = at_y \cdot l_y \tag{27}$$

9. Simplified Formulas for Leakage Flux.—The leakage flux φ that enters into the equation

$$\nu = 1 + \frac{\varphi}{\Phi}$$

includes the fluxes in all stray paths which are associated with the main flux Φ but which, unlike the main or useful flux, do not complete their circuits through the armature. The general distribution of the stray fluxes has already been illustrated in Fig. 9, Chap. VIII. In order to calculate the magnitude of each of these leakage fluxes, it is necessary to divide the m.m.f. acting upon a given path by the corresponding reluctance or to multiply the m.m.f. by the corresponding permeance. In many cases that occur in practice, the geometrical shapes of the leakage paths are so irregular that accurate calculations of their permeances are difficult, if not impossible, and the type of mathematical analysis* required is beyond the scope of this book;

* Those who are interested in the details of the subject will find a useful summary, together with a good bibliography, in a paper entitled The Reluctance of Some Irregular Magnetic Fields, by JOHN F. H. DOUGLAS, *Trans. A.I.E.E.*, **34**, Pt. I, 1067, 1915.

but, for most practical cases, sufficiently accurate results may be obtained by means of approximations which result in formulas

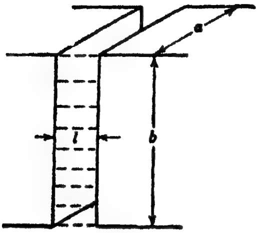


FIG. 15.

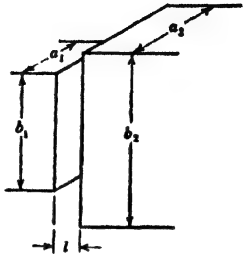


FIG. 16.

that are simple enough for rapid calculation. The cases that usually occur are as follows:

Case 1. *Parallel and Equal Plane Surfaces* (Fig. 15).—The fringing of the flux that occurs at and near the edges of the adjacent surfaces being neglected, the permeance of the path is

$$P = \mu_0 \frac{\text{area}}{\text{length}} = \mu_0 \frac{ab}{l} \quad (28)$$

where all dimensions are in centimeters. This formula is applicable only when l is small in comparison with a and b.

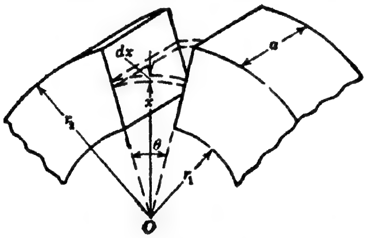


FIG. 17.

Case 2. *Parallel and Unequal Plane Surfaces* (Fig. 16).—Provided that l is small in comparison with the linear dimensions of the opposing areas and provided also that these areas are not widely different, the permeance of the flux path between them may be taken to be

$$P = \frac{\mu_0 \times \text{mean area}}{\text{perpendicular separation}} = \frac{\mu_0}{2} \frac{a_1 b_1 + a_2 b_2}{l} \quad (29)$$

all dimensions being in centimeters.

Case 3. *Nonparallel Equal Plane Areas* (Fig. 17).—Assume that the lines of force are arcs of circles concentric about the axis O at the intersection of the inclined plane surfaces. An elementary tube of force distant x cm. from the axis and having

a cross-section $a\,dx$ sq. cm. will have a permeance

$$dP = \frac{\mu_0 a\,dx}{x\theta}$$

and the total permeance will be

$$P = \mu_0 \int_{r_1}^{r_2} \frac{a\,dx}{x\theta} = \frac{\mu_0 a}{\theta} \ln \frac{r_2}{r_1} \tag{30}$$

where the angle θ is expressed in radians.

Case 4. *Coplanar Equal Areas* (Fig. 18).—Assume that the lines of force are made up of quadrants of circles, of radius x, and

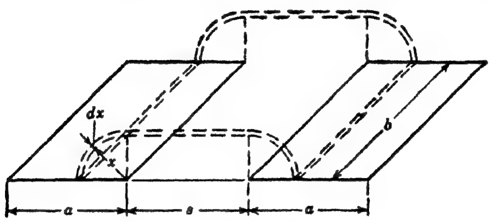


FIG. 18.

linear portions of length s. The permeance of the elementary tube represented in the diagram is

$$dP = \frac{\mu_0 b\,dx}{s + \pi x}$$

and the permeance of the entire path is

$$P = \mu_0 \int_0^a \frac{b\,dx}{s + \pi x} = \mu_0 \frac{b}{\pi} \ln \left(1 + \frac{\pi a}{s}\right) \tag{31}$$

provided that all dimensions are in centimeters.

This formula, which is known as Forbes's formula, gives results that are too small, since the actual paths of the lines of force spread out to a greater extent than is implied in Fig. 18; the spreading of the flux increases the lengths of the tubes, but the enlargement of their cross-sections more than offsets this tendency toward a decreased permeance. A closer approximation to the truth is given by Finnis's formula, which is based upon the assumption that the lines of force are ellipses with foci at the

points F_1, F_2, Fig. 19. This leads to the result

$$P = \frac{\mu_0 b}{\pi} \ln \left[2\frac{a + \sqrt{a^2 + sa}}{s} + 1 \right] \tag{32}$$

Case 5. Coplanar Areas, Varying Potential Difference (Fig. 20).—This is a simplified case of the type that occurs in actual machines in considering the leakage flux between the side flanks

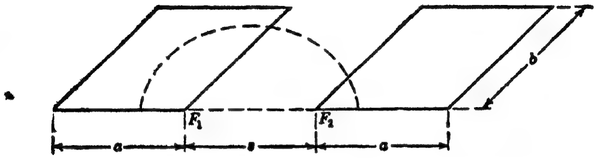


FIG. 19.

of the pole cores and the lateral surface of the yoke. In Fig. 20 the core a_2 is wound with a uniformly distributed winding, so that the difference of magnetic potential between a_2 and a_1 increases linearly in the manner represented by the line OM.

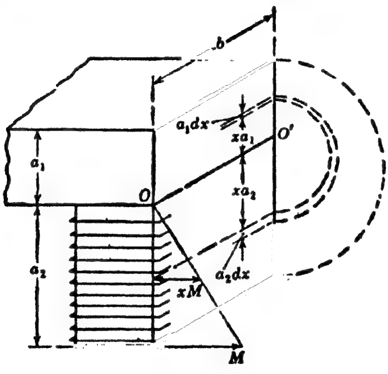


FIG. 20.

It will be assumed that the magnetic potentials of point O and of all points on the yoke a_1 are zero.

Assume that the lines of force are semicircles and that a line issuing from a_2 at a distance xa_2 below OO' reenters a_1 at a distance xa_1 above OO', where x is a fraction that varies between zero and unity. Then, in the elementary tube indicated in Fig. 20, the active difference of magnetic potential (or m.m.f.) is xM, the length of path is $(\pi/2)x(a_1 + a_2)$, and the cross-section

may be taken as the mean of ba_1dx and ba_2dx. The elementary flux is

$$d\varphi = xM\frac{\mu_0(b/2)(a_1 + a_2)dx}{(\pi/2)x(a_1 + a_2)}$$

and the total flux is

$$\varphi = \frac{Mb\mu_0}{\pi}\int_0^1 dx = \frac{Mb\mu_0}{\pi} \tag{33}$$

10. Coefficient of Dispersion in Machines Having Few Poles.—In cases where the number of poles is so limited that the simplifications discussed in the preceding article are not directly applicable, recourse may be had to a process of mapping the leakage paths in the manner illustrated in Fig. 9, Chap. VIII, and in greater detail in Fig. 21. Judgment and experience must be relied upon in drawing the lines of force, but some guidance is afforded by the considerations that the lines are substantially perpendicular to the surface of the iron at the points where they enter or leave and that of the various ways in which the lines may be drawn that arrangement is most nearly accurate which results in the highest permeance.

All the leakage lines are closed curves that link with some or with all of the turns of the exciting winding. The part of the reluctance of a complete tube which is due to the air part of the path is so much greater than the reluctance of the iron path that the latter is negligible in comparison. Those tubes which link with only a part of the exciting winding are acted upon by a correspondingly reduced part of the total m.m.f. due to all the exciting ampere-turns. For instance, the m.m.f. between the tips of adjacent poles is the m.m.f. required to drive the useful flux across the double airgap, two sets of teeth, and the armature core, or it is equivalent to

$$X = AT_a + 2AT_g + 2AT_t \quad \text{amp.-turns} \tag{34}$$

Consequently, points on adjacent pole cores that are each halfway between the yoke and the pole shoe will have between them a difference of magnetic potential approximately equivalent to $\frac{1}{2}X$ amp.-turns.

The leakage fluxes indicated in Fig. 21 may be computed one by one, as follows:

Leakage between Inner Surfaces of Pole Shoes, φ_1.—Using c.g.s. units of length,

$$\varphi_1 = \frac{4\pi}{10} X \frac{a_1 l \mu_0}{l_1} \tag{35}$$

which becomes

$$\varphi_1 = 3.2 X \frac{a_1 l \mu_0}{l_1} \tag{35a}$$

if dimensions are expressed in inches.

The dimensions a_1 and l may be scaled from the drawing, or they may be computed if it is assumed that the lines of force are arcs of circles concentric with the axis of the armature.

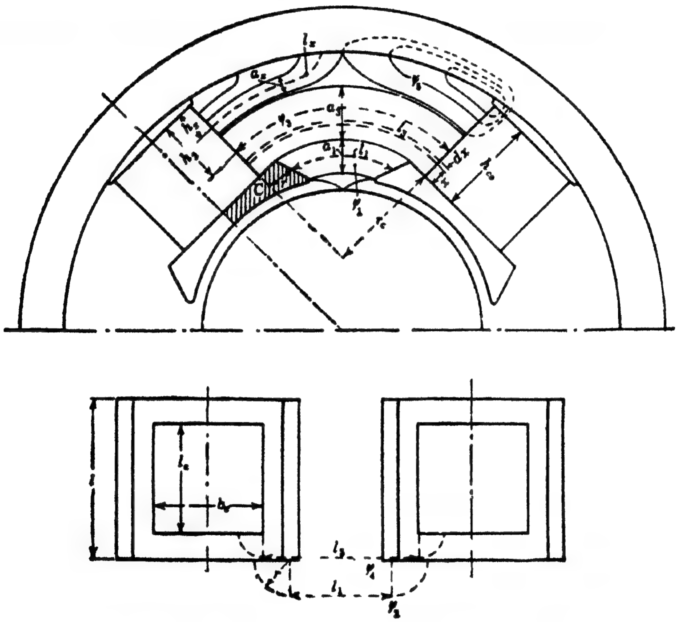


FIG 21.

Leakage between Lateral Surfaces of Pole Shoes, φ_2.—The leakage flux φ_2 issuing from half of the lateral area of a pole shoe (indicated by the crosshatching in Fig. 21) finds its way to the adjacent half of the adjoining pole. The cross-section of the path may be taken equal to the half area of the lateral surface of the pole shoe; the mean length of path may be taken equal to two

quadrants of circles, each of radius r, plus the curved length l_1. The radius r is measured from the pole tip to the center of gravity of the crosshatched area. If the half area of the pole shoe (lateral surface) is A_s,

$$\varphi_2 = \frac{4\pi}{10} X \frac{\mu_0 A_s}{l_1 + \pi r} \tag{36}$$

if lengths are expressed in centimeters, and

$$\varphi_2 = 3.2 X \frac{\mu_0 A_s}{l_1 + \pi r} \tag{36a}$$

if lengths are in inches.

Leakage between Inner Surfaces of Pole Cores, φ_3.—If the pole cores are round, of diameter d_c, they may be assumed to have been replaced by square poles of equal cross-section: in that case,

$$b_c = l_c = \frac{d_c}{2}\sqrt{\pi} = 0.89\, d_c \tag{37}$$

The mean length of path, scaled from the drawing, is l_3, and the mean cross-section is $a_3 l_c$. The mean m.m.f. is $(4\pi/10)X(h_3/h_c)$; hence, if lengths are in centimeters

$$\varphi_3 = \frac{4\pi}{10} X \mu_0 \frac{h_3}{h_c} \frac{a_3 l_c}{l_3} \tag{38}$$

or

$$\varphi_3 = 3.2 X \mu_0 \frac{h_3}{h_c} \frac{a_3 l_c}{l_3} \tag{38a}$$

if lengths are in inches.

If the lines of force comprising φ_3 are assumed to be arcs of circles concentric with the axis of the armature, the calculation of φ_3 becomes a special form of case 3 (Art. 9), the difference being due to the fact that the m.m.f. acting upon elements like those of Fig. 17 varies from point to point along the pole core. Thus, in Fig. 21, the m.m.f. acting upon the elementary tube dx is $\frac{4\pi}{10} X \frac{h_c - x}{h_c}$, the cross-section of the tube is $l_c\, dx$, and the length is $(2\pi/p)(r_c + x)$; hence,

$$\varphi_3 = \frac{4\pi}{10} X \mu_0 \int_0^{a_3} \frac{h_c - x}{h_c} \frac{p l_c\, dx}{2\pi(r_c + x)} =$$

$$\frac{4\pi}{10} X \mu_0 \frac{p l_c}{2\pi h_c}\left[(h_c + r_c) \ln \frac{r_c + a_3}{r_c} - a_3\right] \tag{39}$$

Leakage between Lateral Faces of Pole Cores, φ_4.—Part of the leakage flux issuing from the lateral flank of a pole core will find its way to the nearest half of the adjoining pole; the remaining part will complete its circuit by way of the exposed lateral surface of the yoke, somewhat in the manner indicated in Fig. 20. The first part will be called φ_4, the latter φ_6. The width of the belt comprising φ_4 may be assumed to be a_3, the same as that of belt φ_3. The mean length of the path of φ_4 may be taken to be two circular quadrants, of radius $b_c/4$, plus the scaled length l_3. Hence,

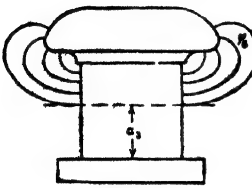


FIG. 22.

$$\varphi_4 = \frac{4\pi}{10} X \frac{h_3}{h_c} \frac{\mu_0 a_3 (b_c/2)}{l_3 + (\pi/4) b_c} \qquad (40)$$

all dimensions being in centimeters.

Leakage Flux between Inner Surface of Pole Core and Yoke, φ_5.—In order to compute this flux, it is necessary to map a sufficient number of tubes of force so that the mean length l_x and the mean depth a_x of each tube may be scaled from the drawing. The mean m.m.f. acting on each tube is $\frac{1}{2} \cdot (4\pi/10) X(h_x/h_c)$; hence,

$$\varphi_5 = \sum \frac{1}{2} \frac{4\pi}{10} X \frac{h_x}{h_c} \frac{\mu_0 a_x l_c}{l_x} = \frac{1}{2} \cdot \frac{4\pi}{10} X \frac{l_c}{h_c} \sum \frac{\mu_0 a_x h_x}{l_x} \qquad (41)$$

the summation being extended to include all the separate tubes

Leakage Flux between Lateral Surface of Pole Core and Yoke, φ_6.—Unless the yoke overhangs the pole core by a considerable amount, this leakage flux can be found by using Eq. (33), the value $\frac{1}{2} \cdot \frac{4\pi}{10} X \frac{h_c - a_3}{h_c}$ being substituted for M in that equation. But if the yoke overhangs the pole core, as it usually does, it is best to compute φ_6 by the method previously described for the leakage flux φ_5. Thus, in Fig. 22 the flux φ_6 has been mapped, and from this drawing the lengths and cross-sections of the individual tubes are to be determined and then substituted in Eq. (41).

$$\varphi_6 = \frac{1}{2} \cdot \frac{4\pi}{10} X \frac{b}{h_c} \sum \frac{\mu_0 a_x h_x}{l_x} \qquad (42)$$

Total Leakage.—The individual parts of the leakage flux having been determined, the total leakage flux will be

$$\varphi = 2\varphi_1 + 4\varphi_2 + 2\varphi_3 + 4\varphi_4 + 2\varphi_5 + 2\varphi_6 \tag{43}$$

and this value may then be used to determine the coefficient

$$\nu = 1 + \frac{\varphi}{\Phi}$$

It is to be noted that the quantity X appears in each of the formulas for φ_1, φ_2, etc. Since X includes the excitation for the teeth, which are usually highly saturated, X will vary more or less with the load, and hence the coefficient ν will not be constant at all loads; generally it increases with the load.

CHAPTER X

OPERATING CHARACTERISTICS OF GENERATORS

1. Service Requirements.—The lamps, motors, or other translating devices supplied with electrical energy from a distribution circuit may be connected to the supply mains in *parallel*, in *series*, or in *series-parallel*, as shown diagrammatically in Fig. 1*a*, *b*, and *c*.

Parallel connection *a* is used when the individual units constituting the load on the system are designed to operate with a

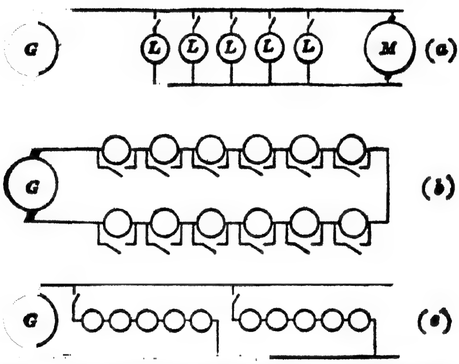


FIG. 1.—Parallel, series and series-parallel circuits.

constant difference of potential between their terminals; in such a system any individual load unit can be disconnected without interfering with the operation of the remaining units. As examples of this class of service may be mentioned the use of incandescent lamps for interior lighting, and street railways operating with constant difference of potential between trolley and rail.

Series connection *b* is used principally in arc lighting and in series-incandescent lighting of streets and alleys, where each lamp requires the same current. If an individual load unit in this system is to be cut out of service, it cannot be disconnected, as in the parallel system, but must be short-circuited by a

"jumper" connection, as in Fig. 1*b*, in order to preserve continuity of service in the remaining units.

Series-parallel distribution *c* is merely a combination of the other two and requires no special consideration, once the principles underlying the series and the parallel systems are understood. A common example of series-parallel connection is found in the lighting circuits of a trolley car, where several strings of five 110-volt lamps in series are each connected across the 550-volt supply circuit.

If in a constant-potential (parallel) system there are N lamps (or other load units) each taking I amp., the total current supply is NI amp., and the power consumed, the loss in the line being neglected, is NIV watts, where V is the line voltage. In a constant-current (series) circuit, if there are N lamps each requiring V volts and I amp., the total impressed voltage must be NV volts and the power consumed will be NVI watts, the line loss being neglected. In the first case (parallel system) the line conductor must have a cross-section at the supply end capable of carrying NI amp. and gradually tapering off as the end of the line is approached. In the second case (series system) the line conductor should have uniform cross-section from end to end, since the current is everywhere the same, but the difference of potential between the supply lines will be large at the generator end (G) and will gradually decrease as the distance from the generator increases. The parallel system requires a much greater weight of copper in the line than the series system, but this disadvantage is offset by the fact that the high voltage required in a series circuit of any considerable power limits the use of series circuits to outdoor service. Thus, if 125 arc lamps, each consuming approximately 50 volts, are connected in series, the total voltage consumed by the lamps will be 6250 volts; if the voltage consumed in overcoming the resistance of the line is added, the e.m.f. required at the generator will be of the order of 7000 to 8000 volts, or much too high for safety in indoor service.

Although the parallel system of distribution is ordinarily called the constant-potential system, it will be apparent that the difference of potential between the conductors will vary more or less from point to point, becoming less as the distance from the generator increases, because of the drop of potential due to the resistance of the line wires. This drop of potential due to

ohmic resistance may be reduced to any desired amount by increasing the cross-section of the conductor, but a limit is set by the rapidly increasing cost of the conductors. If the lamps (or other translating devices) are grouped at a distance from the generator, the voltage at the lamps may be kept constant, irrespective of the current in the feeder circuit, provided that the voltage of the generator is raised, as the load increases, to a sufficient extent to compensate the drop of potential in the line.

2. Characteristic Curves.—In view of the various types of service requirements described in Art. 1, it becomes important to investigate the characteristic behavior of the usual types of generators in order to determine the kind of service to which each is adapted. Probably the simplest way to study and compare the several kinds of machines is to construct *characteristic curves* which show the relations between the variables involved in the operation of the machine. For example, the *external characteristic* of a generator is a curve showing the relation between terminal voltage, plotted as dependent variable, and external (line) current, plotted as independent variable. Other characteristic curves are discussed in following articles.

3. Regulation.—In the case of a generator, the terminal voltage at full load is generally different from that at no load. The difference between the two values is then a measure of the closeness with which the machine regulates for constant voltage.

The regulation, "is usually stated by giving the numerical values of the voltage at no load and rated load, and in some cases it is advisable to state regulation at intermediate loads. The regulation of direct-current generators refers to changes in voltage corresponding to gradual changes in load, and does not relate to the comparatively large momentary fluctuations in voltage that frequently accompany instantaneous changes in load."*

Voltage regulation is frequently expressed as a percentage, computed by dividing the change in voltage from no load to full load by the particular voltage (generally the full-load value) that is considered to be the normal voltage of the machine and multiplying the result by 100.

* A.I.E.E. Standards, C-50, American Standards for Rotating Electrical Machinery.

4. Characteristic Curves of Separately Excited Generator.—The following symbols will be used:

E = generated e.m.f.
V = terminal voltage.
R_a = resistance of armature, including brushes and brush contacts.
R_f = resistance of field winding.
R = resistance of external load circuit.
I = current taken by load.
I_f = current in field winding.
N_f = field turns per pair of poles.
n = speed of rotation in revolutions per minute (r.p.m.).

No-load Conditions.—Under no-load conditions, the armature being driven at its rated speed, the relation between the e.m.f. generated in the armature winding and the field excitation is given by the saturation curve discussed in Chap. VIII (see also Fig. 2 in that chapter). Since the generated e.m.f. is given by the equation

$$E = \frac{p}{a} \cdot \frac{\Phi Z n}{60 \times 10^8}$$

it follows that, if Φ (in maxwells) is kept constant (by keeping I_f constant), the generated e.m.f. will be directly proportional

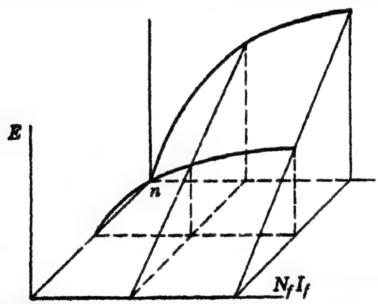


FIG. 2.—Effect of variation of speed upon saturation curve.

to the speed. If, then, E, $N_f I_f$, and n are plotted along three axes of coordinates, there will result a surface of the kind illustrated in Fig. 2, which is the same as that of Fig. 3, Chap. VIII. Sections of the surface cut by planes parallel to the E, n plane are straight lines the slope of which increases as the distance of the section from the E, n reference plane increases—at first

rapidly, then more slowly. Sections cut by planes parallel to the E, N_fI_f plane are saturation curves corresponding to the speed represented by the distance of the secant plane from the origin of coordinates.

External Characteristic. Load Conditions.—With the connections shown in Fig. 3, let the machine be driven at its rated speed, the field excited by a constant current I_f, and the brushes set with an angular lead α most favorable for good commutation. The line current (which is here the same as the armature current) will vary as the external load resistance R is varied, and the terminal voltage V will be less than the generated e.m.f. E by IR_a volts, the latter being consumed in the internal resistance of the armature. That is,

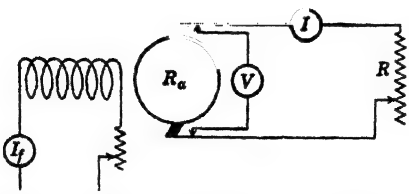


FIG. 3.—Connections for determining no-load characteristic. Separate excitation.

$$V = E - IR_a \tag{1}$$

In this expression, R_a comprises not only the resistance of the armature winding itself, but the resistance of the brushes and their connections, including the contact resistance between commutator and brushes. Although the resistance of the armature is constant when the steady working temperature has been reached, the contact resistance is not constant but varies approximately inversely as the current; that is, the total drop of potential at the contact surface between commutator and brushes is approximately constant and is of the order of 2 volts with ordinary grades of carbon brushes, provided that the current per square inch of contact area does not exceed 45 amp. (or 5 to 7 amp. per sq. cm.).

In the case of copper brushes, which are used only with very low voltage machines, the contact drop is of the order of 0.04 volt with current densities ranging from 65 to 160 amp. per sq. in. (10 to 25 amp. per sq. cm.).

Strictly, therefore,

$$V = E - IR'_a - \Delta e \qquad (2)$$

where R'_a is the constant part of the armature resistance and Δe is the drop over the variable contact resistance. In what follows this correction is ignored for the sake of simplicity.

In Fig. 4 let $O'M$ be the saturation curve at rated speed, plotted with ampere-turns per pair of poles (N_fI_f) as abscissas and generated e.m.f. E as ordinates. If the excitation is adjusted by means of the field rheostat (Fig. 3) until it has the value represented by the abscissa OF_0, the generated e.m.f. will be $F_0G_0 = OP_0$, and

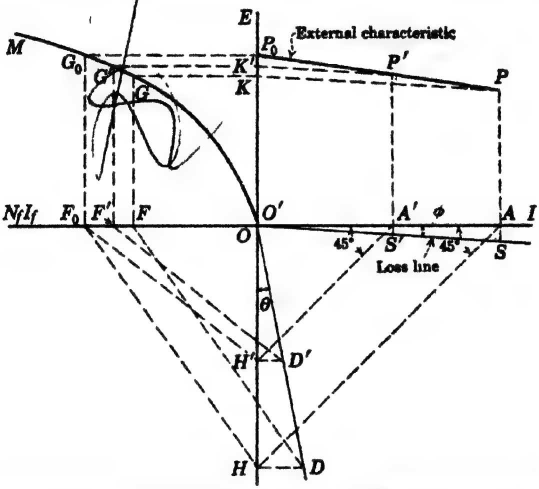


FIG. 4. Construction of external characteristic of separately excited generator.

this will also be the terminal voltage when the machine is running on open circuit ($I = 0$, $R = \infty$).

On closing the external circuit through a load R of such resistance that a current of I amp. is drawn from the machine, the terminal voltage drops below its no-load value: (1) because the demagnetizing action of the armature reduces the useful flux Φ and therefore also the generated e.m.f. E; (2) because of the drop of potential IR_a in the armature. In Fig. 4 let OA represent to scale any value of load current I, and draw AH at an angle of 45 deg. so that $OH = OA = I$; or, what amounts to the same thing, lay off a scale of line current along OH as well as along OA.

At the point H draw a horizontal line HD of length

$$HD = \frac{\alpha ZI}{180a} = \text{demagnetizing amp.-turns per pair of poles}$$

to the same scale previously adopted in drawing the saturation curve M. If the straight line OD is then drawn, the intercept $H'D'$ corresponding to any other value of line current such as $OA' = OH'$ will represent to scale the demagnetizing ampere-turns per pair of poles, since that quantity is directly proportional to the armature current for a fixed setting of the brushes. On subtracting HD from the fixed field excitation OF_0, the remainder OF is the net excitation, and the corresponding value of generated e.m.f. is $FG = OK$; the subtraction can be effected graphically by joining points H and F_0 and drawing DF parallel to HF_0; or else by transferring the length HD to F_0F by means of a pair of dividers.

Corresponding to the current $I = OA$, the drop of potential in the armature is $IR_a = AS$, where AS is laid off to the scale of voltage previously chosen in drawing curve M. If the line OS (sometimes called the *loss line*) is then drawn, the intercept $A'S'$ corresponding to any other value of current such as OA' represents to scale the ohmic drop due to that current.

Since $FG = OK$ is the generated e.m.f. corresponding to the current OA, the terminal voltage can be found by subtracting AS from OK; this is accomplished graphically by drawing KP through the point K parallel to OS until it intersects the ordinate through A in the point P. The latter point is then on the *external characteristic* of the machine. Additional points, such as P', are readily found by the construction indicated in the figure.

This method is subject to small errors because of the fact that it neglects a possible demagnetizing effect due to cross-magnetization (Chap. VIII) and also due to the short-circuit currents in the coils undergoing commutation. The former may be taken into account, if necessary, by slightly increasing the angle θ.

It will be observed that the form of the external characteristic P_0P is dependent upon the form of the saturation curve $O'M$, as well as upon the angles θ and φ. There will be a different characteristic corresponding to each setting of the field excitation, OF_0. The student will find it very instructive to run through

the construction using a value of field excitation such that the point G_0 falls on, or slightly below, the knee of the saturation curve.

The particular construction shown in Fig. 4 is designed to bring out clearly the relations between the magnitudes of the several variables concerned in the determination of points on the external characteristic. It is in reality only a geometrical device for performing graphically those multiplications, additions, or subtractions which are demanded by the fundamental theory. A

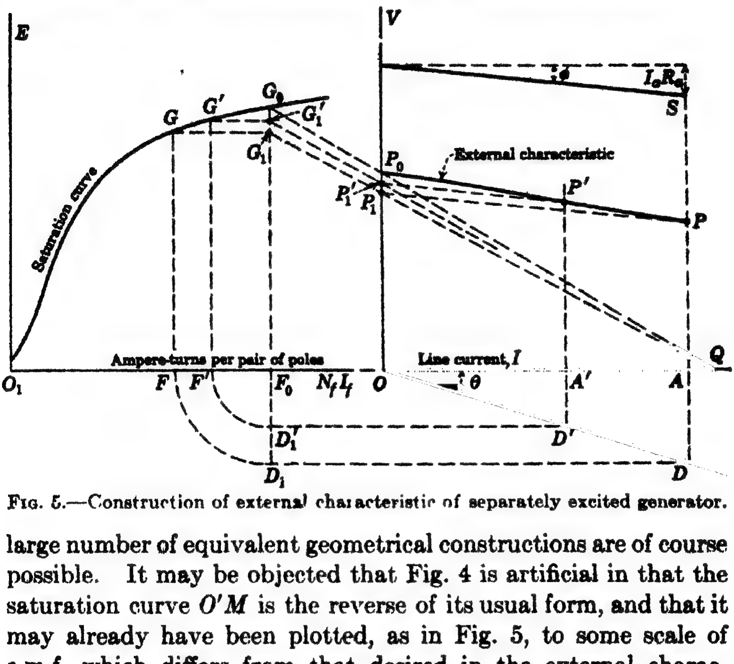


FIG. 5.—Construction of external characteristic of separately excited generator.

large number of equivalent geometrical constructions are of course possible. It may be objected that Fig. 4 is artificial in that the saturation curve $O'M$ is the reverse of its usual form, and that it may already have been plotted, as in Fig. 5, to some scale of e.m.f. which differs from that desired in the external characteristic. To overcome such objections, the construction of Fig. 5 is applicable.

The line OD is so drawn that the intercept AD corresponding to a particular value of line current ($OA = I$) is equal to $AT_d = \alpha ZI/180a$ amp.-turns, to the scale of ampere-turns previously selected for the saturation curve. If the latter curve has been drawn on a separate sheet of coordinate paper, it may be tacked to the drawing board so that the two horizontal axes are in line

with each other. If, then, the field rheostat of the generator is so adjusted as to give the fixed excitation O_1F_0, the demagnetizing ampere-turns AD may be subtracted from O_1F_0 by drawing the straight line DD_1 and the quadrant FD_1, the latter having F_0 as center. The net excitation is then O_1F, and the corresponding generated e.m.f. is FG. Similarly, $F'G'$ is the net generated e.m.f. corresponding to line current OA'.

Suppose, now, that the open-circuit voltage of the machine, F_0G_0, is to appear on the external characteristic as point P_0. Join G_0 and P_0 by a straight line and extend it to intersect the horizontal axis at some point Q. Then any other generated e.m.f., such as FG, may be converted to the scale of OV by drawing GG_1 horizontally to intersect F_0G_0, and connecting G_1 and Q. The intersection of G_1Q with OV will fix a point P_1 such that OP_1 will be the generated voltage to the scale of OV. To find the corresponding point P on the external characteristic, all that is necessary is to draw the line P_1P parallel to VS; the latter is the same as line OS, Fig. 4, being so drawn that its slope (to the scale of the drawing) is R_a.

Applications of Separately Excited Generator.—The constructions of Figs. 4 and 5 show that the separately excited generator has an inherent tendency to regulate for constant voltage. The external characteristic is necessarily drooping, but the change in voltage from no load to full load can be made small by keeping down the demagnetizing action of the armature and by designing the armature winding to have a sufficiently low resistance to limit the ohmic drop to a small percentage of the rated voltage. For these reasons the separately excited generator is suitable for constant-potential service but is seldom used in heavy power installations because equally satisfactory results can be secured from self-excited machines of the shunt and compound types. In laboratory and commercial testing, however, the use of separate excitation is often very convenient and is frequently used.

5. Effect of Speed of Rotation on External Characteristic.—For a given value of the excitation and, therefore, of the flux, the generated e.m.f. will be proportional to the speed. In case the machine is operated at a speed other than the rated speed, the saturation curve of Fig. 4 or Fig. 5 must be replaced by a new curve the ordinates of which bear the same relation to those of the

original curve that the new speed bears to the rated speed. This is shown in the three-dimensional diagram of Fig. 6, which is a direct development of Fig. 4. Figure 6 also shows that the locus of the external characteristics for varying values of speed (but with a fixed value of field excitation) is a wedge-shaped surface OP_0P.

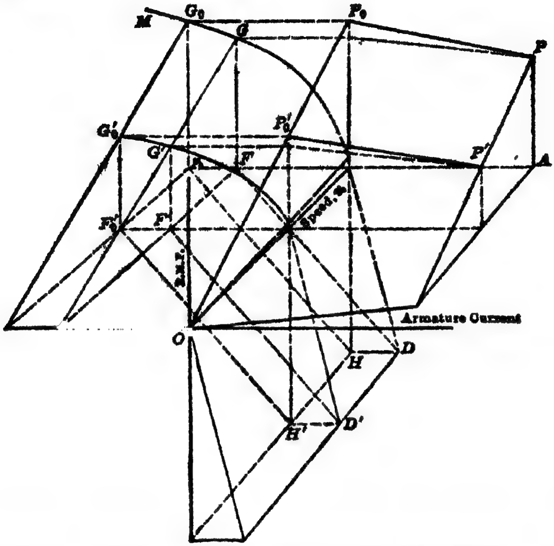


FIG. 6.—Effect of variation of speed upon external characteristic of separately excited generator.

6. Load Characteristic.—By a load characteristic is meant a curve showing the relation between terminal voltage (as ordinates) and field excitation (as abscissas), subject to the condition that the current supplied to the load is constant. If this load current happens to be zero, the curve becomes the no-load characteristic or saturation curve OM, Fig. 7. Now suppose that the resistance R (Fig. 3) of the external circuit is varied and that the field excitation is so adjusted that the current is maintained at its normal full-load value I. The terminal voltage will then be $V = IR$, represented in Fig. 7 by the ordinate OA. At no load this e.m.f. would require an excitation $AB = OC$. But under the assumed conditions the generated e.m.f. must be greater than OA by an amount DF, where $DF = IR_a$; that is, the e.m.f. required to be generated to yield a terminal

voltage $V = OA$ is DG, corresponding to an excitation OG. Finally, because of armature demagnetizing effect, the field excitation must be still further increased to OK, where $GK = (AT)_d$. An ordinate through K then intersects AB (extended) in the point P, which point accordingly lies on the load characteristic corresponding to full-load current.

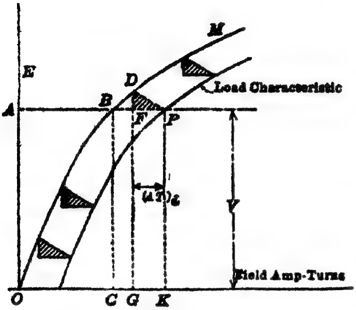


FIG. 7.—Construction of load characteristic, separately excited generator

Since DF and $GK = FP$ remain constant in magnitude when I is constant, the load characteristic is the same in shape as the no-load characteristic, but shifted downward and to the right by the constant length DP, as shown by the series of shaded triangles.

This construction is not strictly accurate, for with increasing excitation the demagnetizing component of cross-magnetization becomes greater, especially if the iron is saturated; in other words, FP should increase in magnitude as the curve rises.

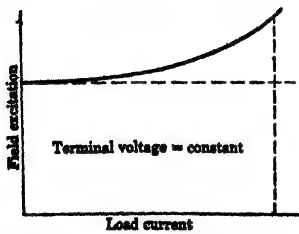


FIG. 8.—Armature characteristic or regulation curve

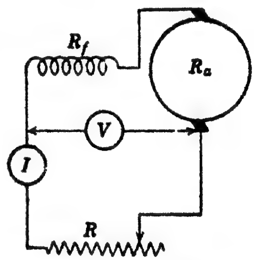


FIG 9.—Connections of series generator, determination of external characteristic.

Further, upon remembering that the coefficient of dispersion ν is itself not constant but that it increases with increasing excitation, it is clear that this feature will contribute to a further increase in FP.

7. Armature Characteristic.—It is evident from Fig. 7 that, if the terminal voltage is to be maintained constant for all values of the load current, the excitation must be increased as the

load increases. Thus, an increase in the current from zero to I requires an increase of excitation from OC to OK in order to maintain a terminal voltage equal to OA. The curve showing the relation between field excitation (as ordinate) and load current (as abscissa) under the condition of constant terminal voltage is commonly called the *armature characteristic* (Fig. 8), though a better name would be "regulation curve."

8. Characteristic Curves of Series Generator. External Characteristic.—Let

E = generated e.m.f.
V = terminal voltage.
I = current in the circuit.
R_a = resistance of armature winding, including brushes and brush contacts.
R_f = resistance of series field winding.
N_f = series field turns per pair of poles.
R = resistance of external circuit.
n = speed in r.p.m.

Since the same current flows through the armature, the field winding, and the load circuit (Fig. 9), it follows that an increase of load causes an increase of excitation and therefore also of generated e.m.f., the speed of rotation being kept constant at its rated value. The external characteristic will have the form of curve III, Fig. 10. If to the ordinates of curve III there are added the ordinates of the loss line, curve II will result. Curve II is the *internal characteristic*, showing the relation between the internally generated e.m.f. and the armature current. This follows because

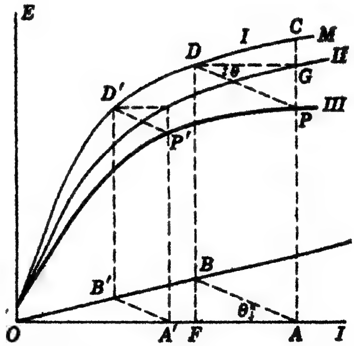


FIG. 10.—External characteristic of series generator.

$$E = V + I(R_a + R_f) \tag{3}$$

If there were no armature reaction, curve II would be the saturation curve of the machine; but in the actual machine, where armature reaction exists, the saturation curve I is dis-

placed from curve II in the manner indicated in the figure. Thus, of the excitation OA required to produce terminal voltage AP and generated e.m.f. AG, a part DG is required to balance the demagnetizing component of armature reaction. The remainder, or OF, is responsible for the generated e.m.f., and hence D is a point on the saturation curve.

Upon repeating this process to find other points, such as D', it will be seen that the size of the triangle PDG must be altered from point to point. But since PG and DG are both proportional to the current I, their ratio, and consequently the slope of DP, will remain constant. This leads to the following simple construction for obtaining the external characteristic from the given saturation curve of the machine.*

The demagnetizing effect AF corresponding to any current $I = OA$ is given by

$$\frac{\alpha ZI}{180a} \cdot \frac{1}{N_f}$$

to the scale of current, and the length FB is equal to $I(R_a + R_f)$ to the scale of volts. These two lengths, when laid off as in Fig. 10, locate the point B, and therefore also the line OB. To find a point P' on the external characteristic corresponding to current OA', draw $A'B'$ parallel to AB, through B' draw $B'D'$ vertically until it intersects curve M in D', and draw $D'P'$ parallel to $B'A'$ until it intersects the ordinate through A' in the point P'.

9. Dependence of Form of the Characteristic upon Speed.—Variation of the speed of a series generator affects the saturation curve in exactly the manner illustrated in Fig. 2 for the case of the separately excited generator. Thus, in Fig. 11, the surface bounded by $OD'D$ is the locus of the saturation curves for various values of speed laid off along the speed axis n. Corresponding to each saturation curve there will be an external characteristic constructed as in Fig. 10, and the locus of all such external characteristics will be a surface indicated by the heavy lines, as OA_0P. The intersection of this surface with the base (n, I) plane is a curve OA_0, which shows the relation between speed and current when the machine is short-circuited ($V = 0$).

* This is the same construction given in Arnold's "Die Gleichstrommaschine," 3d ed., Vol. I, p. 469.

10. Condition for Stable Operation.—With reference to Fig. 9, it will usually be found that when the machine is driven at its rated speed there is a critical value of the load resistance R above

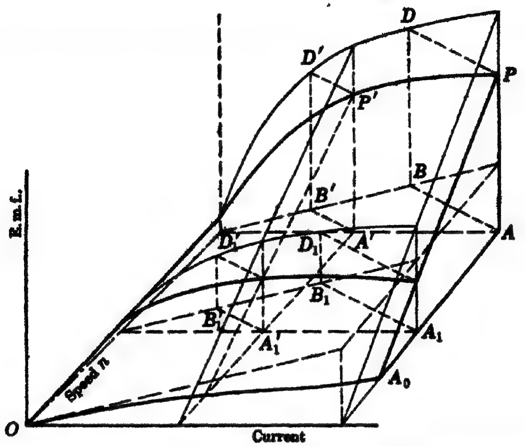


FIG. 11.—Effect of variation of speed upon external characteristic of series generator.

which the machine will fail to generate, or to "build up." When the load resistance has been lowered slightly below this critical value, the terminal voltage and current will at first rise rapidly and then more slowly until a condition of equilibrium is reached, but between the initial and final conditions the machine is in a state of unstable electrical equilibrium. Further reduction of R will cause the current and e.m.f. to change, but there is no further evidence of instability.

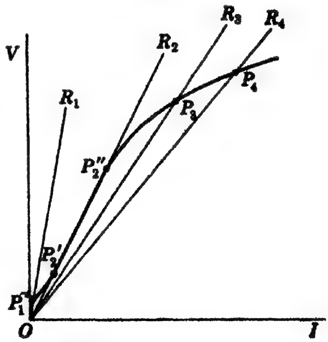


FIG. 12.—Condition for stable operation of series generator.

The reason for this behavior will be evident from a consideration of Fig. 12, in which the curve represents the external characteristic of the generator. It is evident from Fig. 9 that Ohm's law must hold for the external circuit, or $V = IR$. This is the equation of a straight line through the origin, the slope of

the line being proportional to R. Thus, OR_1, OR_2, OR_3, . . . , correspond to successively smaller values of R, and these lines are characteristic of the external circuit. Since the points representing simultaneous values of V and I must satisfy the characteristics of both generator and external circuit, the point of equilibrium will be at their intersection. Thus, when the external resistance is high, as R_1, the terminal voltage and current will be represented by the coordinates of point P_1. When $R = R_2$, the line OR_2 coincides for a greater or lesser range with the external characteristic; hence there is unstable equilibrium between the points P_2' and P_2''. Values of R such as R_3 and R_4 will give stable equilibrium at points P_3 and P_4, respectively.

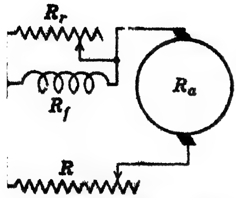


FIG. 13.—Connections of constant-current generator.

11. Regulation for Constant Current.—Inspection of the construction of Fig. 10 shows that the external characteristic of a series generator will droop more and more as the armature reaction and internal voltage drop become greater. In fact, by purposely exaggerating the magnitudes of these two quantities, the characteristic may be made to bend over to such an extent as to become nearly vertical, in which case the machine has an inherent tendency to regulate for constant current. Such a characteristic is desirable for series arc circuits, and some of the older types of Brush arc-light generators were designed on this principle.

Another method of regulating a series generator to make it deliver a constant current is to provide a variable resistance R_r shunted around the series field winding, as in Fig. 13. It is clear that a portion of the exciting current is then by-passed around the winding, the generated e.m.f. being thus reduced Every variation of the resistance R of the load must be accompanied by a corresponding change of the regulating shunt. The relation between the resistance of the regulating shunt and the corresponding terminal voltage of the machine is shown in a simple manner in Fig. 14.

Let OM be the saturation curve, with abscissas equal to the current in the field winding (instead of ampere-turns), and let $OA = I$ be the constant current that the machine is required to

develop; in other words, the external characteristic is to be the vertical line AY. Lay off AB to any convenient scale equal to the resistance R_f of the series field winding, and to the same scale lay off BC equal to any arbitrarily selected value of the resistance R_r of the regulating shunt. AB will be fixed in magnitude whereas BC will be variable. Vertically upward from A, set off $AD = IR_f =$ constant, also $DN = IR_a =$ constant; and downward from A, set off $AG = OA = I =$ constant.

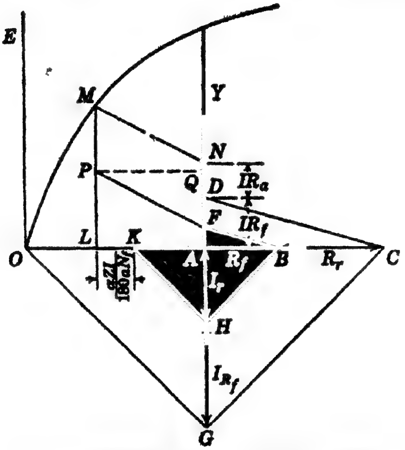


FIG. 14.—Regulation for constant current.

The total current will divide between R_f and R_r in such a manner that

$$I_r + I_{R_f} = I \tag{4}$$

and

$$\frac{I_r}{I_{R_f}} = \frac{R_f}{R_r} \tag{5}$$

Hence, if C and G are joined by a straight line and BH is drawn parallel to CG, point H will divide $AG = I$ into two parts, such that $AH = I_r$ and $HG = I_{R_f}$. Upon joining G and O and drawing HK parallel to GO, OK will be the current through the series field winding. Upon setting off $KL = \alpha ZI/180aN_f$, or the equivalent demagnetizing current of the armature, OL will be the net excitation of the machine, and LM the corresponding generated e.m.f.; if MP is the ohmic drop in the machine, or

$I\left(R_a + \frac{R_rR_f}{R_r + R_f}\right)$, the terminal voltage will be $V = LP = AQ$. To find MP graphically, proceed as follows: Connect C and D, and draw BF parallel to CD; then $DF = I\frac{R_rR_f}{R_r + R_f}$, and $NF = I\left(R_a + \frac{R_rR_f}{R_r + R_f}\right)$. Therefore, point P is found by joining M and N and drawing FP parallel to MN. Finally, therefore, AQ is the terminal voltage corresponding to the shunt resistance $R_r = BC$. Other points may be found by exactly similar construction.

It will be evident from the above discussion that the voltage of a series generator can also be controlled by varying the position of the brushes, angle α being changed thereby and the length KL in Fig. 14 being affected.

Applications of Series Generator.—The characteristic curves of the series generator show that in general an increase of current drawn by the load causes a corresponding, but not proportional, increase of voltage, but that it is possible to make the machine regulate for constant-current output by the use of suitable auxiliary devices. Accordingly the series generator is adapted to series arc-lighting service and is still used to some extent for that purpose, but it has been practically superseded, so far as this use is concerned, by later developments in the art of illumination; series-arc circuits are now usually supplied with current from mercury-arc rectifiers which are in turn supplied from a-c circuits through constant-current transformers. In Europe, series-wound generators, regulated for constant-current service, find a limited application in the Thury system of long-distance transmission of power (see Art. 19).

If a series machine is designed with an unsaturated magnetic circuit, both the magnetization curve and the external characteristic may be made to be nearly straight lines through the origin, so that the machine will develop a terminal voltage practically proportional to the current output. This feature makes the machine useful as a booster, in the manner explained in Art. 2, Chap. XIV.

12. Characteristics of Shunt Generator. *Open-circuit Conditions.*—Let

E = generated e.m.f.
V = terminal voltage.
I_a = armature current.
I = external or line current.
I_s = shunt-field current.
R_a = armature resistance.
R_s = shunt-field resistance, including regulating rheostat.
R = resistance of external load circuit.
N_s = field turns per pair of poles.
n = speed in r.p.m.

When the load or receiver circuit of a shunt generator is disconnected, as in Fig. 15, the armature and shunt field constitute

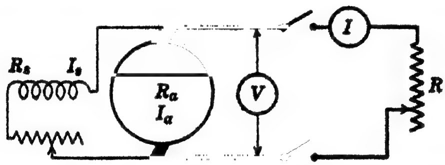


FIG. 15.—Determination of external characteristic, shunt generator.

a simple series circuit identical with that of Fig. 9. It is therefore easily seen that variation of the shunt-field rheostat will give rise to changes in V and I_s in the manner already discussed in Art. 10 in connection with the requirements for the stable operations of a series generator. That is to say, if a shunt generator is driven at rated speed, the external circuit being disconnected, it will not build up its terminal voltage if the resistance of the regulating rheostat is too high, even if the connections are otherwise correct and the residual magnetism has the proper polarity.

Although the manner of building up is the same in an open circuited shunt generator as in a series generator, there is this difference between the two types, that the high resistance of the shunt field winding will limit the flow of current I_s to values that are small compared with the current-carrying capacity of the armature; therefore, the observed readings of V under the conditions assumed will not differ appreciably from E, the total generated e.m.f. Thus, with the external circuit open,

$$E = V + I_s R_a \cong V \tag{6}$$

since both I_s and R_a are small. Moreover, the small flow of cur-

rent in the armature means negligibly small armature reaction; hence the relation between V and I_s will be closely represented by the saturation curve of the machine.

Since $I_s = V/R_s$, the effect upon the terminal voltage of changing the value of R_s will be given by a construction identical with that of Fig. 12; generally, there will be a region of unstable equilibrium in the building up of the generated e.m.f.

External Characteristic.—The form of the external characteristic showing the relation between terminal voltage V and line current I can be determined by the following method:* From Fig. 15 it is evident that

$$E = V + I_a R_a \tag{7}$$

$$I_a = I + I_s \tag{8}$$

$$I = \frac{V}{R} \tag{9}$$

$$I_s = \frac{V}{R_s} \tag{10}$$

The relation between the generated e.m f. E and the field excitation $N_s I_s$ (ampere-turns per pair of poles) is given by the satura-

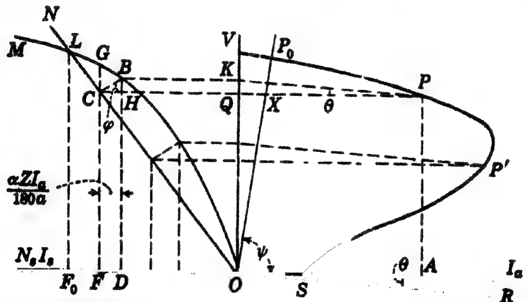


FIG. 16.—Construction of external characteristic of shunt generator

tion curve M, Fig. 16. If the machine is running on open circuit ($R = \infty$), let the resistance of the field rheostat be so adjusted that the excitation is represented by OF_0, the generated e.m.f. then being F_0L, which is nearly equal to the terminal voltage on open circuit if the small drop I_sR_a in the armature is neglected. The line ON is the "field-resistance" line, its slope being V/N_sI_s

* W. S. FRANKLIN and W. ESTY, "Elements of Electrical Engineering," Vol. I.

$= R_f/N_f$, that is, proportional to R_f; it corresponds to the lines OR_1, OR_2, . . . , of Fig. 12. It will be clear that coordinates of all points on the line ON represent simultaneous values of terminal voltage and field excitation.

Now let the external circuit be closed, R being so adjusted that a moderate current will flow. Then, even were the excitation to remain constant, as in the separately excited generator, the terminal voltage would fall because of the ohmic drop in the armature winding, and from Eq. (7),

$$V = E - I_aR_a \tag{11}$$

But, in the case of the shunt machine, a decrease of terminal voltage is accompanied by a proportional decrease of excitation, since $I_f = V/R_f$; hence, when there is an appreciable load current flowing, the flux and the generated e.m.f. are reduced, and a further reduction of V thereupon ensues. Consequently, the greater the load current, the less will be the terminal voltage, and the drop of terminal voltage will be greater in the shunt machine than in the separately excited machine, other things being equal.

Suppose that the load has been increased to such a value that the terminal voltage has fallen to the value OQ, Fig. 16; the problem is then to locate the point P on the horizontal line CQP such that P is a point on the external characteristic.

Through Q draw the horizontal line QC intersecting ON in C; OF then represents to scale the new value of N_fI_f. If there were no armature reaction, the ordinate FG would be the total generated e.m.f. corresponding to the excitation $N_fI_f = OF$; and therefore, since

$$I_aR_a = E - V$$

CG would represent to scale the value of I_aR_a. But since armature reaction does exist, the net excitation is less than OF by an amount FD, where

$$FD \quad \frac{\alpha ZI_a}{180a}$$

Actually, therefore, the net excitation is OD, and the generated e.m.f. is $E = BD$.

$$\therefore I_aR_a = E - V = BD - HD = BH$$

and

$$\frac{BH}{CH} = \frac{BH}{FD} = \frac{I_a R_a}{\alpha Z I_a/180a} = \frac{180aR_a}{\alpha Z} = \tan\varphi = \text{constant}$$

It follows that, when a point C on the field resistance line ON has been fixed, point B is found by drawing through C a line CB making the fixed angle φ with the horizontal.

Through B draw the horizontal line BK, and through K draw KP at an angle θ with the horizontal, this angle being so chosen that $\tan\theta = R_a$ to the scale of the figure. It follows that

$$QP = \frac{KQ}{\tan\theta} = \frac{I_a R_a}{R_a} = I_a$$

Hence, P is a point on a curve of which the ordinates are terminal voltage V and the abscissas are total armature current I_a. Corresponding values of line current I can be found by subtracting I_s; this can be done graphically by drawing the line OP_0 at an angle ψ such that $\tan\psi = R_s$ to the scale of the figure; for it is easily seen that

$$QX = \frac{OQ}{\tan\psi} = \frac{V}{R_s} = I_s$$

Hence, $AP = V$ and $XP = I$ are simultaneous values of terminal voltage and line current. Similar construction serves to locate additional points, such as P', as illustrated in Fig. 16.

The construction shows that the external current at first increases as the load resistance is lowered but that eventually a critical point is reached beyond which a further lowering of the external resistance causes the current to decrease rapidly. The terminal voltage falls steadily throughout the entire process, becoming zero when the machine is completely short-circuited ($R = 0$); under this condition the external current is not zero but has a small value OS due to the fact that residual magnetism generates a small e.m.f. that is entirely consumed in driving the current through the armature resistance. It might be inferred from these facts that a shunt generator can be short-circuited without danger, but this is not the case except in very small machines; for the critical point at which the line current begins to decrease is generally far beyond the safe current-carry-

ing capacity of the armature, and the winding will burn out before the current has had time to decrease to a safe value.

Applications of the Shunt Generator.—Within the range of load determined by consideration of safe heating limits, a properly designed shunt generator, when driven at rated speed, has an inherent tendency to regulate for nearly constant voltage. The drop in voltage between no load and full load, though somewhat greater than in a separately excited generator of the same design, may be kept quite small. Shunt generators are therefore suitable for constant-potential circuits where the load is so close to the generator that the drop of potential in the line resistance is of no consequence; for example, a shunt generator may be used as an exciter for the field circuit of an alternator. Control of terminal voltage to meet changes of load may be made by hand adjustment of the field rheostat if the changes of load are not too rapid; and if the load is of a rapidly fluctuating character, the terminal voltage can still be kept practically constant, or can even be made to rise with increasing load, by an automatic device such as the regulators described in Arts. 22 and 23.

A shunt generator is well adapted to such service as charging storage batteries, for as the battery approaches the condition of full charge its e.m.f. rises and so tends to reduce the amount of charging current; but because of the drooping characteristic of the shunt machine, the decrease of current is accompanied by an increase of the generator e.m.f., and hence there is an automatic balance that prevents the battery from discharging back through the generator.

13. Dependence of Form of Characteristic upon Speed.—The diagram of Fig. 16 was drawn subject to the condition that both the speed and the resistance of the shunt circuit remain constant. A change in speed (R_s remaining the same) will alter the form of the characteristic, and the new relations between V, I_a, and n can be most easily shown by a three-dimensional diagram such as Fig. 17. In this figure the surface OO_1M, drawn to the left of the speed axis, is the locus of the saturation curves for various values of speed, and to each saturation curve there will correspond a characteristic L_1, L_1', . . . , the locus of which has the tubular form shown in the diagram.

If the shunt-field resistance has a constant value, the locus of the field resistance lines ON will be the plane OO_1N, and the

intersection of this plane with the saturation surface OO_1M will be a curve $OL'L$. The projection of this curve on the V, n plane will give curve OL'_1L_1, which shows the relation

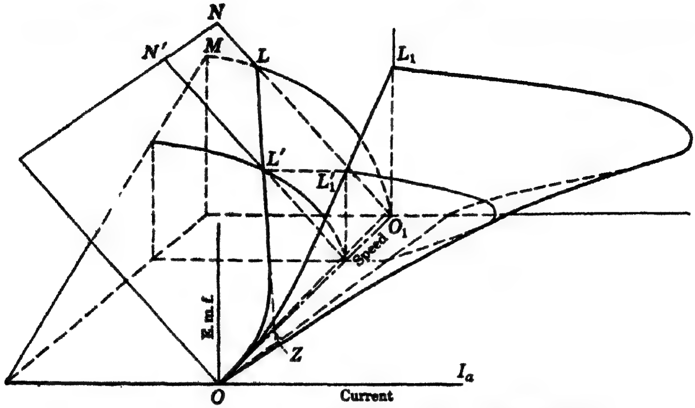


FIG. 17.—Effect of variation of speed upon external characteristic of shunt generator.

between terminal voltage and speed when the generator is operating on open circuit. If there were no residual magnetism, curve $OL'L$ would not pass through the origin but would intersect the speed axis in a point Z; that is, if there were no residual

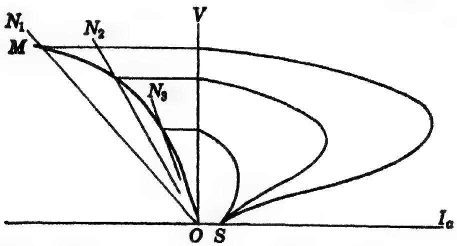


FIG. 18.—Effect of variation of shunt-field rheostat upon external characteristic.

magnetism, the machine would fail to build up for any speed below a critical speed OZ.

14. Dependence of Form of Characteristic upon Resistance of Shunt-field Circuit.—If the speed of a shunt generator is kept constant and the resistance of the field circuit is varied by means of the regulating rheostat, the size and shape of the characteristic will be affected in the manner shown in Fig. 18. OM is

the saturation curve corresponding to the speed at which the machine is driven, and ON_1, ON_2, . . . , are the field-resistance lines corresponding to the setting of the rheostat. The construction of the several characteristics is carried out in the manner described in connection with Fig. 16.

15. Approximate Mathematical Analysis of Shunt-generator Characteristics.—It will be evident from the preceding articles that the form of the external characteristic is in all cases dependent upon that of the saturation curve, and hence an equation representing the relations between the variables V, I, and n must depend upon the equation representing the saturation curve. Since the latter would necessarily involve a relation between B and H for the iron comprising part of the magnetic circuit and since such a relation is entirely unknown, the best that can be done is to represent the saturation curve by an empirical equation originally due to Froelich, which can be written

$$E = \frac{anI_s}{b + I_s} \tag{12}$$

where a and b are constants and n is the speed. If the speed is held constant, this equation represents a hyperbola, with asymptotes as shown in Fig. 19. A suitable choice of the constants a and b will make this hyperbola agree very well with the actual saturation curve within the working range of the machine; but it cannot be made to follow the irregularities in the actual curve at low magnetizations, and it does not take account of residual magnetism unless the equation is modified in the manner discussed in Art. 3, Chap. VIII, by shifting the origin of coordinates to the right.

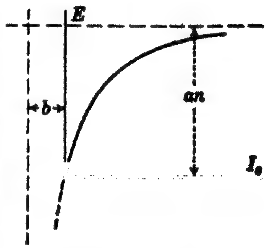


FIG. 19.—Empirical form of saturation curve.

If both numerator and denominator of Eq. (12) are multiplied by N_s (the number of field turns per pair of poles), it becomes

$$E = \frac{an \times \text{amp.-turns per pair of poles}}{bn_s + \text{amp.-turns per pair of poles}}$$

which can also be written

$$E = \frac{An \times \text{amp.-turns per pole}}{B + \text{amp.-turns per pole}}$$

Under load conditions, the number of ampere-turns per pole (or per pair of poles) to be substituted in the above equation is the *net* number obtained by subtracting the armature demagnetizing ampere-turns from the number of ampere-turns supplied by the field winding.

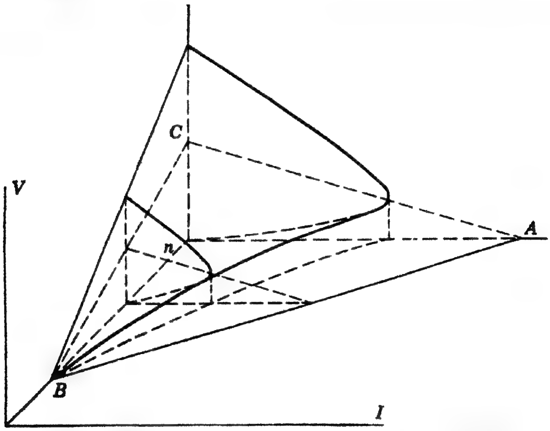


FIG. 20.—Idealized external characteristic of shunt generator.

Using Froelich's equation in the form of Eq. (12) and *ignoring armature reaction,* we have the following relations [see Eqs. (7), (8), (10)]:

$$E = \frac{anI_s}{b + I_s} = V + I_aR_a \tag{13}$$

$$I_a = I + I_s$$

$$I_s = \frac{V}{R_s}$$

whence

$$V = \frac{anV}{bR_s + V} - \left(I + \frac{V}{R_s}\right)R_a \tag{14}$$

On solving for V and simplifying by assuming that R_a is small compared with R_s (i.e., $R_a + R_s \cong R_s$), there is obtained

$$V = \frac{an - bR_s - IR_a}{2} \pm \sqrt{\left(\frac{an - bR_s - IR_a}{2}\right)^2 - IR_aR_sb}$$

$$= \frac{an - bR_s - IR_a}{2} \pm$$

$$\frac{1}{2}\sqrt{[an - (\sqrt{IR_a} - \sqrt{bR_s})^2][an - (\sqrt{IR_a} + \sqrt{bR_s})^2]} \tag{15}$$

This is an equation of the second degree between the three variables V, I, and n; hence, it represents a surface (Fig. 20), plane sections of which are either conics or straight lines. Moreover, the surface is symmetrical with respect to the plane

$$V = \frac{an - bR_s - IR_a}{2} \tag{16}$$

which is shown in the figure as ABC.

If, in the general equation of the surface, Eq. (15), there is substituted $I = 0$, it is seen that

$$\left.\begin{aligned} V &= an - bR_s \\ V &= 0 \end{aligned}\right\} \quad \text{and} \tag{17}$$

The first of these two equations represents open-circuit conditions; the second, short-circuit conditions. From the former it appears that $V = 0$ when $n = bR_s/a = OB$; hence, OB is the critical speed below which the machine would fail to build up if residual magnetism were not present.

Upon inserting in the general equation for V the condition $n =$ constant, there will result the equation of the external characteristic corresponding to the selected value of speed. It is obvious that there are two values of I that will reduce the radical to zero, and hence the characteristic intersects the plane ABC in two points; one of these values of current is $(\sqrt{an} - \sqrt{bR_s})^2/R_a$ the other is $(\sqrt{an} + \sqrt{bR_s})^2/R_a$. Between these values of current the radical becomes imaginary, and hence the theoretical external characteristics are hyperbolas. If, instead of neglecting armature reaction, it is desired to take it into account, Eq. (13) would have to be changed to read

$$E = V + I_aR_a = \frac{an\left(N_sI_s - \dfrac{\alpha ZI_a}{180a}\right)}{b + \left(N_sI_s - \dfrac{\alpha ZI_a}{180a}\right)} \tag{18}$$

Example.—A four-pole, 120-volt shunt generator, rated at 25 kw. at 900 r.p.m., has a saturation curve represented by the equation

$$E = \frac{180 \times \text{amp.-turns per pole}}{2000 + \text{amp.-turns per pole}}$$

The field winding has 800 turns per pole and a hot resistance, not including the field rheostat, of 20 ohms. The armature has a simplex wave winding of 194 face conductors, one turn per element, and the armature resistance is 0.0245 ohm. The angle of brush lead is equivalent to two segments of the commutator. If the field rheostat is adjusted to give an open-circuit voltage of 125 volts, what will be the terminal voltage when the machine is delivering full-load current?

Solution.—The field ampere-turns at no load can be found from the relation

$$125 = \frac{180 \times \text{amp.-turns per pole}}{2000 + \text{amp.-turns per pole}}$$

whence

Field ampere-turns per pole at no load $= 4550$

Shunt field current at no load $= {}^{4550}/_{800} = 5.7$ amp.

$$\text{Shunt field resistance (total)} = \frac{125}{5.7} = 22 \text{ ohms}$$

Resistance in field rheostat $= 22 - 20 = 2$ ohms

Again,

$$\text{Full-load current} = I = \frac{25{,}000}{120} = 208 \text{ amp.}$$

Number of commutator segments $= S = {}^{194}/_{2} = 97$

$$\alpha = 2 \times {}^{360}/_{97} = 7.43 \text{ deg.}$$

Demagnetizing amp.-turns per pole =

$$\frac{\alpha Z I_a}{360a} = \frac{7.43 \times 194}{360 \times 2}\left(208 + \frac{V}{22}\right)$$

$$= 416 + \frac{V}{11}$$

$$\text{Drop in armature} = I_a R_a = \left(208 + \frac{V}{22}\right) \times 0.0245 = 5.1 \text{ (nearly)}$$

Generated e.m.f. $= V + 5.1$

$$\therefore V + 5.1 = \frac{180\left(800\frac{V}{22} - 416 - \frac{V}{11}\right)}{2000 + 800\frac{V}{22} - 416 - \frac{V}{11}}$$

whence

$$V = 110.5 \text{ volts or } 20.7 \text{ volts}$$

The larger value of V is the full-load voltage, and the smaller value is that which corresponds to a line current of 208 amp. after the reverse bend in the external characteristic has been passed.

16. Characteristic Curve of Compound Generator. *Long-shunt Connection.*—The drop in terminal voltage between no load

and full load inherent in a shunt generator can be compensated, partly or wholly, or even over-compensated, by the addition of a series field winding excited by the armature current. It has been pointed out previously that the object of over-compounding is to keep the line voltage constant at a distant point, at or near the center of distribution of the load, the increase in the voltage at the generator terminals being consumed by the resistance of the line. In a general way the compound-wound generator (Fig. 21) may be considered as combining the rising characteristic of the series generator with the drooping characteristic of

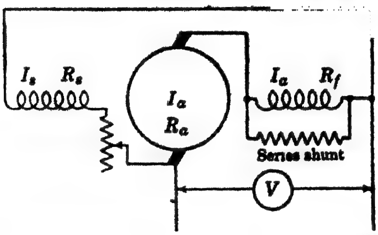


FIG 21 —Connections of long-shunt compound generator.

the shunt generator, the slope of the resulting curve depending upon the relative slopes of the components

Upon starting as before with the saturation curve $O'M$, Fig. 22, the external characteristic can be constructed in a simple manner as follows:

Let ON be the shunt-field resistance line, its equation being

$$V = \frac{R_s}{N_s}(N_sI_s) \tag{19}$$

The intersection of this line with the saturation curve determines a point L of which the ordinate is (very nearly) the terminal voltage at no load. Assuming that we are dealing with an over-compounded machine, let $F_1G_1 = AP$ be the terminal voltage corresponding to a value of $I_a = OA$ (the latter being supposed to be known). The field excitation due to the shunt turns is given by OF_1, and the total field excitation is OF_2, where $F_1F_2 = N_fI_a$ is the excitation supplied by the series turns. The net excitation, or OF, is less than this by an amount $FF_2 = \alpha ZI_a/180a$ = demagnetizing ampere-turns per pair of poles; hence, the e.m.f. actually generated in the armature is FG. The difference between FG and F_1G_1, or GH, must therefore

be the drop in the armature and series field, or $I_a(R_a + R_f)$. Summarizing,

$$F_1F_2 = G_1G_2 = N_fI_a$$
$$G_2H = \frac{\alpha Z}{180\ a} \cdot I_a$$
$$GH = (R_a + R_f)I_a$$

It follows, therefore, that all three sides of the triangle GG_1G_2 are proportional to I_a; hence, their ratios remain fixed no matter what the value of I_a may be, and the angles at the vertices of the triangle are also constant. In particular, the slope of the side GG_1 is constant, and its length is proportional to I_a. Hence, if it is desired to find a point P' on the characteristic curve

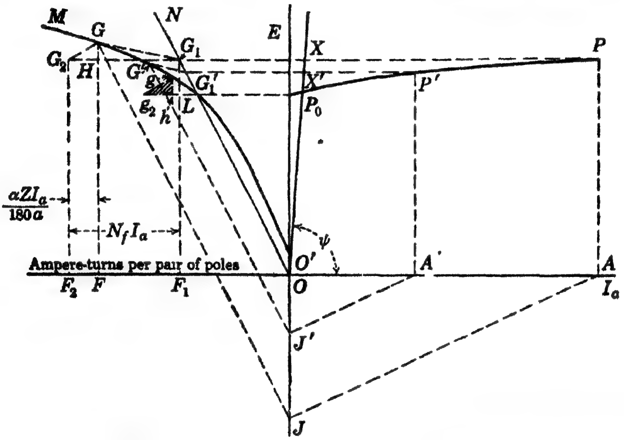


FIG. 22.—Construction of external characteristic of compound generator.

corresponding to any other current OA, all that is necessary is to fit a line $G'G_1'$ between curve M and the line ON, in such manner that $G'G_1'$ is parallel to GG_1 and so that the ratio of their lengths shall be equal to the ratio of OA' to OA. The following construction is thus indicated:

Through the point G draw GJ parallel to ON, J being on the axis of ordinates (prolonged downward); and join J with A Draw $A'J'$ parallel to AJ, draw $J'G'$ parallel to JG until it inter sects curve M in G', and through G' draw $G'G_1'$ parallel to GG_1, the point G_1' being on the line ON. Draw $G_1'P'$ horizontally until it intersects ordinate $A'P'$ in the point P', which is a point on the curve showing the relation between V and I_a.

If it is desired to show the relation between V and I, it is necessary to subtract the value of I_s from each corresponding value of I_a. To do this, draw the line OX at an angle ψ with the horizontal so that $\tan \psi = R_s$ to the scale of the drawing. Then PX and $P'X'$ are the values of I corresponding to I_a equal to OA and OA', respectively.

This method presupposes a knowledge of the coordinates of at least one point on the characteristic. The chief value of the construction lies in the clearness with which it shows the intimate relation between the saturation curve and the external characteristic. Thus, it becomes evident from the diagram that the external characteristic will approach a linear form more and more nearly as the saturation curve flattens out (that is, when the figure LG_1G approaches triangular shape). If the point L is so placed that it is below the knee of the curve, the external characteristic will become more and more convex (from above), the curvature being considerable for small values of the load and less pronounced as the load increases. Considerations of this kind become important when the specifications of a machine call for a compounding that shall not depart from a linear relationship by more than a limited amount.

A study of Fig. 22 shows that a reduction in the number of series turns will shorten G_1G_2 and will make the characteristic more nearly horizontal. If it is desired to make the terminal voltage at full load equal to that at no load, that is to say, to make the machine flat-compounded, GG_1G_2 (assumed to correspond to full-load conditions) will degenerate to Lgg_2, where $g_2gh = G_2GH$. It is clear from the construction that the characteristic of a flat-compounded generator, like that of an over-compounded generator, cannot be exactly a straight line because of the curvature of the saturation curve.

Short-shunt Connection.—In this case the current through the series winding is $I = I_a - I_s$; hence, the construction of Fig. 22 is not strictly applicable. But at or near full load the difference between I and I_a will be relatively small, especially in large machines, so that the above method will give a very close approximation to correct results.

Application of Compound Generators.—The cumulative compound generator is used more than any other type of d-c generator for the reason that its characteristics adapt it to all classes of

service which require constant voltage at the point of application of the load. The rise of voltage from no load to full load can be made to have any desired value, from zero up to any reasonable limit, so that the drop of potential in the transmission circuit can be compensated. Compound generators are used for supplying current to incandescent lamps, for heavy power service such as electric railways, and in general for motor drives requiring d-c supply at constant voltage.

Compound generators of the differential type, in which the series field opposes the shunt field, will obviously have a steeply drooping external characteristic—similar to that of a shunt generator with exaggerated armature resistance and demagnetizing action. Such a characteristic finds a useful application in connection with electrically operated shovels; for if the shovel suddenly encounters a considerable obstacle, it is desirable to reduce the voltage impressed upon the motor drive in order to prevent the complete stalling of the motor and the opening of its circuit breaker. Another useful application of the differentially compounded generator is in connection with arc welding, where increased current flow should be accompanied by correspondingly reduced voltage.

17. Series Shunt.—In practice it is quite common to design the series field windings of compound generators with a sufficient number of turns to produce the maximum percentage of compounding that may reasonably be specified. If a lesser degree of compounding is required, the magnetizing effect of the series winding is then reduced by connecting a shunt across the terminals of the series winding, as indicated in Fig. 21. This shunt is made of German-silver strip and serves to by-pass a portion of the main current. The total current will divide between the series winding and its shunt in the inverse ratio of their resistances, provided that the load current remains steady or changes very slowly; but in cases where the load is subject to sudden fluctuations, as in street-railway service, the current will not divide properly between the inductive series winding and the noninductive series shunt while the current is changing, for the self-inductance of the series winding tends to retard any change of current therein, whereas the noninductive series shunt introduces no such effect. For example, if there is a sudden increase of load current, the series winding will have less than its pro-

portionate amount of the total current during the time the change occurs, and thus delay the increase of excitation required to build up the voltage to the larger value demanded by the increased load. The remedy for this difficulty is to make the series shunt inductive by threading the resistor material of which it is made through a laminated iron core somewhat in the manner indicated in Fig. 47, Chap. III; to secure proper division of the total current through the series winding and its diverting shunt under all conditions, both must have the same time constant.

18. Connection of Generators for Combined Output.—When the load on a circuit exceeds the capacity of a single generator, one or more additional units must be connected to supply the excess. Thus, in a constant-current system in which the voltage varies in proportion to the load, additional generators must be connected in series when the voltage limits of the machine or machines already in service have been reached. Similarly, in constant-potential systems, additional generators must be put in parallel with those already in service when the safe current-carrying capacity of the latter has been reached.

19. The Thury System.*—The series system, in which series-wound generators, regulated to give constant current, are connected in series, has thus far found no application in the United States, save in those now obsolete plants in which constant-current motors were supplied from high-voltage arc circuits. But in Europe this system was developed to a high state of perfection through the work of R. Thury, who installed a number of plants operating on this principle, most of them in Switzerland, Hungary, and Russia.†

In the Thury system the series-wound generators are driven at constant speed, and the current is kept constant by a regulating device that shifts the brushes (though it may be arranged to vary the speed). The regulating device is actuated by a solenoid through which the main line current flows. Sufficient generators are connected in series to develop the voltage required by the load. The load consists of series motors, also connected in series, which in turn drive generators (generally alternators)

* See also Chap. XI.

† *Jour. Inst. Elec. Eng.* (*London*), **38**, 471, 1906–07.
Elec. World, **63** (No. 11), 361, 583, 1914.

for the supply of current at the receiving or distributing end of the line. This system is generally used for the transmission of power over considerable distances, as distinguished from merely local distribution. The original generating station of the Moutiers-Lyon installation in the Rhone valley was equipped with four sets of generators, each set consisting of four generators grouped in pairs, that is, with 16 generators in all. Each generator developed 3600 volts per commutator at 300 r.p.m., and the total station voltage was 57,600. Three additional sets of generators were later installed at Roziere, each of these sets delivering 18,000 volts at 150 amp. with 4500 volts per commutator. This 275-mile system transmits a maximum power of about 20,000 kw. at 125,000 volts.*

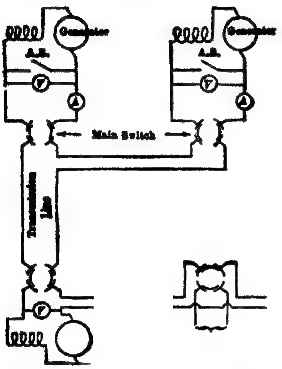


FIG. 23.—Diagram of connections, Thury system.

The starting and stopping of the generators in the Thury system is very simple. Each generator is equipped with an ammeter, a voltmeter, and a switch, as in Fig. 23, the switch being so arranged that when it is in the "off" position the generator is short-circuited and when in the "on," or running position, the machine is in series with the line. To start the machine, the switch being in the off position, the prime mover is brought up to normal speed, and the switch thrown to the running position when the ammeter reads normal current. To shut down the machine this process is reversed.

Since all the machines are in series when under load, the potential of the circuit rises from generator to generator; hence, the machines must be carefully insulated from earth to prevent breakdown of the insulation.

20. Parallel Operation of Generators. 1. *Series Generators.*—Series generators connected in parallel as in Fig. 24 will not operate satisfactorily; for if one of them suffers a momentary reduction of its output (as from a momentary drop in speed), both its voltage and current will be reduced, as may be seen

* *Elec. World,* **106,** 1341, 1936; *Elektrotech. Z.,* **51,** 114, 1930.

from the form of the characteristic (Fig. 10). The other machine will then assume the part of the load dropped by its mate, and its current and voltage will accordingly rise. The increased voltage will cause a further increase of current; hence, an additional increment of load is thrown on the second machine, and the load, current, and voltage of the first will be still further reduced. This process will tend to continue until the first machine is driven as a motor by the second machine; moreover, the direction of rotation of the former will reverse when it becomes a motor, so that the connecting rod of its driving engine will tend to buckle. Series generators connected in parallel are, therefore, in unstable equilibrium, there being no inherent tendency to bring about a proper division of the load between the two units under consideration. This behavior is a consequence of the rising form of the external characteristic of the series generator.

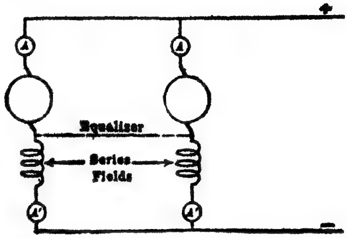


FIG. 24.—Series generators in parallel.

The natural instability of series generators in parallel can be overcome by the *equalizing connection* shown in Fig. 24 as a dashed line. The effect of this connection is to put the series field windings in parallel with each other. If one machine assumes more than its proper proportion of the total load, the excess current will divide between the two field windings, and the effect is to raise the excitation and voltage of the machine which has momentarily dropped its load and automatically to readjust the division of the load.

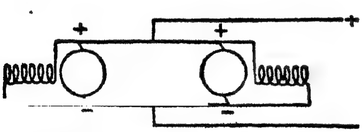


FIG. 25.—Shunt generators in parallel.

2. *Shunt Generators.*—The drooping form of the external characteristic of the shunt generator shows that, if such a machine drops its load, its voltage will automatically rise. Consequently, if two shunt generators are connected in parallel, as in Fig. 25, their operation will be stable. Any tendency that causes one machine to lose its proper share of current, and thus to shift an equal amount of current to the other, will result in a rise of voltage of the first machine and a drop in the

voltage of the second. If the prime movers are properly governed, the original conditions will be restored.

It is, of course, not necessary that the two (or more) generators thus connected in parallel shall have the same ratings. But it is essential to good operation that the machines should divide the total load, whatever that may happen to be, in proportion to their ratings. Suppose, for instance, that two shunt generators that are to be connected in parallel have external char-

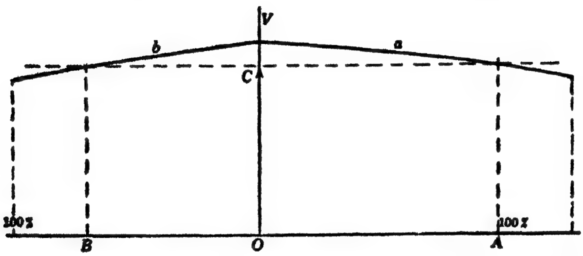


FIG. 26.—Division of load between shunt generators in parallel.

acteristics as shown in Fig. 26, curve *a* representing the characteristic of one machine, curve *b* that of the other. In this figure, ordinates are plotted in volts and abscissas in percentage of full-load current. Since the machines are in parallel, their terminal voltages must necessarily be equal; hence, if the load is such that the terminal voltage is OC, machine *a* will deliver OA per cent of its rated current and machine *b*, OB per cent. The diagram has been drawn to represent an unequal division of load between the two machines; but since proportionate division is desired, it follows that the characteristics, when plotted in percentage of full-load current, must be identical.

3. *Compound Generators.*—Inasmuch as the compound generator partakes of the characteristics of both shunt and series generators, two or more of them, if over-compounded, will operate satisfactorily in parallel only when the series fields are provided with equalizer connections as in Fig. 24. The equalizer is needed because of the rising characteristic. But if the machines have drooping characteristics, that is, if they are under-compounded, the equalizer is not necessary. The diagrammatic scheme of connections of two compound generators in parallel is shown in Fig. 27. It is clear that if ammeters were connected as at

A', they would not indicate the true current actually delivered by the machines to the external circuit, for the readings would be affected by the equalizing current. Thus, a heavily loaded generator might be supplying an equalizing current of large magnitude to the other lightly loaded machines, and at the same time the ammeter of the loaded machine would read low while those of the other machines would read high. For this reason the individual ammeters must be placed as at A, that is, in the lead that connects to the armature on the side *opposite* to the equalizing connection. For the same reason, if single-pole circuit breakers are used, they should be placed in the same lead as the ammeters; thus, if two machines in parallel are each delivering full-load current and one of them should develop a momentary drop in speed, the heavy equalizing current might open its circuit breaker, if incorrectly placed, with the result that the entire load would be thrown on the other machine and so open its circuit breaker also.

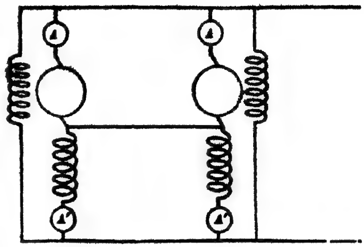


FIG. 27.—Compound generators in parallel.

The complete switchboard connections of two compound generators are shown in Fig. 28. The main switch and the equalizer switch are usually combined in a triple-pole switch.

The process of paralleling a compound generator with one or more that are already running is as follows: The main switch of the incoming machine being open and its circuit breaker closed, the prime mover is brought up to speed and the voltage of the incoming machine adjusted to equality with the bus-bar voltage by manipulation of the shunt-field rheostat; the main switch is closed, and proper division of the load is then secured, if necessary, by further adjustment of the field rheostat. To shut down a machine running in parallel with others, its load is shifted to the others by weakening its shunt-field current, and the main switch is opened when the ammeter indicates a small, or zero, current.

If two compound generators are to divide the load in proportion to their ratings, their characteristics must obviously be identical

as explained in connection with shunt machines. Moreover, since the series fields are in parallel by virtue of the equalizer connection, the resistances of the series windings, including the resistances of the respective equalizing leads, must be inversely proportional to the rated currents of the two machines. Neglect of this feature will result in a disproportionate division of load. For example, if the machines are at unequal distances from the switchboard, the resistance of the series field of the more remote machine will be unduly high because of the longer equaliz-

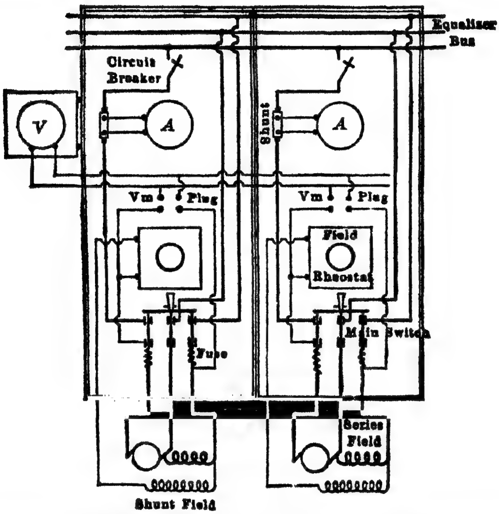


FIG. 28.—Diagram of switchboard connections, compound generators in parallel.

ing connection, and this machine will, therefore, not take its full share of the load.

The division of load between two over-compounded generators in parallel cannot be determined in the manner indicated in Fig. 26 for the case of shunt generators. For if the individual characteristics of the machines are not the same the adjustment of load between them is brought about by the equalizer connection, one machine carrying more than its normal series excitation, the other less. This internal adjustment alters the external characteristics of both machines, as may be seen by referring to the construction of Fig. 22, the change being due to the modified value of the series ampere-turns. Using the construction of

Fig. 26 in the case of over-compounded generators would make it appear that the machine having the more steeply rising characteristic would take the smaller load, whereas in reality the reverse is true.

The series field of a compound generator may be connected either to the positive or to the negative terminal of the armature. In street-railway generators built by a well-known company the series field is connected on the negative, or grounded, side. In this case it is not sufficient to use one single-pole circuit breaker A on the positive, or feeder, terminal; another circuit breaker B must be put in the lead to the grounded bus, as shown in Fig. 29. For if circuit breaker B were not present and the armature winding were to become grounded to the core, the short-circuit current through the armature and series field would sustain the excitation and maintain the short circuit without the possibility of protection by circuit breaker A.

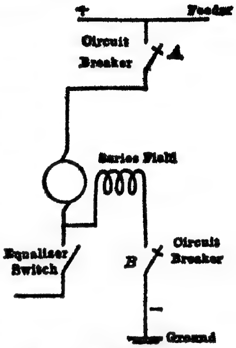


FIG. 29.—Diagram of connections of railway generator supplying grounded circuit.

21. Three-wire Generators.—Economy in the use of copper in distributing circuits for lighting and power dictates the selection of high voltage and moderate current, rather than low voltage and large current; but, in incandescent lighting, lamps designed for 110 to 115 volts are more efficient than those operating at higher voltages. To get the benefit of the high-efficiency 110- to 115-volt lamp and at the same time to obtain the copper economy of higher voltage, the three-wire system of distribution diagrammatically illustrated in Fig. 30 is extensively used. The reason for the greater economy of the three-wire system is evident from the following considerations:

In a two-wire distributing line, let V be the voltage at the generator end, R the resistance of the two line wires, and I the current transmitted. The total power supplied by the generator is VI watts, and the power lost in the line is I^2R watts, so that the ratio of loss to power developed is $p = I^2R/VI = IR/V$. If the same amount of power is to be transmitted over a three-wire line, the load being equally distributed on the two sides of the circuit, the generator voltage will be $2V$ and the current in each of the

two outer wires will be $I/2$. Since there will be no current in the middle wire under the assumed condition of balanced load, the entire loss will be due to the current $I/2$ flowing through the resistance R' of the two outer wires; and if this loss is to be the same percentage of the generated power as in the two-wire line, it follows that $(I/2)^2R' = I^2R$, or $R' = 4R$. Hence the cross-section and, therefore, the weight of the outer wires of the three-wire line need be only one-fourth that of the wires of the two-wire system, the length of the lines being the same in both cases. If the middle wire has the same section as the outer wires, the

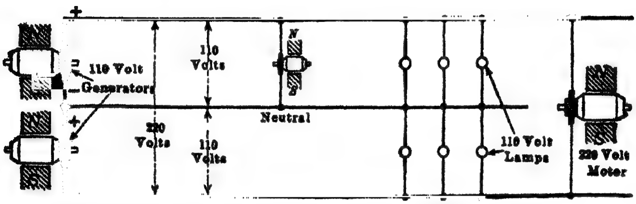


FIG. 30.—Three-wire system, two generators in series.

total amount of copper in the three-wire line will be only three-eighths as much as in the two-wire line.

In the three-wire system, the individual lamps, small motors, and other translating devices are connected between the outer wires and the middle, or *neutral*, wire, and larger motors, designed to operate on the higher voltage of the system, are connected between the outer wires. In the early forms of three-wire systems, the splitting of the moderately high voltage between the outer wires was accomplished by using two generators in series (Fig. 30), the neutral being tapped into their common junction.

A later arrangement, shown in Fig. 31, consisted of a main two-wire generator wound for the voltage between the outside wires, with a *balancer set* connected across the outside wires. If the load on the two sides of the system, that is, between neutral and outer wires, were exactly balanced, no current would flow in the neutral, and the neutral might then be omitted; this is sometimes done in 220-volt systems, the lamps being connected in series in pairs and connected across the main wires. But if the load is not exactly balanced, the neutral will carry a current equal to the difference between the currents supplied to the two sides of the system. The attempt is always made

to balance the system as completely as possible, but provision is usually made for an unbalancing of about 10 per cent, that is, 10 per cent of full-load current in the neutral wire. When a system employing a balancer set becomes unbalanced, the voltage on the more lightly loaded side tends to be higher than on the more heavily loaded side. In this case, the machine on the side having the lighter load operates as a motor and drives the other

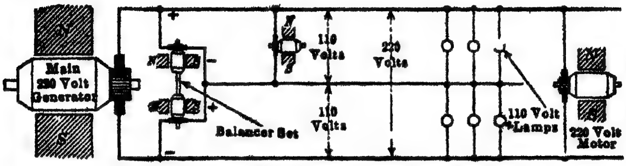


FIG. 31.—Three-wire system with balancer set.

as a generator; the latter then supplies current for the excess load on its side of the system and thus automatically tends to balance the system. With perfect balance of load both machines of the balancer set operate as motors running without load.*

Systems of the kind shown in Figs. 30 and 31 are open to the objection that they involve the use of more than one piece of running machinery and so require extra attendance and main-

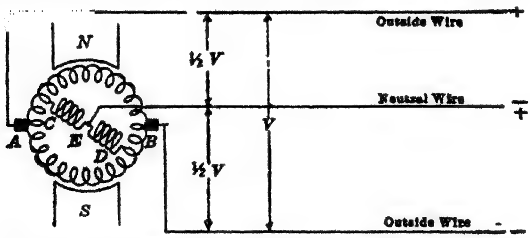


FIG. 32. Three-wire generator with coil mounted inside armature core.

tenance, in addition to being higher in first cost and lower in efficiency than a single machine of the same capacity. These objections are overcome by a system originally devised by Dobrowolsky and shown diagrammatically in Fig. 32. A coil of wire, *CED*, wound on an iron core, is tapped into the main armature winding of the generator at the points *C* and *D*, 180 electrical degrees apart, that is, points that occupy homologous positions with respect to poles of opposite polarity. The difference

* For a further discussion of balancer sets, see Chap. XIV.

of potential between C and D is alternating, so that the coil is traversed by an alternating current which goes through one cycle (two alternations) per revolution per pair of poles; this alternating current is small because of the large self-inductance due to the iron core on which the coil is wound. The middle

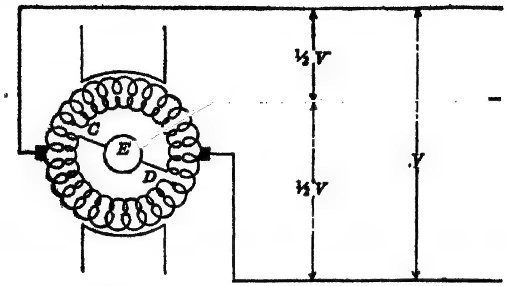


FIG. 33.—Three-wire generator with auxiliary winding in slots

point of the coil E has a potential midway between the potentials of C and D, and therefore also midway between the potentials of the brushes A and B, since the potentials of C and D are always symmetrically related to those of A and B. A tap brought out from the point E may then be used as the neutral

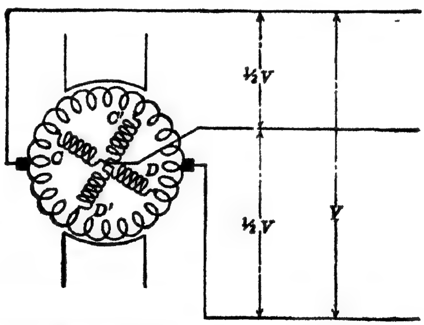


FIG. 34.—Three-wire generator with two coils tapped into armature winding.

wire of a three-wire system. In machines of this kind that have been built by the General Electric Company, the coil CD is wound on a core mounted inside the armature core, and the connection from the middle point E to the outside circuit is made through a single slip ring mounted on the main shaft of the generator. The Burke Electric Company designed a three-wire

generator with the coil *CD* wound in the same slots that carry the main armature winding, in the manner indicated in Fig. 33.

The balance coil *CD* may also be placed outside of the generator, connection to the armature winding being made in that case through two slip rings; or two balance coils may be used, con-

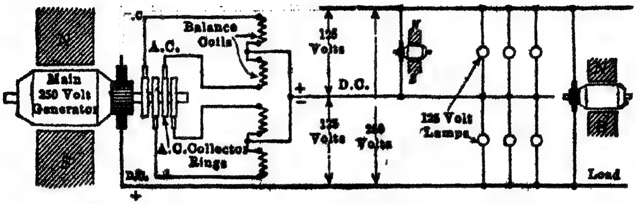


FIG. 35.—Three-wire generator with two auxiliary coils mounted externally.

nected to the armature winding as in Fig. 34. The alternating voltages between the points *C*, *D* and *C'*, *D'* are 90 electrical degrees apart, that is, one of them is a maximum when the other is zero, and vice versa. Figure 35 shows the connections when two balance coils, mounted externally to the generator, are used.

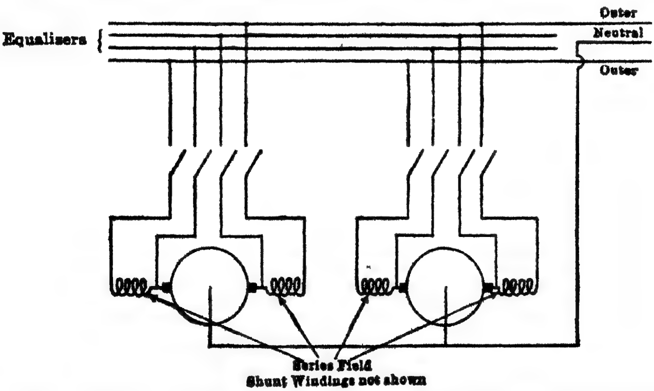


FIG. 36.—Diagram of connections of compound three-wire generators in parallel.

If three-wire generators are to be compounded, the series field winding must be in two equal parts, half of the turns being in series with one of the outer wires, the other half in series with the other outer wire, as in Fig. 36. If two or more three-wire generators are to be operated in parallel, two equalizer connections must be used; hence, the main switch of a three-wire generator is usually constructed with four blades.

22. Tirrill Regulator.—It has been shown in preceding articles how the voltage of shunt and compound generators may be regulated either by manual adjustment of the rheostat in the shunt field circuit or by the automatic compounding effect of the series field winding. In lighting circuits where steady voltage is of the greatest importance, accurate and automatic regulation of voltage may be obtained by the use of the Tirrill regulator; this device makes it possible to maintain a steady voltage at the generator terminals irrespective of changes in the load or of fluctuations of speed, and also to compensate for line drop by increasing the generator voltage as the load increases.

The regulator maintains the desired voltage by rapidly opening and closing a shunt circuit connected across the terminals of

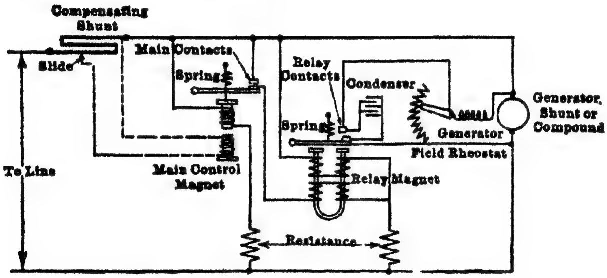


FIG. 37.—Diagram of connections of Tirrill regulator.

the exciter field rheostat. The field rheostat is so adjusted that when the regulator is disconnected the generator voltage is about 35 per cent below normal; on closing the regulator circuit the rheostat is short-circuited, and the generator voltage rises. When the voltage reaches a predetermined value, the short circuit around the rheostat is opened and the voltage again falls. The opening and closing of the short circuit around the rheostat is so rapid that the voltage does not actually follow the changes of the field-circuit resistance but merely tends to do so, with the result that incipient changes of voltage are immediately checked.

An elementary diagram of connections of the regulator is shown in Fig. 37. The opening and closing of the by-pass around the exciter rheostat is accomplished by means of contacts on the armature of a differentially wound relay magnet of U shape. One winding of the relay magnet is connected directly across the main bus bars, in series with a current-limiting resistor; the other winding is also connected across the bus bars, but

through a pair of main contacts actuated by the main control magnet. The latter is wound with a potential coil, connected directly across the bus bars, and a current coil (which may or may not be used) whose magnetizing action opposes that of the potential coil.

The operation of the regulator is as follows: If the generator voltage falls, the current through the potential coil of the main control magnet is weakened and the spring closes the main con-

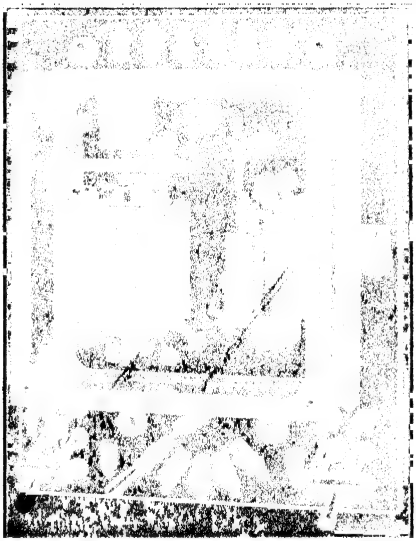


FIG. 38.—Voltage regulator. (*General Electric Company.*)

tacts. Current then flows through both windings of the relay magnet, which becomes demagnetized, and the spring closes the relay contacts, the field rheostat being thus short-circuited. As the voltage rises, the armature of the main control magnet is again pulled down, the main contacts are opened, the relay magnet is again energized, and the rheostat is thus again inserted in the field circuit. If the current coil of the main control magnet is used, its differential action will cause the voltage to rise higher, before the main contacts are opened, than would otherwise be the case, a compounding action thus resulting. The degree of compounding may be varied by means of the sliding contact on the compensating shunt with which the current coil is in parallel. The condenser shown in the figure is for the pur-

pose of reducing sparking at the relay contacts. A perspective view of a simple regulator built by the General Electric Company is shown in Fig. 38.

When several compound generators of moderate capacity are operated in parallel, a simple regulator may be connected to one of them and the others allowed to "trail." The generator provided with the regulator will take the fluctuations of load, and the load on the others will be equalized through the series windings. Regulators are also built for controlling the voltages of two or more generators operating in parallel; instead of using a single relay magnet, from two to ten are employed, part of them serving to short-circuit sections of the field rheostat of one generator, the others performing a like function for the other machines. Regulators of this kind are suitable for two-wire or three-wire generators with shunt or compound windings and will compensate for line drops up to 15 per cent.

In the case of very large machines, it is advisable to use separate excitation and to connect the regulators so that they act upon the exciter fields. The object of this modification is to limit the magnitude of the current which must be interrupted at the relay contacts, since it is not practicable to provide a condenser sufficiently large to take care of heavy current. In such a case the main control magnet would be actuated by the main bus-bar voltage and line current.

23. Counter E.M.F. Regulator.—The regulator described in the preceding article operates by the alternate increase and decrease of the resistance of the field circuit of a generator—that of the main generator if it has moderate capacity, otherwise the field of an exciter of which the armature supplies the field of the main generator. But the current in a circuit can be controlled not only by varying the resistance of the circuit but also by keeping the resistance fixed and inserting into the circuit a variable e.m.f. If this auxiliary e.m.f. acts in the same direction as the principal impressed voltage, the current will increase, just as though the resistance had been lowered; conversely, a decrease of the auxiliary e.m.f., equivalent to a reversal of part or all of the original "booster" voltage, acts as though the resistance had been increased. In other words, a booster e.m.f. is equivalent to a negative resistance, and a counter e.m.f. is equivalent to a positive resistance.

This principle has been embodied in a form of regulator developed by the General Electric Company. The diagram of Fig. 39 illustrates the general scheme of connections. The armature of a small motor M is connected in series with the field winding F of the main generator G, and a fan is attached to the shaft of motor M. The field winding f of motor M is in series with a current-limiting resistor R_1 across the bus bars, and the terminals of f are connected to contacts C. These contacts are operated by a solenoid S which is connected across the main line, but in series with a current-limiting resistor R_2.

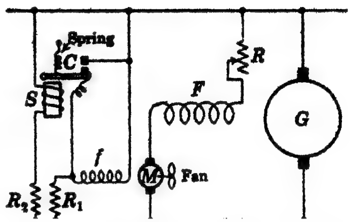


FIG. 39.—Counter e.m.f. regulator.

If the voltage of the main line rises, the solenoid S overcomes the restraining spring and opens the contacts C, current from the line being thus allowed to flow through the winding f. This current strengthens the field of motor M and thus increases the counter e.m.f. unless the drop in speed that accompanies the stronger field is too pronounced. But if the speed decreases, the frictional resistance of the fan decreases at a still greater rate, and this decrease of mechanical load tends to keep the speed up, the net result being that the flux due to f increases more than the speed falls, so that there is a resultant increase of counter e.m.f. Accordingly, the current through F is reduced and there is an immediate check upon the original incipient increase of main line voltage.

Similar reasoning shows that a drop in bus-bar voltage will cause contacts C to close, weakening the flux due to f and reducing the counter e.m.f. of M. The speed of M tends to increase, but this tendency is held within bounds by the fan. The decreased counter e.m.f. of M admits more current to F, and the line voltage is thus raised.

The opening and closing of the contacts C takes place at a rapid rate, any tendency toward change of voltage being checked in its incipience. It will be noted that the counter-e.m.f. regulator holds down the excitation of the main generator; that is, the field rheostat of the latter is set to give an excitation somewhat above normal. This condition is just the reverse of that obtaining in the case of the Tirrill regulator.

CHAPTER XI

MOTORS

1. Service Requirements.—In the industrial application of the motor drive, there are three principal classes of service, characterized by *constant speed, adjustable speed,* and *variable speed.* Constant-speed motors, of which the shunt motor is an example, maintain an approximately constant speed at all loads when supplied from constant-potential mains and are used for such purposes as driving line shafting, fans, etc. In the case of adjustable-speed motors, the speed can be fixed at any one of several values between a minimum and maximum value and when so set will remain substantially constant for all loads within the limits of the machine's capacity, the impressed voltage remaining constant throughout; motors of this kind are used, for example, in individual drives for machine tools. Variable-speed motors include those types in which the speed is inherently variable, changing as the load changes, with constant impressed voltage; examples of this class are the series motor and the cumulative compound-wound motor; their speed characteristics make them especially suitable for that class of service in which it is desirable to reduce the speed as the load increases, as in street-railway and in hoisting service.

Intelligent operation of motors requires a knowledge of the relations between speed, torque (turning moment), load (output), and the electrical and magnetic quantities involved. These relations determine the operating or *mechanical characteristics,* which will be discussed for the different types of motors. The principal types are designated in terms of the connections of the armature and field windings, as explained in Chap. VI; they include separately excited, shunt, series, and compound motors.

2. Counter E.M.F., Torque, and Power.—It has been shown in Chap. VI that when a current is sent into the winding of an armature which is under the influence of a magnetic field the individual conductors of the winding are subjected to a lateral

thrust. Motion of the conductors will result unless the resisting torque is greater than that developed by the reaction of the current upon the magnetic field. The immediate effect of this motion is to generate in the conductors an e.m.f. which has a direction opposite to that of the current. This *counter-generated e.m.f.* is called the *back e.m.f.* or the *counter e.m.f.*, and its magnitude is given by Eq. (7), Chap. VI.

The effective development of torque in a motor is dependent upon a proper space relation between the field flux and the

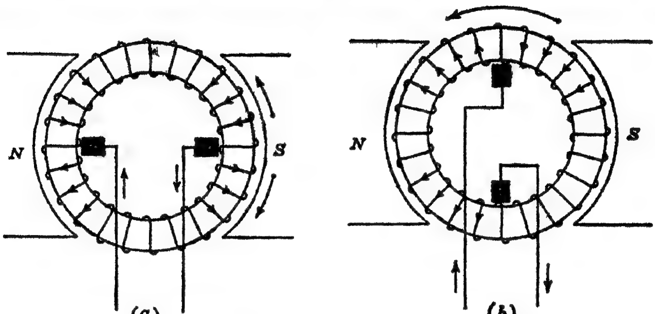


FIG. 1—Effect of brush position on the torque.

armature current. If, for instance, the brushes are so set that the axis of the armature current coincides with the axis of field flux, as in Fig. 1*a*, there is no resultant tendency to rotation; but if the axes of armature current and field flux are at right angles to each other, as in Fig. 1*b*, the torque will be a maximum for a given current in the winding.

Counter E.M.F. and Speed—In a separately excited or a shunt motor, the voltage impressed upon the armature terminals must be consumed in overcoming the back e.m.f. plus the ohmic drop due to the resistance of the armature winding and the brush contacts.

$$\therefore V = E_a + I_aR_a \tag{1}$$

where

$$E_a = \frac{p}{a}\frac{\Phi Z n}{60 \times 10^8}$$

In the series and long-shunt compound motor, there is an additional drop due to the resistance of the series field winding; hence,

$$V = E_a + I_a(R_a + R_f) \tag{2}$$

In the short-shunt compound motor the relation is

$$V = E_a + I_aR_a + IR_f \tag{3}$$

In general,

$$V = E_a + I_aR' = \frac{p}{a}\frac{\Phi Zn}{60 \times 10^8} + I_aR' \tag{4}$$

or

$$n = \frac{V - I_aR'}{\Phi Z'} \tag{5}$$

where R' = resistance of armature and circuits in series therewith, and

$$Z' = \frac{p}{a}\frac{Z}{60 \times 10^8} \tag{6}$$

The product $\Phi Z'$ is obviously equal to the number of volts generated in the armature winding when the speed of rotation is 1 r.p.m.

It is also seen that

$$I_a = \frac{V - E_a}{R'} \tag{7}$$

an equation that is of importance in connection with the starting of motors, as explained later.

In Eqs. (1), (2), (3), and (4), the term or terms representing the ohmic drop in the armature and any part of the field winding in series with it are quite small in comparison with the impressed voltage, within the safe limits of load; for if they were not, the loss of power represented by the product of the drop of potential and the current would injuriously lower the efficiency of the motor and would at the same time result in undue heating and a reduction in the safe load of the motor. The ohmic drop in the armature, expressed in percentage of the impressed voltage, is the smaller the larger the machine. It follows, therefore, that to a first degree of approximation *the counter e.m.f. is practically equal and opposite to the impressed voltage.* This is a fundamental principle of all types of motors, including a-c as well as d-c motors. Since most motors operate with constant impressed voltage, it follows that the counter e.m.f. is nearly constant within the working range for which the motor is designed. But it will be observed that for a given design

the counter e.m.f. is proportional to the product of the flux per pole and the speed; hence, if the flux is constant (or nearly so) under running conditions, the speed of the motor will likewise be substantially constant. On the other hand, if the flux is variable under running conditions, the speed will also be variable in nearly inverse ratio.

These conclusions should be compared with those readily deducible from the generator characteristics already discussed. Thus, in generators, the terminal voltage will be substantially constant if the flux is constant, provided that the speed is held fixed, as in separately excited, shunt and flat-compound machines; but where the flux varies, as in series generators, the terminal voltages also varies. It will be seen, therefore, that there is a reciprocal relation between generators and motors, voltage in generators being the inverse of speed in motors.

It will be seen from Eq. (5) that, if the flux Φ is reduced to a small value while the voltage impressed upon the armature remains constant, the speed will rise to a dangerously high value; the motor will "run away" and may wreck itself. This contingency may arise in the case of a shunt motor if the field circuit is opened, as by a broken wire or loose connection, and in the series motor by an accidental short-circuiting of the terminals of the series winding. This behavior is due to the tendency of any motor to run at such a speed that the back e.m.f. is nearly equal to the impressed voltage, the lowering of the flux demanding an increased speed.

Torque and Power.—Upon multiplying Eq. (4) by I_a and transposing, there results

$$VI_a - I_a^2R' = E_aI_a \qquad (6)$$

The term VI_a represents the power supplied to the armature, and I_a^2R' is the power dissipated as heat in the ohmic resistance of the armature circuit. The difference between these two terms, or E_aI_a, must therefore be the amount of *mechanical power developed by the armature.* Not all this developed power is available at the shaft or pulley, for some is lost in overcoming the friction of the bearings and brushes, windage (air resistance), and hysteresis and eddy currents in the armature core and pole faces

If P = total mechanical power, in watts, developed in the armature,
T = torque in dyne-centimeters ÷ 10^7,

$$P = E_a I_a = 2\pi \frac{n}{60} T \quad \text{watts} \tag{9}$$

or

$$T = \frac{60}{2\pi n} E_a I_a = \frac{60}{2\pi n} \frac{p}{a} \frac{\Phi Z n}{60 \times 10^8} I_a$$
$$= \frac{60}{2\pi} Z' \Phi I_a \tag{10}$$

where Z' has the meaning shown in Eq. (6). The foregoing unit of torque is inconvenient for practical application; expressing torque in kilogram-meters, pound-feet, and pound-inches, respectively,

$$\left.\begin{aligned} T &= \frac{60}{2\pi} \times \frac{10^7}{980 \times 10^3 \times 10^2} Z'\Phi I_a = 0.975 Z'\Phi I_a \quad \text{kg.-m.} \\ &= \frac{60}{2\pi} \frac{10^7}{980 \times 453.6 \times 30.48} Z'\Phi I_a = 7.05 Z'\Phi I_a \quad \text{lb.-ft.} \\ &= 84.6 Z'\Phi I_a \ \text{lb.-in.} \end{aligned}\right\} \tag{11}$$

It is clear from these equations that the torque is dependent only upon the flux and the armature current* and is independent of speed. It is to be understood, of course, that the brushes are properly set, in such a way that the axis of commutation is perpendicular to the axis of the flux; otherwise, the foregoing equations will not hold true.

The four equations, (4), (5), (7), and (11), may be said to summarize in analytical form the physical facts involved in the operation of any motor of the usual types. These facts may be stated as follows:

Assume that the motor has been started in some suitable manner, and that it is running with only a small load, the source

* For given values of the flux per pole, Φ, and the armature current I_a, the torque is found to be the same for both smooth-core and slotted armatures, notwithstanding the fact that the conductors embedded in slots lie in a field the flux density of which is much less than in the case of the smooth-core construction. This apparent paradox is explained by the fact that, whereas the reaction between current and flux, proportional to BlI, is actually less in slotted cores than in smooth cores, the reaction between the iron of the teeth and the iron of the pole face exactly makes up for the difference.

of supply being a circuit which maintains constant voltage at the motor terminals. Since the load, or mechanical output, is small by hypothesis, the electrical input need only be sufficient to supply the small power demanded by the load plus the losses in the motor itself; and since the losses must necessarily be only a small percentage of the rated capacity of the motor, the input will be small under the conditions stated, and the armature current will be correspondingly small. The current is kept small by reason of the fact that the counter e.m.f. developed by the rotation of the armature through the field flux opposes the impressed voltage. The speed of the motor, which together with the flux determines the counter e.m.f., automatically adjusts itself to such a value that the difference between the impressed voltage and the counter e.m.f. permits the flow of just enough current, as fixed by Eq. (7), to develop the torque [Eq. ([illegible])] required to carry the load. If the mechanical load on the motor is increased, the electrical power input to the motor must also increase, and hence more current must be supplied from the line, but the current can increase only in case the counter e.m.f. is decreased, and thus a decrease of the product of flux and speed is required. If the motor is a constant-flux machine, the speed will therefore fall, a conclusion that seems entirely logical as a mere result of the increased load; the speed will again become constant at the lower value when the adjustment of the magnitude of the counter e.m.f. permits the flow of just enough current to develop the increased torque required by the increased load. The sequence of reactions in this case is very similar to what happens in a steam engine controlled by a flyball or inertia governor; an increase of load momentarily checks the speed, and the automatic response of the governor admits more steam until a condition of equilibrium is again established.

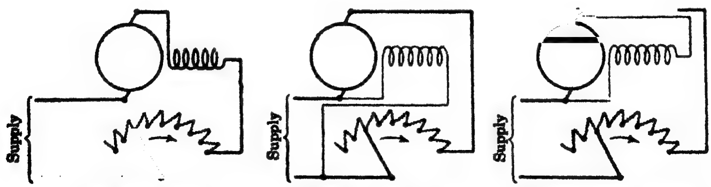
It is quite possible, however, to design a motor in such a way that an increase of load brings about an increase of speed. As was explained in the paragraph above, the increased mechanical load demands an increase in the amount of current supplied by the line, and this in turn requires a decrease in the counter e.m.f.; the latter can be decreased by decreasing the flux as well as by lowering the speed, and consequently if the flux is caused to be decreased in a proportion greater than the necessary decrease in counter e.m.f. the speed will actually increase as a result of the

increased load. This case arises in the differentially compounded motor discussed more in detail in Art. 7.

3. Starting of Motors.—If a motor is called upon to start a heavy load from rest, the starting torque may be as large as, or even larger than, the full-load running torque. If the flux at starting has its normal full-load value, the starting current, by Eq. (11), will have to be equal to, or perhaps somewhat greater than, its full-load value. Other things being equal, the greater the value of the flux the smaller will be the value of the starting current. But since $E_a = 0$ when the armature is stationary, it is clear from Eq. (7) that at the moment of starting $I_a = V/R'$, and, therefore, that the normal small running resistance of the armature circuit (R_a or $R_a + R_f$) must be increased during the starting period by the insertion of a starting rheostat in order to limit the flow of current to a reasonable value. Thus, a 10-hp. 220-volt shunt motor takes an armature current of approximately 40 amp. when carrying its rated load and has an armature resistance of about 0.5 ohm. If the full voltage were impressed directly upon the armature, the initial current would be 440 amp., or more than ten times normal full-load current. To limit the starting current to the full-load value, the resistance that must be put in series with the armature is

$$^{220}\!/_{40} - 0.5 = 5 \quad \text{ohms}$$

The resistance of the starting rheostat is usually so adjusted that the initial current is somewhat greater than that giving full-load torque.



(a) Series motor (b) Shunt motor, correct (c) Shunt motor, incorrect

FIG. 2.—Connections of starting rheostats.

Figure 2*a* shows diagrammatically the connections of the starting rheostat in a series motor, and Fig. 2*b* those in a shunt motor. It should be carefully noted that in Fig. 2*b* the rheostat is in series with the armature only, and so the shunt field winding receives the full line voltage at all times, including the starting

period. Figure 2c shows an incorrect set of connections, since here the shunt-field current is seriously reduced during the whole of the starting period, the flux and also the torque being thus reduced and, if the motor is unloaded, the speed being caused to rise dangerously high.

If an ordinary rheostat of the kind illustrated in Fig. 2 is used in industrial installations, there is danger of burning out the armature if, after an interruption of the service and the consequent stopping of the motor, the voltage is again applied to the supply line; for in that case the full line voltage would be thrown directly across the low resistance of the armature (or armature and series field) and a very heavy current thus produced. For this reason most motor-starting rheostats are provided with a "no-voltage release" which automatically restores the rheostat to the starting position when the line voltage is removed; quite frequently there is also an "overload release," which opens the circuit and automatically cuts in the starting resistance if the current becomes excessive for any reason. The operation of the no-voltage release is usually made to depend upon an electromagnet connected directly across the line terminals (the winding is generally in series with a current-limiting resistor so that the number of turns in the coil of the electromagnet may not be excessively large). So long as the line voltage remains at its normal value, this electromagnet continues to be energized and it is therefore enabled to hold the starting lever (or its equivalent) in its running position; but if the voltage falls below a predetermined limit, the electromagnet will be deenergized, and the starting lever will automatically return to the starting position, either by the action of a spring or by gravity.

The overload release is usually actuated by an electromagnet connected directly in series with the main line (or armature) current. If the current becomes excessive, the pull of the electromagnet on its plunger is made to operate a tripping device which restores the starter to its off position.

Auxiliary starting devices are used with all except very small motors (fractional horsepower under ½ hp). In such motors the armature resistance consumes a considerably larger percentage of the impressed voltage than in motors of higher rating, so that to a certain extent the resistance of the armature itself serves to hold down the starting current; but a more important reason for

dispensing with the starting rheostat is that the inertia of the armature of a fractional-horsepower motor is so small that the armature accelerates with great rapidity, the counter e.m.f. being thus increased at a corresponding rate and the relatively large starting current limited to such a short interval that the winding does not have time to heat up to a destructive temperature.

A more detailed description of starting devices and an analysis of their characteristics are reserved for a later article. For the present, let it be assumed that the motor has been brought up to speed, and that its running characteristics are to be studied.

4. Characteristics of the Separately Excited Motor. 1. *Speed Characteristics.*—Let it be assumed that both the impressed voltage V and the field-exciting current are constant. It follows from the speed equation, (5), which here takes the form

$$n = \frac{V - I_aR_a}{\Phi Z'}$$

that were it not for the demagnetizing action of the armature current the denominator of the fraction would be constant and the speed would decrease slightly and uniformly with increasing values of I_a, as in Fig. 3, provided that R_a is assumed to be constant, or that the temperature of the armature is maintained at its normal running value. The separately excited motor with constant excitation is therefore inherently self-regulating as regards speed.

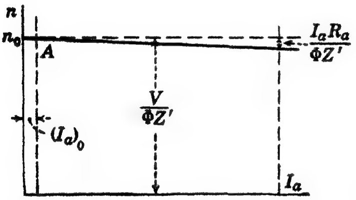


FIG. 3.—Approximate speed characteristic of separately excited motor.

Both Eq. (5) and Fig. 3 indicate that, if I_a were to become zero, the speed would be $n = n_0 = V/\Phi Z'$. Obviously, the armature current I_a cannot be zero under running conditions; for even if there is no mechanical load on the shaft or pulley, it is still necessary to supply sufficient power to overcome the internal losses due to windage, friction, hysteresis, and eddy currents. Consequently, there is a minimum running value of armature current $(I_a)_0$ which is indicated by the point A in the figure. The speed $n_0 = V/\Phi Z'$ may be called the ideal zero-load speed; it is the speed that would be reached if there were no losses, in which case, $E_a = V$ and $I_a = 0$.

It is obvious that the speed may be varied through wide limits by varying either Φ or V, or both of them. Thus, the speed can be raised by reducing Φ or by increasing V. However, the possible range of speed due to the adjustment of the excitation is restricted unless special devices are used, for there are limits to the field strength above or below which there are serious commutation difficulties. Variation of V gives little or no trouble so far as commutation is concerned, provided that the flux is originally adjusted to about its normal value unless, indeed, V is raised to too great an extent. The fact that the field excitation and the armature impressed voltage are independently variable in the separately excited motor gives to this type its chief advantage.

2. *Effect of Armature Reaction.*—The form of the speed equation shows at a glance that the effect of armature reaction, since it reduces Φ, will be to raise the speed, thereby partially neutralizing the slowing-down effect of armature resistance and improving the speed regulation. The armature of a motor may therefore be designed magnetically more powerful than the armature of an otherwise identical machine intended for use as a generator, but within limits that are determined by the effect of the armature reaction upon the airgap flux in the commutating zone.

The curve showing the relation between speed and armature current can be constructed in the following manner:

Let $O'G$, Fig. 4, be the magnetization curve of the machine, abscissas (drawn downward from O) representing ampere-turns per pair of poles ($=N_fI_f$) and ordinates (drawn to the left of O) representing values of $\Phi Z'$. It is important to realize that this curve is entirely independent of the speed of the machine. Select any convenient scale of armature current along OA and a scale to represent the impressed voltage along OV. Assume that the field excitation is constant and equal to OF_0 and that the voltage impressed upon the armature is likewise constant and equal to $OE = V$.

Through the point E draw the straight line EC at an angle φ with the horizontal line EB, so that the intercept BC, corresponding to a value of $I_a = OA$, is equal to I_aR_a to the scale of voltage laid off along OV. The counter e.m.f. is

$$E_c = V - I_aR_a = AB - BC = AC$$

Similarly, draw F_0D making an angle θ with OA, such that the intercept DR, corresponding to $OA = I_a$, is equal to $\alpha ZI_a/180a$ = demagnetizing ampere-turns per pair of poles, to the scale previously adopted along OF_0.

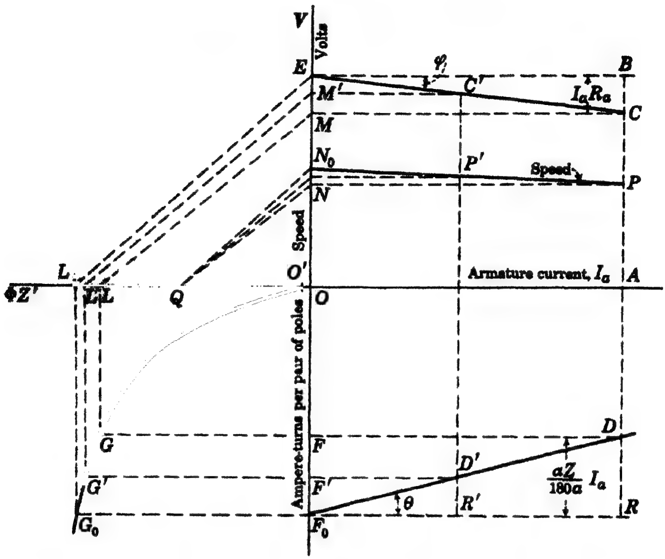


FIG. 4.—Construction of speed characteristic of separately excited and shunt motor

If the armature were currentless ($I_a = 0$) as in the ideal no-load condition, the value of $\Phi Z'$ would be $F_0G_0 = OL_0$, and the ideal no-load speed is therefore

$$n_0 = \frac{V}{\Phi Z'} = \frac{OE}{OL_0}$$

At any general load, as when $I_a = OA$, the demagnetizing effect in ampere-turns per pair of poles is given by DR. The net excitation is reduced to OF and $\Phi Z'$ becomes $FG = OL$. At the same time the counter e.m.f. is $AC = OM$, and hence the speed is

$$n = \frac{V - I_aR_a}{\Phi Z'} = \frac{OM}{OL}$$

In this way values of speed can be computed for various values of I_a and the results plotted to obtain the desired curve N_0P.

But the diagram lends itself readily to a complete graphical solution, as follows:

Select any convenient point Q on the $\Phi Z'$ axis, and draw QN_0 parallel to L_0E; then,

$$n_0 = \frac{OE}{OL_0} = \frac{ON_0}{OQ}$$

Similarly, join L and M, and draw QN parallel to LM. Then,

$$n = \frac{OM}{OL} = \frac{ON}{OQ}$$

Since OQ is constant, it follows that ON_0 and ON are, respectively, proportional to n_0 and n and may be made *equal* to the speed by a suitable choice of scale. Upon projecting N across to P, the latter being on the ordinate through A, P will be a point on the required curve. In precisely the same manner, P' is a point corresponding to $I_a = OA'$. It is clear that the speed curve cannot be exactly straight because of the curvature of the magnetization curve.

3. *Torque Characteristic.*—From Eq. (11), the torque is

$$T = 7.05\Phi Z' I_a \quad \text{lb.-ft.}$$

With reference to Fig. 4, this becomes

$$T = 7.05 FG \cdot OA = 7.05 OL \cdot OA$$

when $I_a = OA$. This may be written

$$\frac{T}{OA} = \frac{OL}{\text{constant}}$$

which suggests the following construction for the curve showing the relation between the armature current and the torque:

In Fig. 5 draw the axes of coordinates, the $\Phi Z'$ curve, and the line F_0D just as in Fig. 4. Proceed as before to locate points G and L. Select any convenient constant length OS, draw LS, and then draw OP perpendicular to LS (using the semicircle on OS as a construction line) until it intersects the ordinate through A in the point P. By construction,

$$\frac{AP}{OA} = \frac{OL}{OS} = \frac{OL}{\text{constant}}$$

and hence AP is proportional to the torque, or may be made equal to the torque by a suitable choice of the length OS, and P is then a point on the desired torque curve. The torque curve is slightly concave downward, but to all intents and purposes it is a straight line through the origin, since Φ is almost constant for all values of I_a within the working range.

5. Speed and Torque Characteristics of the Shunt Motor.- A plain shunt motor operated on constant-potential mains and

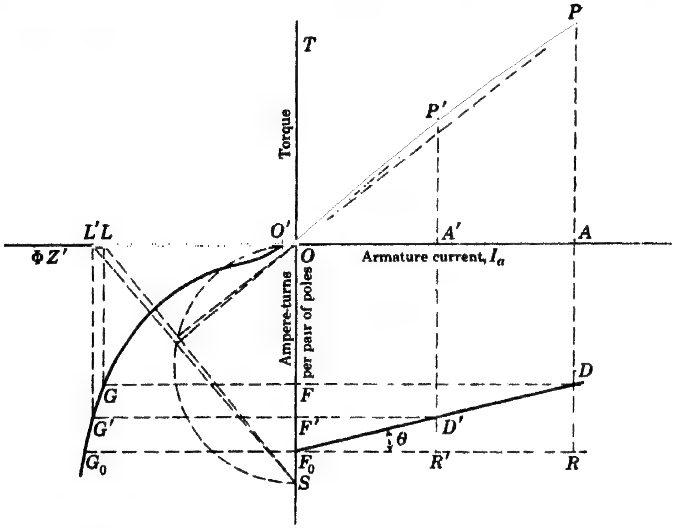


FIG. 5 — Construction of torque characteristic of separately excited and shunt motor

having fixed field resistance differs in no way from a separately excited machine with constant impressed voltage and constant excitation. The determination of its operating characteristics may therefore be carried out in exactly the manner described in the preceding article.

The chief point of difference between the shunt and the separately excited motor is that in the former the field excitation and the impressed voltage are not independently variable, as in the latter. The possible range of speed variation is therefore less in the shunt motor than it is in the separately excited motor, as may be seen by referring to Figs. 6 and 7. Figure 6 shows in a single diagram the several widely separated speed character-

istics that may be obtained by operating a separately excited motor with fixed field excitation and applying different voltages to the armature. Figure 7 indicates the more restricted range of speed that can be obtained with a shunt motor, the reasons being in part that if the impressed voltage is too high the flux

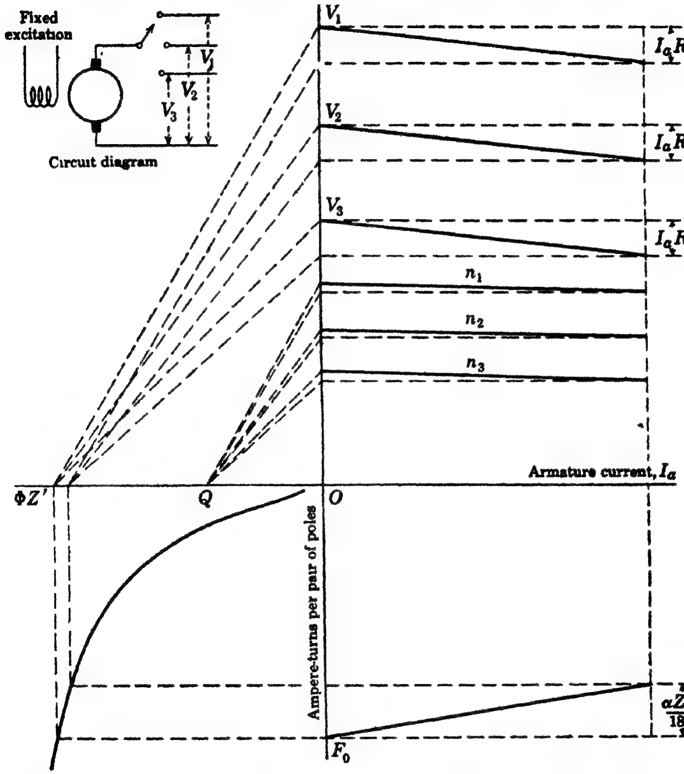


FIG. 6.—Speed characteristics of separately excited motor with different impressed voltages.

approaches the saturation limit, and if it is too low the unstable part of the magnetization curve is reached. Other reasons, which still further restrict the possible range of speed, are concerned with unsatisfactory commutating conditions. The construction of Fig. 7 will be understood when it is remembered that the shunt-field current (and therefore also the field excitation) is proportional to the impressed voltage; consequently, if OF_{0_1} is the field

excitation corresponding to voltage V_1, the excitation OF_{0_2} corresponding to voltage V_2 may be found by first drawing the line $B_1F_{0_1}$, which intersects OA at point K, and then connecting B_2 with K and extending this line to an intersection with the vertical axis of reference.

An interesting and important fact in industrial applications of the shunt motor is that the speed rises perceptibly as the tem-

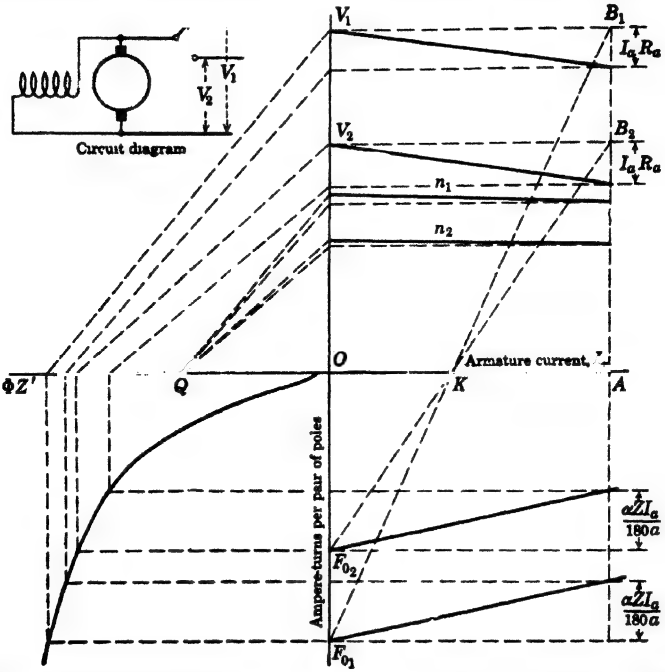


FIG. 7.—Speed characteristics of shunt motor with different impressed voltages

perature of the motor is increased. This is due to the fact that the higher temperature of the field winding increases its resistance and so reduces the exciting current and the flux; at the same time the increased resistance of the armature winding tends to decrease the speed, but the influence of the field preponderates. Shunt motors for industrial service are not provided with regulating field rheostats as are generators; hence, the field winding must be designed to have the proper resistance to give the required speed at the operating temperature. In machines of standard

design, the difference between full-load speed, running cold, and full-load speed, running hot, may be 10 per cent of the higher value.

6. Characteristics of the Series Motor. 1. *Speed Characteristic.*—Inspection of the general equation for the speed

$$n = \frac{V - I_a(R_a + R_f)}{\Phi Z'}$$

shows that the speed of the series motor must decrease rapidly with increasing load for the reason that Φ increases with increasing I_a. In other words, while the numerator of the fraction decreases, the denominator increases. Theoretically, if $I_a = 0$, $\Phi = 0$; hence, at no load the speed would be infinite. Practically, although the flux does not become zero because of residual magnetism, actually it becomes so small that the speed reaches a dangerously high value, on the assumption that V remains constant. For this reason, a series motor must always be so installed as to be positively connected to its load, by gearing or direct connection, never by belting, and the minimum load must be great enough to keep the speed within safe limits; such is the case, for instance, in railway motors, hoists, rolling mills, etc.

Assuming that the motor is to be operated on constant-potential mains, its speed characteristic can be determined by a modification of the methods described in the case of the separately excited motor, as follows:

Let $O'G$, Fig. 8, be the magnetization curve, that is, the curve showing the relation between $\Phi Z'$ and the field excitation, the latter in ampere-turns per pair of poles. Let OE represent to scale the constant impressed voltage V, and draw EC at an angle φ such that the intercept BC is the ohmic drop $I_a(R_a + R_f)$ volts, corresponding to the armature current $I_a = OA$. The length AC then represents to scale the counter e.m.f. corresponding to this current. Also, draw OD at an angle θ so that the intercept $AD = \alpha Z I_a/180a$ = demagnetizing ampere-turns per pair of poles.

When the armature (and field) current is $I_a = OA$, the field excitation is $N_f I_a$ where N_f is the number of turns in the field winding per pair of poles. Let OF represent the particular field excitation, in ampere-turns per pair of poles, corresponding to $I_a = OA$. The net excitation OH is then found by subtracting

AD from OF by drawing the line DF and then AH parallel thereto. The value of $\Phi Z'$, corresponding to excitation OH, is $HK = OL$; consequently, the speed is

$$n = \frac{V - I_a(R_a + R_f)}{\Phi Z'} = \frac{AC}{HK} = \frac{OM}{OL}$$

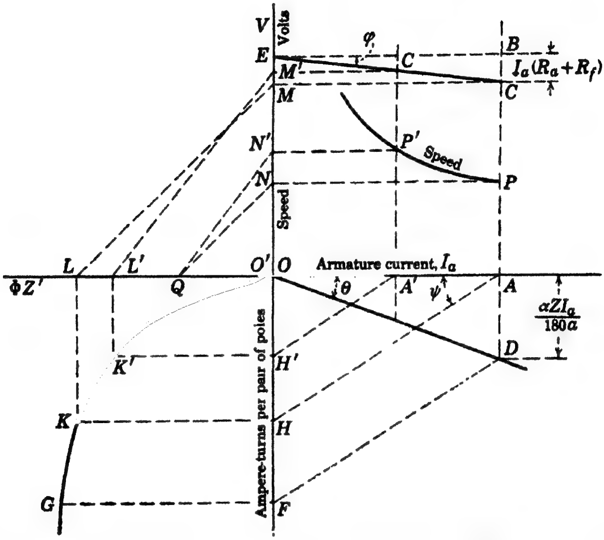


FIG. 8.—Construction of speed characteristic of series motor.

Upon selecting a point Q on the $\Phi Z'$ axis and drawing QN parallel to LM, it is seen that in the similar triangles OLM and OQN

$$n = \frac{OM}{OL} = \frac{ON}{OQ}$$

and hence ON is proportional to the speed. If N is projected upon the ordinate at A, the resulting point P is a point on the speed-current characteristic.

Since $FH = AD = (\alpha Z/180a)I_a$ and $OH = OF - FH = (N_f - \alpha Z/180a)\ I_a$ it follows that

$$\frac{OH}{OA} = \tan\psi = N_f - \frac{\alpha Z}{180a} = \text{constant}$$

Consequently, to find the net excitation OH' corresponding to any other current, such as $I_a = OA'$, all that is necessary is to

draw $A'H'$ parallel to AH in the process of locating point P' on the speed characteristic.

2. *Torque Characteristic.*—As before, the torque is

$$T = 7.05\Phi Z'I_a = 7.05\ OL \cdot OA \quad \text{lb.-ft.}$$

when $I_a = OA$. Select a point S, Fig. 9, such that $OS =$ constant, and draw OP_T perpendicular to LS; it will then be true that

$$\frac{AP_T}{OA} = \frac{OL}{OS}$$

whence AP_T is proportional to the torque and P_T is a point on the torque-current curve.

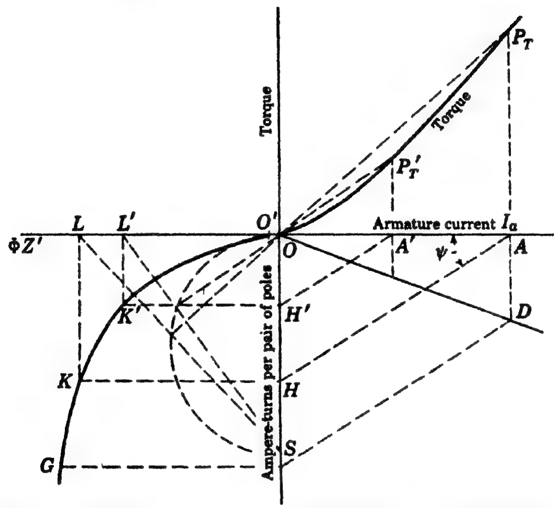


FIG. 9. Construction of torque characteristic of series motor.

The torque curve of a series motor deviates considerably from linear form owing to the fact that the flux varies with the current. If the magnetization curve were a straight line, that is, Φ proportional to I_a, the torque would be proportional to I_a^2, and the curve would be a parabola; actually it lies between the linear and parabolic curves.

Lines such as OP_T can readily be drawn perpendicular to LS by constructing a semicircle on OS as a diameter and drawing a line through O and the point where LS cuts the circle.

The torque curves discussed in connection with the separately excited, shunt, and series motors refer to the total developed torque, as given by Eq. (11). The actual torque at the pulley that would be measured by a brake test is less than the total torque by an amount that corresponds to the torque required to overcome internal friction and iron losses. The curve of useful torque may be obtained from that of total torque by subtracting from the ordinates of the latter the "lost torque"; the useful torque passes through zero value when I_a has an appreciable value (see Fig. 47).

7. Characteristics of Compound-wound Motor.—If the shunt and series windings of a compound-wound (long-shunt) machine are so connected that their magnetizing effects cooperate, or

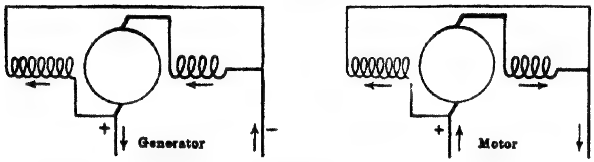


FIG. 10.—Relative directions of shunt and series exciting current in compound machine

are *cumulative* when the machine is used as a generator, then, if the machine is used as a motor, the two windings will oppose each other, and a *differential* effect results. This is illustrated diagrammatically in Fig. 10. If the machine is designed to over-compound as a generator, the differential motor action will be considerable, and there results a decided decrease of flux under load conditions and a speed higher than would obtain without the series winding. In general, the case is similar to that of a shunt motor with exaggerated armature demagnetizing effect.

In the same way a differentially wound generator, having a drooping e.m.f. characteristic when driven at constant speed, becomes a cumulative-compound motor with a drooping speed characteristic when supplied with constant terminal voltage.

1. *Differential-compound Motor.*—The only difference between this case and the one discussed in connection with Fig. 4, is that now

$$\tan \varphi = R_a + R_f$$

and

$$\tan \theta = \frac{\alpha Z}{180a} + N_f$$

on the assumption that the resistance of the shunt field winding is constant. The construction of the speed characteristic has been carried out in Fig. 11, from which it appears that if N_f is sufficiently large the speed rises with increasing load. It is clear that there is a particular value of θ for which the speed will be the

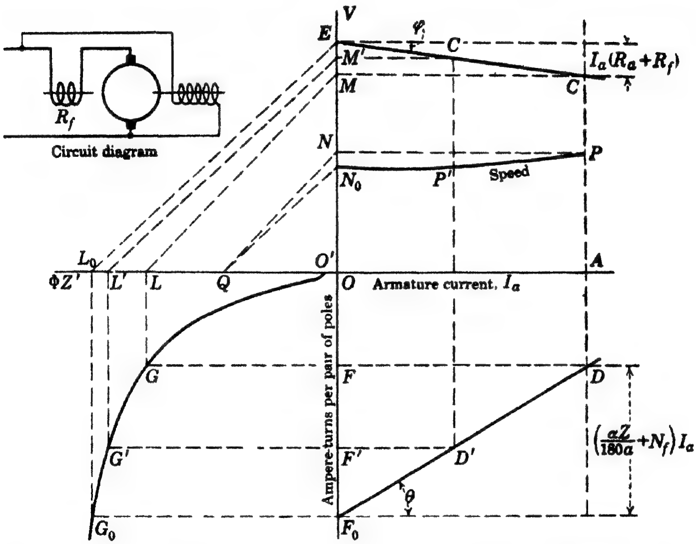


FIG. 11. Construction of speed characteristic of differential-compound motor

same at full load as at no load but that it cannot be made absolutely constant at all loads (in the absence of special regulating devices) because of the curvature of the magnetization curve $O'G$.

The construction of the torque curve is shown in Fig. 12, the fixed point S being used as in previously discussed cases. The curve is concave downward, owing to the fact that the flux decreases with increasing current. Differentially compound-wound motors are not used for the reason that their rising speed characteristics make them unstable under load conditions.

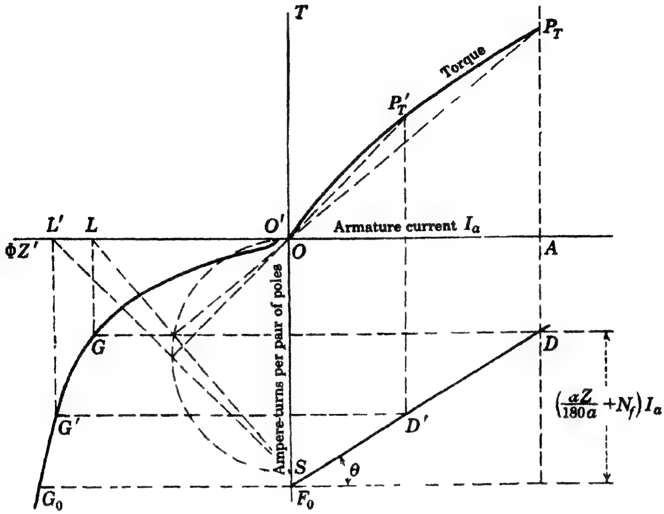


FIG. 12. Construction of torque characteristic of differential-compound motor

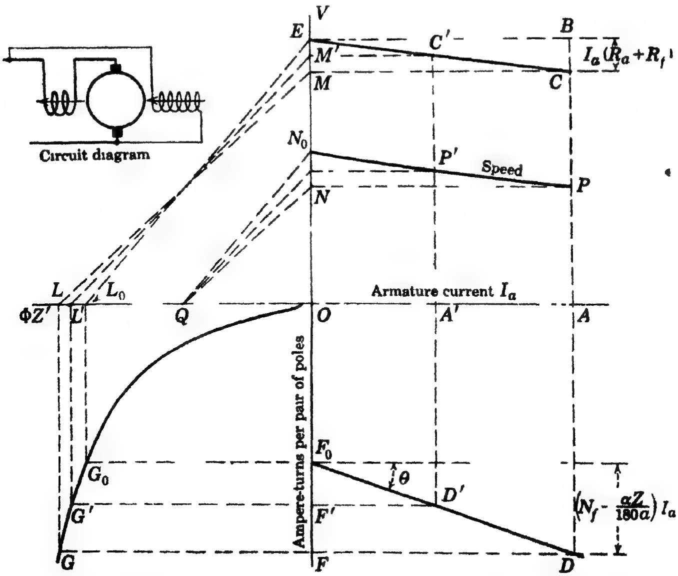


FIG. 13.—Construction of speed characteristic of cumulative-compound motor

2. *Cumulative-compound Motor.*—In this case, which is represented by the constructions of Figs. 13 and 14,

$$\tan \varphi = R_a + R_f$$

and

$$\tan \theta = N_f - \frac{\alpha Z}{180a}$$

but here the line F_0D, the slope of which is fixed by angle θ, is drawn in the direction of increasing excitation, instead of in the

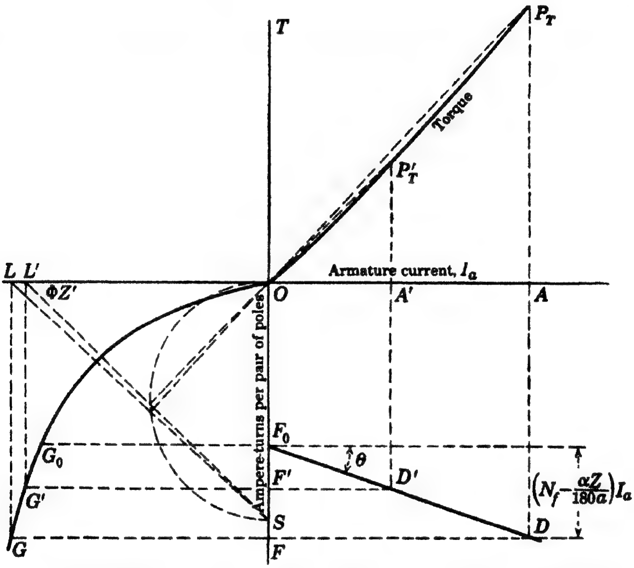


FIG. 14.—Construction of torque characteristic of cumulative-compound motor.

opposite direction as in Fig. 11, because of the fact that the excitation due to the series field winding (which is greater than the demagnetizing action of the armature) is in the same direction as the shunt-field excitation. Except for this difference, the procedure followed in making the drawings of Figs. 13 and 14 is the same in all essential features as in the cases previously explained. The speed of the cumulative-compound motor falls considerably with increasing load, and the torque curve is concave upward. The cumulative-compound motor has characteristics that are intermediate between those of the shunt and

series types. It differs from the series motor in that its speed rises to a definite limit when the load is suddenly removed, whereas the series motor under similar circumstances would run away.

8. Comparison of Motor Characteristics.—It is obvious that a given frame may be wound so as to make the machine a series,

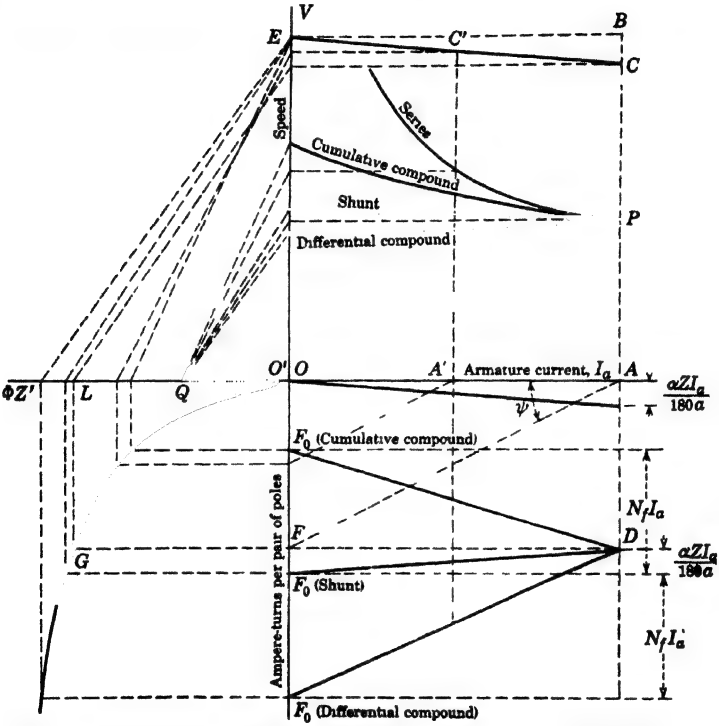


FIG. 15.—Comparison of speed characteristics Speed at full load same for all types

a shunt, or a compound motor and that by proper adjustment of the windings, or of the field rheostat controlling the flux, the full-load speed may be made the same in each case. At all other loads the speeds will differ, the speed-current curve following in each case the particular course that is characteristic of the corresponding type of motor. By constructing in a single diagram the speed-current curves of each of the types discussed

in the preceding articles, their comparative performances under operating conditions stand out clearly. Such a construction is given in Fig. 15. To simplify the diagram, it has been assumed that the ohmic drop I_aR_a in the shunt motor may be represented by the same line which gives the ohmic drop $I_a(R_a + R_f)$ in the series and compound types. It has also been assumed that at full load the net excitation OF, and the actual value of

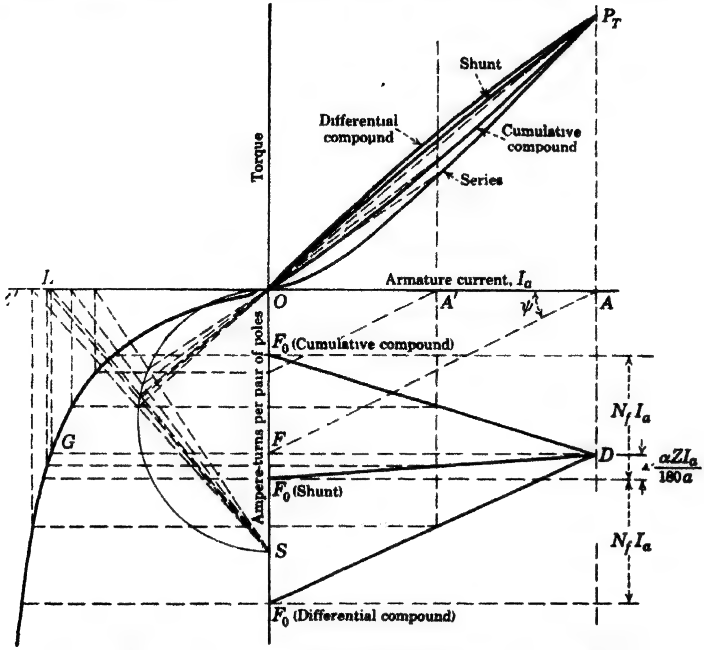


FIG. 16.—Comparison of torque characteristics. Torque at full load same for all types.

$\Phi Z'(FG)$, are the same for all the motors, the full-load speed being thus the same for all.

These assumptions also make it possible to compare the torque curves on the basis of equal torque under full-load conditions. The construction is indicated in Fig. 16.

9. Subdivision of Starting Rheostats.—In accelerating a motor from standstill to its rated speed, it is usually desirable to make the speed increase at a uniform rate, that is, to make the angular acceleration constant during the starting period. Since for a

given moment of inertia of the rotating masses the angular acceleration is proportional to the net torque, it follows that the latter must be constant if the acceleration is to be constant. The net torque is the difference between the total torque and the resisting torque due to the load and the rotational losses in the motor itself; these resisting torques may be so nearly constant (except in the case of loads like fans or blowers) that it is approximately correct to say that constant acceleration will result if the total torque developed by the motor is constant. Since the torque developed by the motor is proportional to the product of flux and armature current (ΦI_a), it follows that to produce constant acceleration the armature current must be constant during the starting period; for whatever the type of motor—series, shunt, compound, or separately excited—the flux Φ will remain fixed if I_a is fixed, provided that the line voltage remains constant.

If the flux Φ were fully established before the current I_a is introduced into the armature circuit, the sudden development of torque might cause serious damage to the mechanism driven by the motor, such as the breaking of gear teeth or the straining of other parts. In a series motor, where flux and current are both initially zero, the two necessarily build up together, so that the physical shock at the moment of starting is cushioned. In shunt and compound motors, the large inductance of the shunt field winding appreciably delays the building up of the flux when this winding is connected to the supply line, and advantage is sometimes taken of this fact to cushion the start of such motors by so arranging the starting device that the field circuit and the armature circuit are closed at the same time. There are cases, however, in which the full field strength is developed before the armature connection is established.

It is seen from Eq. (4) that

$$E_a = \Phi Z' n = V - I_a R'$$

so that if Φ and I_a are both constant during the starting period, the resistance R' must be varied in such a way that, as n increases, the above relation is continuously satisfied; or

$$R' = \frac{V}{I_a} - \frac{\Phi Z'}{I_a} n \tag{12}$$

which means that R' must vary linearly and continuously as a function of n, subject to the assumption that $\Phi Z'$ and I_a remain fixed. Continuous variation of resistance is possible with liquid or carbon-pile rheostats, but with rheostats of the ordinary type, made up of discrete sections, the variation of resistance, as successive steps are cut out, will necessarily be discontinuous. It follows that step-by-step reduction of the starting resistance

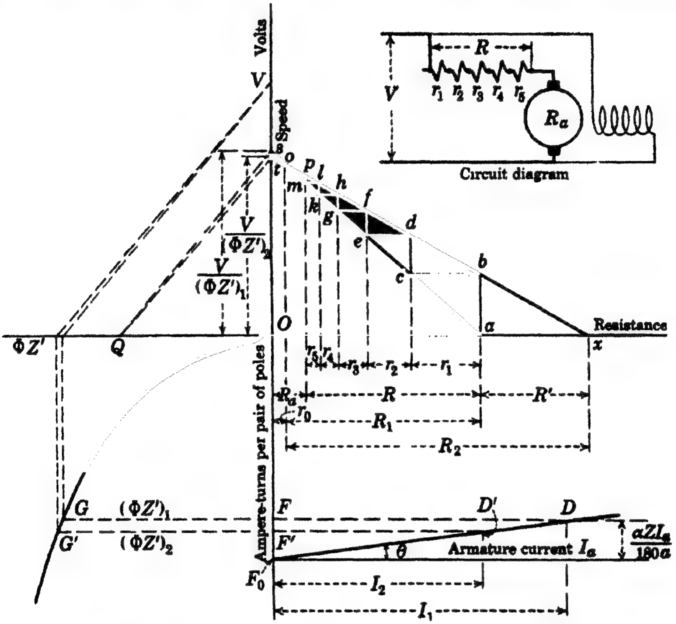


FIG. 17.—Steps of starting rheostat, shunt or separately excited motor.

will introduce variations in the magnitude of the starting current, and the magnitude of $\Phi Z'$ will in turn be affected. The problem, therefore, is to proportion the successive steps of the starting rheostat so as to keep the variation of I_a and $\Phi Z'$ within limits that will not seriously affect the torque and the rate of acceleration.

1. *Consider the case of a shunt (or a separately excited) motor*, the circuit diagram being shown in Fig. 17. The magnetization curve is drawn as in Fig. 4, and the line F_0D is drawn at an angle θ such that the intercepts between F_0D and a horizontal

line through F_0 give to scale the demagnetizing ampere-turns per pair of poles. The constant excitation contributed by the field excitation is represented by the length OF_0.

Let the starting resistance have a total resistance R, so that at the moment of starting the armature current is

$$I_a = I_1 = \frac{V}{R_a + R}$$

If the current remained steady at this value, the relation between the speed and the resistance in the armature circuit would be given by the equation [obtained from Eq. (5)]

$$n = \frac{V}{(\Phi Z')_1} - \frac{I_1}{(\Phi Z')_1} \cdot r \tag{13}$$

which is represented in Fig. 17 by the straight line sa. The intercept on the speed axis (corresponding to $r = 0$) is $V/(\Phi Z')_1$, which is the ideal zero-load speed corresponding to the magnetization $(\Phi Z')_1$. The intercept on the resistance axis (corresponding to $n = 0$) is $r = V/I_1 = R_a + R$. But as soon as the current I_1 is established the motor speeds up, the counter e.m.f. begins to build up from its initial zero value, and the armature current begins to decrease. Suppose that the armature current falls to the value I_2; if it remained at that value, the relation between speed and resistance would be given by

$$n = \frac{V}{(\Phi Z')_2} - \frac{I_2}{(\Phi Z')_2} \cdot r \tag{14}$$

represented in the diagram by the line tx The intercept $Ot = V/(\Phi Z')_2$ is the ideal zero-load speed corresponding to the magnetization $(\Phi Z')_2$; and the intercept $Ox = r = V/I_2 = R_a + R + R'$, where R' is a fictitious resistance, not actually present in the circuit, but which would have to be in series with R to make the starting current equal to I_2.

To sum up these considerations, the line sa is the locus of points of which the coordinates represent speed and resistance when $I_a = I_1$, and line tx is a similar locus when $I_a = I_2$. Let it now be specified that the rheostat is to be so manipulated that the armature current is to fluctuate between the maximum value I_1 and the minimum value I_2, during the starting period.

When the motor is started, the current being I_1, the relation between speed and starting resistance is represented by point a. As the speed rises, the resistance retaining its initial value $(R_a + R)$, the relation between speed and resistance is represented by the vertical line ab. At the moment the speed attains the value ab, the current will have fallen to I_2. If now the first step of the rheostat r_1 is suddenly cut out before the speed has had time to change appreciably, the current will jump to the value I_1, and the conditions will be represented by point c. Thereafter, as successive sections r_2, r_3, . . . are cut out of circuit. the relation between speed and resistance is given by the stepped line $abcd$ . . . p.

The condition to be satisfied is that when all the starting resistance has been cut out, only R_a remaining, the speed will have attained the value corresponding to the ordinate of point p and satisfying the relation

$$n = \frac{V - I_2R_a}{(\Phi Z')_2} \tag{15}$$

Since the number of steps in the rheostat is a whole number, it is clear that I_1 and I_2 cannot be chosen arbitrarily and that the relation between them is a function of the number of steps. To determine this relation, it may be noted that the stepped line $abcd$ . . . p divides the triangle oax into a series of similar triangles which are so related that the magnitudes of the segments r_1, r_2, r_3, . . . constitute a geometrical progression. Thus,

$$\frac{r_1}{R'} = \frac{cb}{ax} = \frac{ob}{ox} = \frac{R_1}{R_2} = \rho \tag{16}$$

$$\frac{r_2}{r_1} = \frac{ed}{cb} = \frac{od}{ob} = \frac{R_1 - r_1}{R_1} = 1 - \frac{r_1}{R_1} =$$

$$1 - \rho\frac{R'}{R_1} = 1 - \rho\frac{R_2 - R_1}{R_1} = \rho \tag{17}$$

Similarly,

$$\frac{r_3}{r_2} = \frac{r_4}{r_3} = \frac{r_5}{r_4} = \frac{r_m}{r_{m-1}} = \rho \tag{18}$$

where m is the number of sections into which the rheostat is divided.

The total resistance R of the rheostat is therefore given by

$$R = r_1 + r_2 + r_3 + \cdots + r_m = r_1(1 + \rho + \rho^2 + \cdots \rho^{m-1})$$
$$= r_1\frac{1 - \rho^m}{1 - \rho} \tag{19}$$

and

$$r_1 = R\frac{1 - \rho}{1 - \rho^m} = \left(\frac{V}{I_1} - R_a\right)\frac{1 - \rho}{1 - \rho^m} \tag{20}$$

In Eq (20) the value of ρ is determined by Eq. (16), which may be written

$$\rho = \frac{R_1}{R_2} = \frac{R_a + R - r_0}{R_a + R + R' - r_0} \tag{21}$$

If I_1 and I_2 are not too widely different, r_0 will be small, and under these conditions the three points s, t, and o in Fig. 17 will be practically coincident; in that case

$$\rho \cong \frac{R_a + R}{R_a + R + R'} = \frac{I_2}{I_1} \tag{22}$$

If r_0 is not sufficiently small to warrant disregarding it, its magnitude may be computed with satisfactory precision by assuming that the magnetization curve may be represented by Froelich's equation. Suppose that

$$\Phi Z' = \frac{A \cdot F}{B + F}$$

where F is the net field excitation in ampere-turns per pair of poles, A and B being constants that can be found quite simply when the magnetization curve is given. The net excitation when the armature current is known is then $(F_0 - DI_a)$, where F_0 is the fixed excitation contributed by the shunt (or separately excited) field winding and $D = \alpha Z/180a$. When the armature current has the values I_1 and I_2,

$$(\Phi Z')_1 = \frac{A(F_0 - DI_1)}{B + F_0 - DI_1} \tag{23}$$

$$(\Phi Z')_2 = \frac{A(F_0 - DI_2)}{B + F_0 - DI_2} \tag{24}$$

But Eqs. (13) and (14) represent the two straight lines *sa* and *tx*, so that, if they are made simultaneous and solved for r, the result will be r_0. Thus,

$$r_0\left[\frac{I_1}{(\Phi Z')_1} - \frac{I_2}{(\Phi Z')_2}\right] = V\left[\frac{1}{(\Phi Z')_1} - \frac{1}{(\Phi Z')_2}\right] \tag{25}$$

and, on substituting Eqs. (23) and (24) and reducing,

$$r_0 = V\frac{B \cdot D}{BF_0 + (F_0 - DI_1)(F_0 - DI_2)} \tag{26}$$

2. *In the series motor* the conditions are represented in Fig. 18; so far as the limiting lines *sa* and *tx* are concerned, their intersection now lies to the left of the origin of coordinates instead of to the right, as in Fig. 17. Their equations are again given by (13) and (14), and r_1 is given by an equation only slightly different from (20), namely,

$$r_1 = R\frac{1 - \rho}{1 - \rho^m} = \left(\frac{V}{I_1} - R_a - R_f\right)\frac{1 - \rho}{1 - \rho^m} \tag{27}$$

where

$$\rho = \frac{R_1}{R_2} = \frac{R_a + R_f + R + r_0}{R_a + R_f + R + R' + r_0} \tag{28}$$

and

$$\left.\begin{aligned} R_a + R_f + R &= \frac{V}{I_1} \\ R_a + R_f + R + R' &= \frac{V}{I_2} \end{aligned}\right\} \tag{29}$$

To find r_0, Froelich's equation may again be used, but in the series motor

$$\Phi Z' = \frac{A(N_f - D)I}{B + (N_f - D)I} = \frac{CI}{B + KI}$$

where $C = A(N_f - D)$ and $K = N_f - D$. Hence,

$$(\Phi Z')_1 = \frac{CI_1}{B + KI_1}$$

$$(\Phi Z')_2 = \frac{CI_2}{B + KI_2}$$

and, on substituting these values in Eq. (25),

$$r_0 = -\frac{VB}{KI_1I_2} \tag{30}$$

the negative sign indicating that r_0 lies to the left of the origin. This sign must be taken as positive before substituting in Eq. (28).

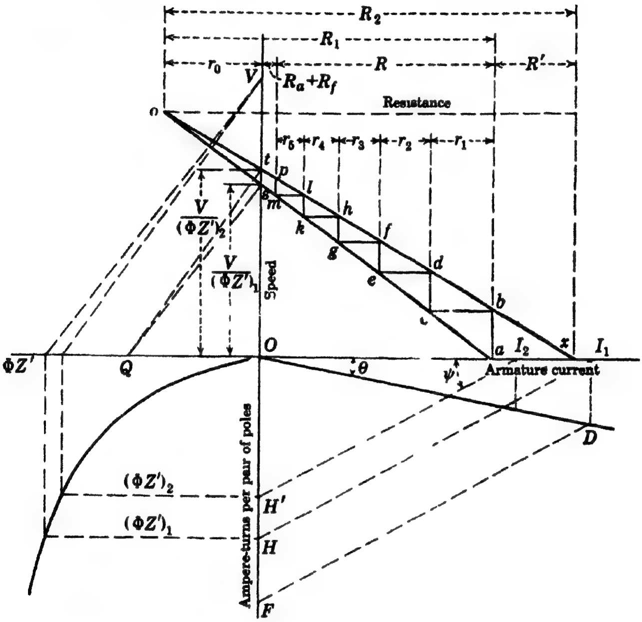


FIG. 18.—Steps of starting rheostat, series motor.

3. *In the case of the cumulative-compound motor,* the conditions are indicated in Fig. 19. The values of r_1 and ρ are determined by Eqs. (27), (28), and (29). To find r_0, it is to be noted that

$$(\Phi Z')_1 = \frac{A[F_0 + (N_f - D)I_1]}{B + F_0 + (N_f - D)I_1}$$

$$(\Phi Z')_2 = \frac{A[F_0 + (N_f - D)I_2]}{B + F_0 + (N_f - D)I_2}$$

and on substituting these relations in Eq. (25) there results

$$= -V\frac{B(N_f - D)}{BF_0 + [F_0 + (N_f - D)I_1][F_0 + (N_f - D)I_2]} \tag{31}$$

the negative sign indicating that r_0 is to be drawn to the left of the origin; this sign must be reversed before substituting r_0 in Eq. (28).

10. Manually Operated, Dial-type Starting Rheostats.—This type of starting rheostat, which at one time was used for all classes of service, is now usually confined to those cases where

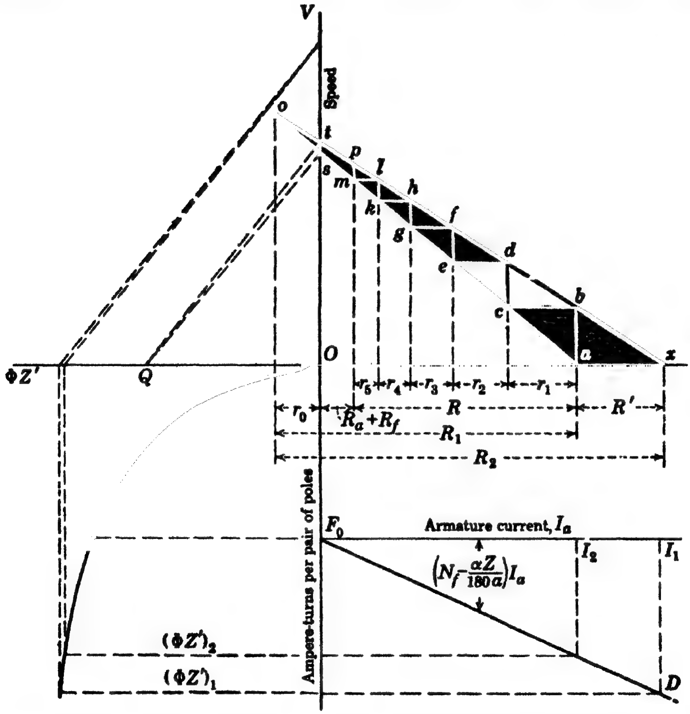


FIG. 19.—Steps of starting rheostat, cumulative-compound motor.

starting requirements are relatively infrequent and not severe. Such starters have the advantage of low first cost and maintenance, but they have the disadvantage that the rate at which the resistance is cut out is left to the judgment of the operator. If this rate is too high, the motor and the driven machine may be damaged; if it is too low, the rheostat itself may be overheated and one or more sections of the resistor be burned out, for the resistor is not designed to carry the starting current for more than

a short time. They are used with motors of moderate rating, up to a maximum of about 400 hp. at 550 volts, or 200 hp. at 230 volts, the limiting factor being the amount of current that can be handled without undue arcing at the sliding metallic contacts.

Dial-type or face-plate starting rheostats are used mainly with shunt- and compound-wound motors. They are available for series-wound motors, but most series motors are used in heavy-duty applications which call for the drum-type or automatic magnetic starters described in later articles.

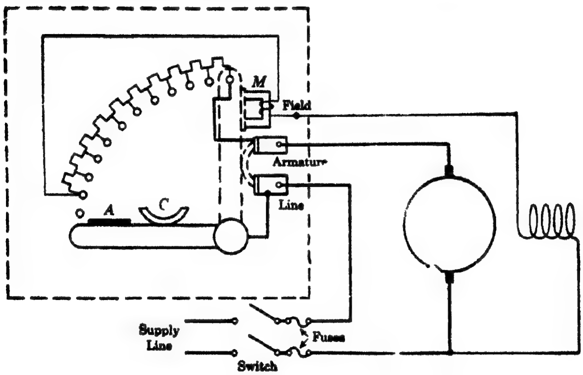


FIG. 20.—Three-point starting rheostat, shunt motor.

Figure 20 is a diagram of the connections of a *three-point* starting rheostat for a shunt or compound motor, and Fig. 21 is a similar diagram for a *four-point* starting rheostat. The terms "three-point" and "four-point" refer to the number of terminals provided on the rheostat. The difference is mainly that in the former the holding magnet M is in series with the field winding, whereas in the latter M is connected directly across the line (but in series with a resistor). The holding magnet M provides protection against low voltage, for if the current through M falls below a definite value the magnet becomes deenergized and the spring-retracted starting lever returns to the starting position. Some starting rheostats, as in Fig. 20, are provided with a contact switch C, made of laminated spring copper, which short-circuits the main armature current that would otherwise have to pass through the starting lever and the last stud of the disk.

When the motor is stopped by opening the main line switch, the starting lever of a rheostat connected as in Fig. 20 will not immediately return to the starting position, for the reason that the motor acts as a generator, sending current through the closed

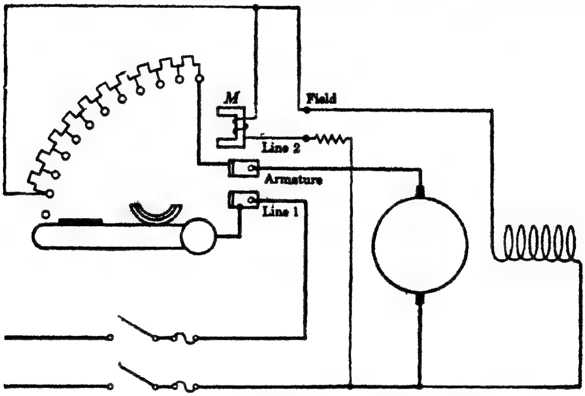


FIG. 21.—Four-point starting rheostat, shunt motor.

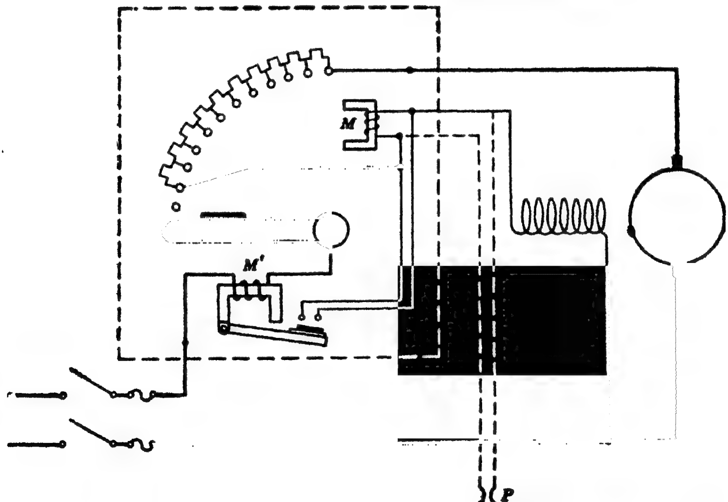


FIG. 22.—Three-point starting rheostat with overload release.

circuit which includes the armature and field winding, and the holding coil. This generator action continues until the energy stored kinetically in the rotating armature, and magnetically in the field structure, has been dissipated to such an extent that the

generated current is no longer sufficient to hold the starting lever. In the case of Fig. 21 the starting lever is released more promptly on opening the line switch, but the energy stored in the magnetic field discharges harmlessly through the armature in the manner already described.

The three-point starting rheostat affords protection against the danger of excessive (runaway) speed if the field circuit of the motor is broken, for any interruption of the field current will restore the starter to the off position.

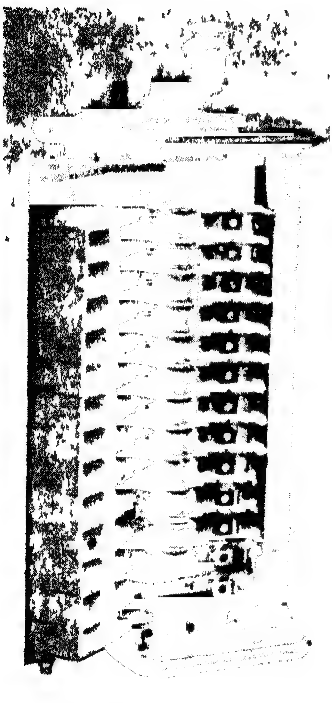


FIG. 23.—Drum controller for crane hoist. (*General Electric Company.*)

Overload protection may be afforded in the manner indicated in Fig. 22 or else by an ordinary circuit breaker in the main line. In Fig. 22 the overload release coil M' actuates a pivoted arm which bridges the terminals of the holding magnet M, thereby releasing the starting handle and allowing the retractive spring to bring the handle back to the starting position. The dashed lines in Fig. 22 represent an auxiliary circuit leading to a push button P which provides a simple method of stopping the motor from the point where P is mounted. If this method is used, the resistance of the wires comprising the auxiliary circuit should be sufficiently smaller than that of coil M with which they are in parallel so that on closing contact P enough current will be diverted from M to release its hold.

11. Drum Controllers.—When service requirements call for frequent starts, stops, and reversals, together with variations of speed at irregular intervals, the dial-type starter and regulator must be replaced by a sturdier construction. The drum con-

shunt winding and so bring about another reversal of rotation. This process, which is another example of instability, may go on indefinitely unless the design constants of the machine are such that the successive impulses are damped out, that is, do not synchronize with the natural period of oscillation of the armature.*

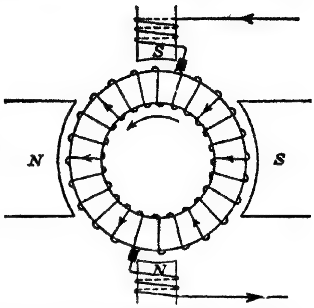


FIG. 37.—Differential effect in interpole motor due to backward shift of brushes.

A similar state of affairs may arise in the case of shunt motors provided with commutating poles if the brushes are not properly placed. Normally the axis of commutation coincides with the axis of the interpoles; but if the brushes are accidentally shifted backward, against the direction of rotation, as in Fig. 37, the interpoles will produce a component of flux in opposition to that of the main poles and so convert the machine into one having the characteristics of a differentially wound motor.

15. Control of Speed of Shunt Motors.—Inspection of the fundamental equation for the speed of a motor

$$n = \frac{V - I_a R'}{\Phi Z'}$$

reveals the fact that there are three principal methods for regulating the speed, namely: *rheostatic control*, by varying the resistance R' which includes the armature resistance R_a; *voltage control*, by varying the impressed voltage V; and *flux control*, by varying Φ. A fourth method occasionally used involves changing Z' by using an armature having two windings and two commutators which may be connected either in series or in parallel.

1. *Rheostatic Control.*—In this method the effective resistance of the armature is increased by connecting in series with it (but not in the main line or field circuit) a variable resistance. This has the effect of imparting a pronounced droop to the speed characteristic, the downward slope of the characteristic being deter-

* L. R. LUDWIG, Effect of Transient Conditions on Application of Direct-current Compound Motors, *Trans. A.I.E.E.*, **47**, 599, 1928.

mined by the combined resistance of the armature winding and external resistor in the manner indicated in Fig. 38. This diagram is based upon the same construction used in Fig. 4. A motor used in this way has poor speed regulation; that is, the speed will fluctuate between rather wide limits as the load changes. Moreover, the method is inefficient because of the loss of power due to the flow of the armature current through the external resistor. It is not to be recommended for general industrial installations but is frequently convenient in laboratory investigations, in special tests. and in industrial applications

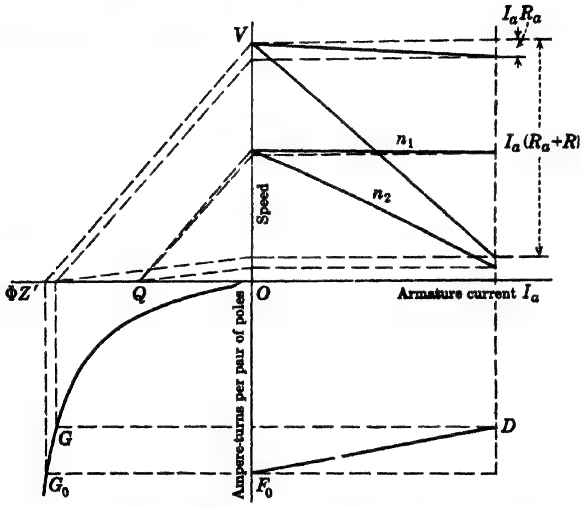


FIG. 38.—Effect of series resistance upon speed.

where the device driven by the motor must be "inched" along to a definite position.

2. *Voltage Control.*- Subdividing the voltage of the main generator or bus bars by means of a balancer set, as in Fig. 39, makes it possible to impress upon the armature of the motor a number of different voltages, to each of which there corresponds a definite speed characteristic, as illustrated in Fig. 6. For any given impressed voltage the speed is substantially constant and approximately proportional to the impressed voltage. The variation in speed between full load and no load with normal voltage is usually between 2 and 10 per cent, the smaller limit

holding for large motors, the larger limit for small motors. It should be understood that the motor connections are such that the voltage impressed on the shunt field winding is not changed when the armature is switched from one circuit to another, in order that the field flux may remain substantially constant. The armature connections are usually changed by means of a drum-type controller.

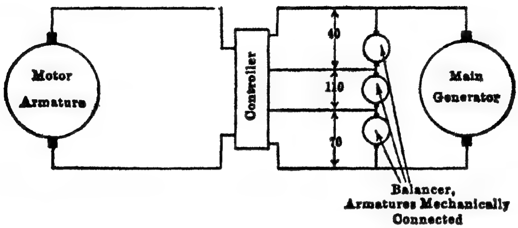


FIG. 39.—Speed regulation of motor by means of voltage control.

With the arrangement indicated in Fig. 39 it is possible to impress six different voltages upon the motor, namely, 40, 70, 110, 150, 180, or 220 volts, giving six different speeds. Intermediate speeds may be secured by adjusting the flux by means of a rheostat in series with the shunt field winding. This method is extensively used for driving machine tools, such as lathes

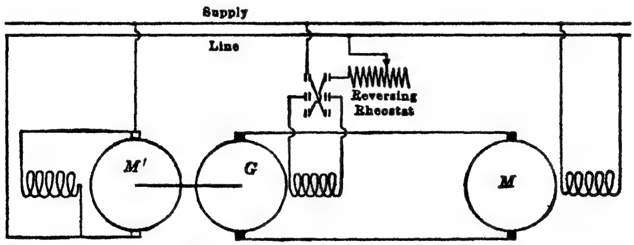


FIG. 40.—Diagram of connections of Ward-Leonard system of speed control.

and boring mills. It has the disadvantage of requiring a considerable investment in copper because of the extra wires of the distributing circuits.

Where uniform gradation of speed in either direction is required, as in the operation of the turrets of battleships or in steering by electrically controlled rudders, the Ward-Leonard system may be used. The motor M, Fig. 40, whose speed is to be regulated, is separately excited from the main supply lines,

and its armature is supplied from an auxiliary generator G, the latter driven at constant speed by a shunt motor M' which takes its power from the line; instead of driving the generator G by a motor, any other form of prime mover may be used. The field of the generator is excited from the constant-voltage supply line and may be adjusted from zero to a maximum, in either direction, by means of a reversing field rheostat; in this way it is possible to obtain a smooth variation of the voltage impressed upon the motor. This method is very effective but is naturally expensive because of the auxiliary motor-generator set.

3. *Flux Control.*—The simplest and cheapest method of regulating the speed of a shunt motor is that in which the flux is varied by means of a rheostat in the shunt field circuit. If the machine operates normally with a nearly saturated magnetic circuit, all resistance of the rheostat being cut out, the speed may be approximately doubled by weakening the field current; beyond this point the field intensity at the pole tips becomes so weakened by armature reaction, especially under load conditions, that commutation is seriously interfered with. Consequently this method is limited to those cases in which a very moderate range of speed will suffice.

The commutating-pole motor affords means whereby a wide range of speed is made possible, a ratio of maximum to minimum speed of 5:1 or 6:1 being fairly common. The principle of the commutating-pole or interpole motor involves the neutralization of the armature reaction of the motor by placing auxiliary poles in the axis of commutation and exciting them by the same current that flows through the armature, the winding of the auxiliary poles being so designed that the m.m.f. of the armature is either exactly balanced or else slightly overcompensated. In this way the main field may be varied through a wide range without producing sparking, the commutating poles always producing a field of the proper strength to reverse the current in the coils undergoing commutation. Interpole motors are used to a very large extent where variable speed is a necessity, as in machine-tool operation. They are generally provided with a controller which serves not only to start the motor and to reverse its direction but also to vary its speed as desired.

The methods thus far described effect the variation of speed by adjustment of the electrical circuits of the machine. But the

flux and, therefore, the speed can be varied by mechanical devices that change the reluctance of the magnetic circuit by varying the length of the airgap. In the Reliance motor, formerly called

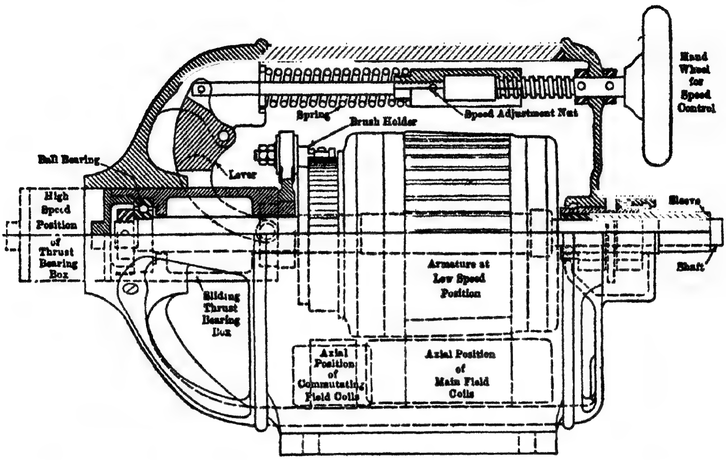


FIG. 41.—Sectional view of Reliance adjustable-speed motor.

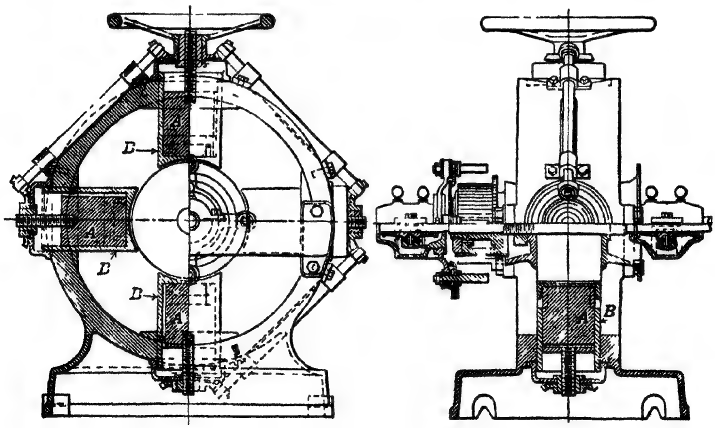


FIG. 42.- Stow multispeed motor.

the Lincoln motor, shown in section in Fig. 41, the armature core is conical, so that as the armature is moved sideways by means of the handwheel the effective length of airgap may be increased or decreased at will. A range of speed of 10:1 is readily

obtained in the smaller sizes. Commutation difficulties at high speeds (and weak field) are avoided by using interpoles.

The reluctance method of speed control was first used in the Stow motor, now no longer manufactured; as indicated in Fig. 42, plungers A were moved in and out of the hollow pole cores B.

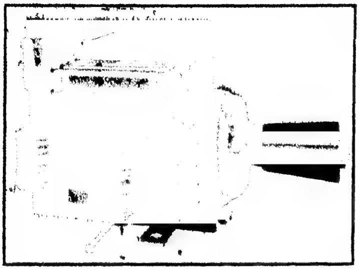


FIG. 43.—Control panel with cover removed, showing assembly of carbon piles, normal- and quick-response lever system, and anti-hunting feature.

The flux-control methods described in the preceding paragraphs relate to adjustment of speed, between wide limits, to meet the requirements of adjustable-speed service. A different type of problem arises when it becomes necessary to hold the speed of a motor constant within close limits when the load fluctuates widely. An ingenious device to accomplish this result, developed by the Westinghouse Electric and Manufacturing Company,* is illustrated in Figs. 43, 44, and 45.

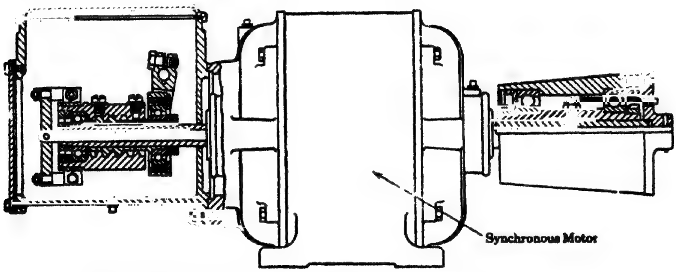


FIG. 44.—Cross-section of differential mechanism.

The basic principle of this regulator involves the use of a carbon-pile rheostat in the shunt field winding of the motor to be regulated, with the compression of the carbon pile (and therefore its resistance) responsive to any change in the speed of

* R. R. BAKER, *Reprint* 424, Westinghouse Electric and Manufacturing Company; J. H. ASHBAUGH, Automatic Speed Regulation, *Elec. Eng.*, May, 1931, p. 363.

the motor. If the speed of the motor increases, the pressure on the rheostat is increased, and thus the field strength is increased and the rise of speed is checked; if the speed falls, the pressure is reduced, and thus the field is weakened and the speed is raised. Variations in the speed of the motor to be controlled are communicated through a belt drive to the cone pulley of Fig. 44, which is mounted on ball bearings carried on the extension of the hollow shaft of a small, fractional-horsepower, synchronous

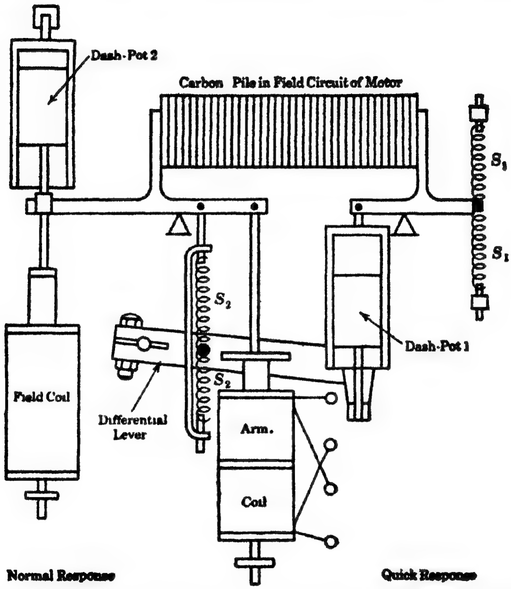


FIG. 45.—Diagram of control panel, showing operating mechanism.

motor. The latter, which runs at constant speed, drives the screw member of a differential mechanism, while the cone pulley drives the nut member. The differential mechanism is connected by suitable linkage to the differential lever (Fig. 45) so that any relative rotation between the cone pulley and the synchronous motor causes the differential lever to rotate in one direction or the other. The synchronous motor, operating on an a-c supply of fixed frequency, simply provides a steady reference speed.

When the differential lever (Fig. 45) moves up, owing to a drop in speed, dashpot 1 communicates the motion to the

quick-response lever, the pressure on the carbon pile is thus relieved and the drop in speed tends to be checked; this response, though drastic in its action, is of short duration because of the action of the balanced springs S_1, which tend to restore the quick-response lever to its normal position. The sudden change in field and armature current causes the field and armature coils (Fig. 45) to move in a direction that anticipates the return to normal conditions; these coils, which are respectively in series with the field winding and armature winding, exert a force on the normal-response lever through springs S_2. Dashpot 2 acts as a stabilizer in regulating this response. The normal-response system tends to prevent overshooting or hunting of the regulation. When the initial movement of the differential lever is slow, dashpot 1 is not moved because the air in it leaks out through an adjustable vent hole; the normal-response lever is then deflected through springs S_2, and the desired change of pressure on the carbon pile is thus produced.

16. Industrial Applications of Shunt and Compound Motors.—The preceding analysis of the characteristics of the shunt motor shows that it is particularly well adapted to those applications where substantially constant speed is required at all loads. Examples are line-shaft drives and individual drives of spinning and weaving machines. For each setting of the field rheostat, if one is used, there will be a definite speed characteristic having the form shown in Fig. 4, that is, drooping slightly between no-load and full-load conditions. By adjusting the field rheostat the speed characteristic will be shifted bodily up or down. The range of such adjustment is limited in the case of plain shunt motors because, if the field is weakened in an attempt to secure a considerable increase of speed, there is a limit beyond which the full-load armature current cannot be commutated without injurious sparking.

The provision of commutating poles removes this difficulty, it thus being possible to secure a wide range of speed control by means of a field rheostat, the speed remaining substantially constant at all loads for each setting of the field resistance. This feature is particularly advantageous in individual drives for machine tools, such as lathes and boring mills.

Cumulative-compound motors are used where inherently variable speed is desirable but where the no-load speed must be

kept within definite limits. Drives for rolling mills and certain types of hoists are examples of such applications. Another example is afforded by electrically operated shovels, where the fall in speed as the load is increased serves to protect against excessive stresses in the structural parts of the shovel.

A useful criterion for studying the suitability of the various types of motors is afforded by a diagram like Fig. 46, which shows the relation between speed and torque. This drawing was made by taking pairs of values of speed and torque from Figs. 15 and 16, each pair corresponding to a particular value of the load current.

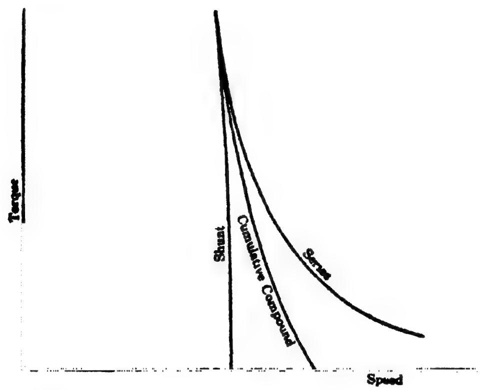


FIG. 46.—Speed-torque characteristics.

17. Applications of the Series Motor.—The rapid drop in speed of the series motor as its load is increased makes this type of machine especially valuable for traction purposes, as in street railways and hoisting service. In railway work, for example, motors having a constant-speed characteristic like the shunt motor are seldom used for the reason that the current taken by such a motor in going up a steep grade is excessive; for since the speed of such a motor remains substantially constant if the impressed voltage is constant, the additional power required to climb the grade demands a proportionally increased current. The series motor, on the other hand, slows down as the load increases, and an excessive load is thus automatically prevented and, to a certain extent, a constant load on the system tends to be maintained; at the same time it develops a torque more than

proportional to the current, whereas in the shunt motor the torque increases less than proportionately to the current. It is not necessarily implied that the torque of a series motor will be greater than that of a shunt motor when I_a is the same in both; for since torque is proportional to the product of flux and armature current, the two types may be made to have the same value of torque with full-load I_a by making the flux likewise the same (see Fig. 16). The advantage of the series motor for tractive purposes depends, rather, upon the form of the speed torque characteristic (Fig. 46).

Series motors for railway, automobile, hoisting, and rolling-mill service are generally of the totally enclosed type. In railway and automobile service, in particular, the motors must be waterproof and of rugged construction to withstand the rough usage to which they are subjected by reason of poor roadbed and improper handling of the starting controller. A too rapid cutting out of the starting resistance results in very heavy current, excessive torque, and a wracking of the armature winding.

18. Equations of Characteristic Curves of the Series Motor.* In Art. 15, Chap. X, there was presented a derivation of the characteristic curves of the shunt generator, based on the use of Froelich's equation for representing the relation between flux and excitation. An analogous study of the characteristic curves of the series motor leading to empirical equations for the speed and torque characteristics is given here, partly because such equations may at times be useful and partly because it furnishes a good example of some of the methods of developing equations to fit experimentally determined graphs.

For example, let it be required to determine the equations of the speed and tractive effort curves of Fig. 47 (tractive effort being proportional to torque).

Upon assuming that the relation between flux and excitation is given by

$$\Phi = \frac{aI}{b + I} \tag{32}$$

it follows that the effect of the demagnetizing ampere-turns of the armature is in this case accounted for; for the net excitation

* A. S. LANGSDORF, Empirical Equations of the Speed and Torque Characteristics of the Series Motor, *Washington University Studies*, 6 (No. 1), 1918.

is expressed by $(1 - k)I$, where k is a constant equal to the ratio of armature demagnetizing turns per pole to the field turns per pole; strictly, therefore, the form of Froelich's equation should be

$$\Phi = \frac{a(1 - k)I}{b + (1 - k)I}$$

but on dividing numerator and denominator of this expression by the term $(1 - k)$, it reduces to the form of Eq. (32).

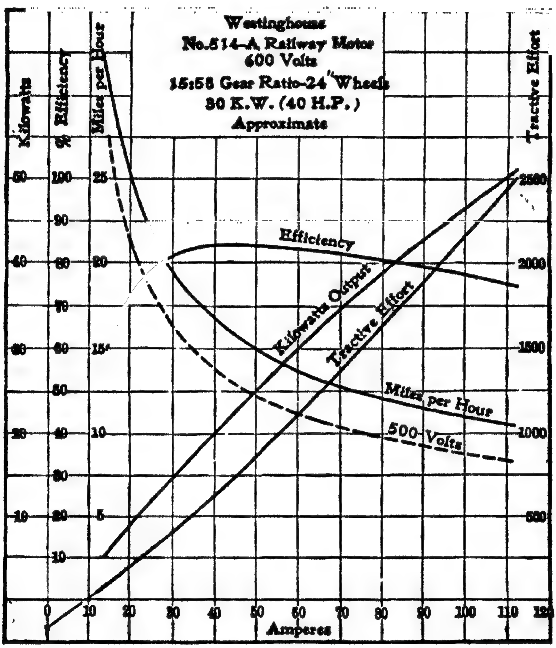


FIG. 47.—Characteristic curves of railway motors.

From Eq. (4),

$$V = \Phi Z'n + IR$$

where R is the total resistance of the motor, and on substituting for Φ from Eq. (32),

$$V = \frac{aZ'nI}{b + I} + IR$$

whence

$$nI = \frac{Vb}{aZ'} + \frac{V - bR}{aZ'} I - \frac{R}{aZ'} I^2 \tag{33}$$

If in Eq. (33) we regard the product nI as a new variable, say y, with the variable I as argument, the equation will represent a parabola with its axis parallel to the y axis. If, therefore, a table of simultaneous values of n and I is prepared by reading the coordinates of a series of points from the speed characteristic of Fig. 47, as has been done in Table I, and if there are then

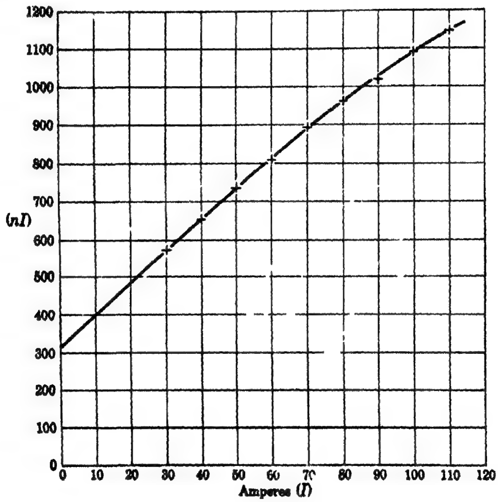


FIG. 48.—Plot of nI and I.

computed from these data the products nI having as factors the corresponding pairs of values of n and I, it may reasonably be anticipated that a curve plotted between nI and I will have a parabolic form. The curve of Fig. 48 has been constructed in this manner, and it shows clearly that the method is thus far justifiable.

It will be observed, however, that if I is made zero in Eq. (33), it becomes,

$$(nI)_{I=0} = \frac{Vb}{aZ'}$$

which means that, if the curve of Fig. 48 is produced backward until it intersects the nI axis, the intercept on that axis will give to scale the numerical value of the constant term Vb/aZ'. In the particular case considered, this turns out to be 320. If the curve between nI and I turns out to be somewhat flat and if the points determinable from the original data, or from Fig. 47, lie considerably to the right of the axis of ordinates of Fig. 48, there may be some uncertainty in the value of Vb/aZ' so determined. In such a case it may be necessary to make several trials before a satisfactory value is found, but in any case the trial value must be tested in the manner indicated in the remaining part of the analysis, as follows:

If the value of Vb/aZ' is used, as found from Fig. 48, Eq. (33) can be written in the form

$$\frac{nI - (Vb/aZ')}{I} = \frac{V - bR}{aZ'} - \frac{R}{aZ'}I \tag{34}$$

and the value of the quantity $\frac{nI - (Vb/aZ')}{I}$ can then be computed from the data of Table I. Inspection of Eq. (34) reveals

TABLE 1

1	2	3	4	5
I amperes	n speed, miles per hour	nI	$\frac{nI-320}{I}$	n computed
30	19.2	576	8.53	19 22
40	16.4	656	8.40	16.43
50	14.7	735	8 30	14.71
60	13 5	810	8.17	13.52
70	12.65	886	8.08	12.64
80	12.0	960	8.00	11.96
90	11.4	1026	7.84	11.39
100	10.9	1090	7.70	10.92
110	10.5	1155	7.59	10.50

the fact that this new quantity bears a linear relation to I, and when these two quantities are plotted there should result a straight line. Such a line is shown in Fig. 49, which is plotted

from the data in columns 1 and 4 of Table I. The fact that a straight line is obtained proves that the theory of the method is sound and incidentally that the value of Vb/aZ' was correctly obtained from Fig. 48; if the resultant line is found to be not straight, either the value of the constant was not correctly determined, in which case another trial should be made, or else the method is at fault.

Upon continuing the analysis of Eq. (34) and its graphical representation in Fig. 49, it will be seen that the intercept of the

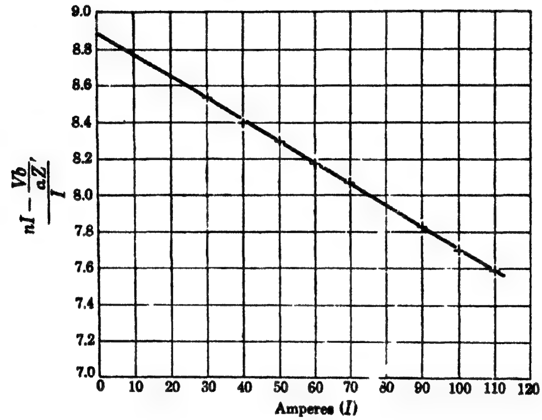


FIG. 49.—Plot of $(nI - Vb/aZ')/I$ and I.

line on the axis of ordinates is $(V - bR)/aZ'$, and that its slope is $-R/aZ'$. It is therefore easy to find the actual numerical values of these quantities, which are found to be:

$$\frac{V - bR}{aZ'} = 8.9$$

$$\frac{R}{aZ'} = 0.0118$$

Upon substituting these values in Eq. (34) and solving for n there results,

$$n = 8.9 + \frac{320}{I} - 0.0118I \qquad (35)$$

which is the desired equation (n being given in miles per hour).

By assigning various values to I, say the values in column 1 of Table I, corresponding values of n may now be computed and compared with the original data. Computed values of n are given in column 5 of Table I, and it will be seen that the agreement is practically absolute, because the ordinates of the curve of Fig. 47 cannot be read with accuracy beyond the third significant figure.

The torque and therefore the tractive effort developed by the motor are proportional to ΦI, but because of friction the net tractive effort available at the drawbar is less than that devel-

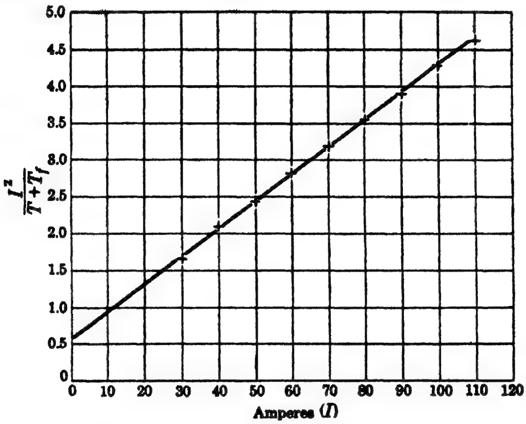


FIG. 50.—Plot of $I^2/(T + T_f)$ and I.

oped by a nearly constant amount. It is therefore legitimate to write

$$T = k\Phi I - T_f \tag{36}$$

where T is the net tractive effort and T_f is that part of the total tractive effort which is lost in friction. Substituting Eq. (32) in Eq. (36) and transposing,

$$T = \frac{kaI^2}{b + I} - T_f = \frac{cI^2}{b + I} - T_f \tag{37}$$

where c is a new constant.

Equation (37) can be written

$$c\frac{I^2}{T + T_f} = b + I \tag{38}$$

The form of Eq. (38) then suggests the following procedure: In Fig. 47 produce the curve of tractive effort backward until it intersects the axis of ordinates; the intercept, which is found to be 150 lb. (in the negative direction) in the case under consideration, may then fairly be taken as the value of T_f. Prepare Table II by reading simultaneous values of T and I from Fig. 47, and compute column 4 of the table by forming the ratio $I^2/(T + T_f) = I^2/(T + 150)$. Equation (38) then indicates that this quantity, when plotted with I, should yield a straight line. If it does not do so, a new trial value of T_f should be used. In the case here considered, the plot of $I^2/(T + T_f)$ and I, where T_f is 150, yields the straight line of Fig. 50 Upon selecting two points on this line and substituting their coordinates in Eq. (38) and solving to find the constants, it is found that

$$T = \frac{I^2}{0.68 + 0.0361I} - 150 \tag{39}$$

TABLE II

1	2	3	4	5
I amperes	T tractive effort, pounds	$T + T_f$	$\dfrac{I^2}{T + T_f}$	T computed
30	380	530	1 698	360
40	600	750	2 13	605
50	860	1010	2 48	855
60	1120	1270	2 84	1114
70	1370	1520	3 22	1380
80	1640	1790	3 58	1640
90	1900	2050	3 95	1910
100	2150	2300	4 35	2180
110	2440	2590	4 68	2450

If the values of I given in Table II are assumed, the values of T computed from Eq. (39) are as shown in column 5. The discrepancies are again very small, the correctness of the formula for T being thus verified.

19. Cycle of Operation of Railway Motors.—The horsepower rating of a railway motor has little significance in determining its suitability for a particular equipment. There are several

ways for specifying motor ratings for railway, service, the one most commonly used being known as the one-hour rating. This is defined* as the output at the motor shaft, measured in horsepower (or kilowatts) that the motor can carry for 1 hr. on stand test, starting cold, at its rated voltage and frequency (in the case of an a-c motor), with the ventilation system as in service, without exceeding specified temperature limits (see Chap. XIII). In any given case the motors must be so selected that they will not overheat, and the heating depends in part upon the average value of the square of the current taken throughout the whole of the working period, including stops. The current has its

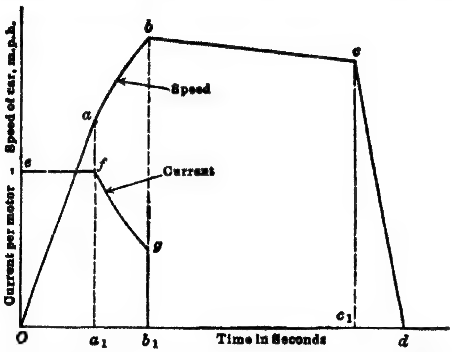


FIG. 51.- Speed-time and current-time curves of railway motor.

largest value during the starting or acceleration period, and hence the heating is largely dependent upon the number of stops in any given schedule.

When the car—or train—is started, the resistance in series with the motors should be cut out step by step in such a manner that the current through each motor remains practically constant until all the resistance is out of the circuit. The torque per motor will then be constant; hence, the drawbar pull and the resulting acceleration will also be constant, and the speed of the car will increase uniformly, as indicated by the line *Oa* (Fig. 51). When the resistance is all out, the speed will continue to increase, but at a steadily decreasing rate, as represented by the curved line *ab*, and during this interval the current decreases from the initial constant value *Oe* in the manner indicated by curve *fg*. The cause of the decreasing current is the increasing counter

* A. I. E. E. Standards.

e.m.f. due to the rising speed. After the time Ob_1, the current is shut off, and the car allowed to coast, the speed accordingly falling in the manner shown by line *bc*. The brakes are then applied and the speed rapidly falls from cc_1 to zero. The broken line *Oabcd* is called a *speed-time* curve, and its area is proportional to the distance traveled by the car in the time *Od*.

Figure 51 has been drawn to represent conditions common in urban service, where starts and stops are so frequent that long runs with the controller in the last running position are rare. But in interurban and trunk-line service the stations may be so far apart that power continues to be used for a longer time than is represented by the interval Ob_1 in Fig. 51. In that case the speed continues to rise and the current to fall, but at a decreasing rate (a level track being assumed), until the power input is equal to the power consumed in friction and windage losses; thereafter the speed and the current input remain constant.

The slope of the line *Oa* is the acceleration of the car; the value ordinarily used varies from 1 to 3 m.p.h. per sec., and this in turn determines the drawbar pull, torque, and current when the weight of the car, gear ratio, and type of motor are known.

20. Series-parallel Control.—In cars having a two-motor equipment the motors and starting resistance are at first all connected in series; and after the resistance has been cut out, the connections are quickly changed so that the motors themselves are in parallel with a resistance between them and the line. This resistance is then cut out, so that finally the motors are in parallel directly across the full voltage of the line The elementary diagram of connections is shown in Fig. 52 In four-motor equipments, the motors are usually connected in parallel in pairs, and the two pairs are then connected in series-parallel just as though each pair were a single machine.

Much greater economy of operation is possible by means of series-parallel control than can be obtained by providing each motor with a separate rheostat or by connecting the motors permanently in parallel, using a single resistance. For example, assume a two-motor equipment with the following data:

V = line or trolley voltage.
I = current per motor during acceleration period.
R = resistance of each motor.
t = duration of acceleration period in seconds.

At the moment of starting, the motors being in series (Fig. 52a), the starting rheostat must have a resistance of R_1 ohms such that

$$I = \frac{V}{R_1 + 2R}$$

or

$$R_1 = \frac{V}{I} - 2R$$

The loss in the rheostat at the first instant is at the rate of I^2R_1 watts; but as the motor speeds up at a uniform rate under the assumption of constant current the counter e.m.f. also increases uniformly, and in order to keep the current constant the resistance must be cut out at a uniform rate. All the resistance should be out of circuit in a time $t/2$ sec., and the motors then switched to the parallel position (Fig. 52b). During the first half of the acceleration period the energy lost in the rheostat is

$$W_{R_1} = \frac{1}{2}I^2R_1\frac{t}{2} = \frac{1}{4}I^2R_1t \quad \text{watt-sec.}$$

At the instant when all the resistance R_1 is out of circuit, each motor consumes half the line voltage, and this condition may be maintained efficiently if it is desired to continue running at reduced speed. But if the speed is to be increased at the original rate, as in Fig. 51, the motors must be put in parallel and a new resistance R_2 inserted between them and the line. In order that there may be no break in the smoothness of the acceleration, each motor must continue to take I amp., and at the first instant after the transition has been made the resistance R_2 must consume $V/2$ volts since the remaining $V/2$ volts are taken up by the motors. The resistance R_2 must therefore have such a value that

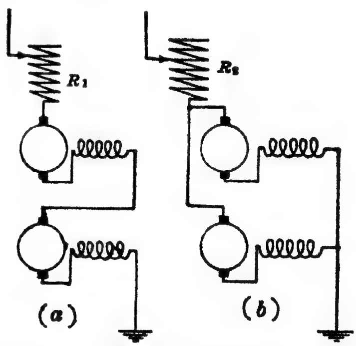


FIG. 52.—Elementary diagram of connections, series-parallel control.

$$R_2 = \frac{\frac{1}{2}V}{2I} = \frac{V}{4I} \quad \text{ohms}$$

and the energy lost in the rheostat during the second half of the acceleration period is

$$W_{R_2} = \frac{1}{2}(2I)^2 R_2 \frac{t}{2} = I^2 R_2 t \quad \text{watt-sec.}$$

The total loss in the rheostat, in watt-seconds, is

$$W = W_{R_1} + W_{R_2} = I^2 t\left(\frac{R_1}{4} + R_2\right) = \frac{1}{2} I^2 t\left(\frac{V}{I} - R\right)$$

If the motors had been originally in parallel, as in Fig. 52*b*, with a resistance of R_3 ohms between them and the line, the value of R_3 would have to be

$$R_3 = \frac{V}{2I} - \frac{R}{2} \quad \text{ohms}$$

in order to allow a current of I amp. to flow through each motor. The loss in the rheostat would then be

$$W_{R_3} = \frac{1}{2}(2I)^2 R_3 t = 2I^2 R_3 t = I^2 t\left(\frac{V}{I} - R\right) \quad \text{watt-sec.}$$

or exactly twice as great as in the case of series-parallel control.

21. Railway Controllers.—Before the general introduction of series-parallel control, the original street cars employed plain rheostatic control, with rheostats of the face-plate type operated from the motorman's position by a chain-and-sprocket drive. This rather crude mechanism was quickly replaced by controllers of the drum type, similar in their general design to the type illustrated in Fig. 23. The changes of connections effected by the movable contacts of typical drum controllers are illustrated in Figs. 53 and 54, which show successive stages in the development of this kind of control for single cars. The typical drum controller has two handles, one for the usual operation of accelerating the car, the other for the reversal of its direction of motion; the two handles are mechanically interlocked in such manner that the reversing handle cannot be moved unless the main handle is in the off position, and the main handle cannot be moved unless the reversing handle is in either the forward or the reverse

position. The reversing handle changes the direction of rotation of the motors by interchanging the connections of the field windings relative to the armature terminals.

A characteristic feature of drum controllers of this type is that the main current passes directly through them. This construction is feasible in the case of a single car or of a motor car and trailer, though at the cost of the heavy wiring from motors to controller; but where several motor cars must be operated as a train, *multiple-unit* control must be used. The controller for this service carries only a small auxiliary current supplied from the line, and this current actuates electromagnets which operate contactors that control the main current. In this system, shown diagrammatically in Fig. 55, a single master controller serves to operate simultaneously the contactors of all the motors in the train, the auxiliary circuit being extended by suitable couplings from end to end of the train.

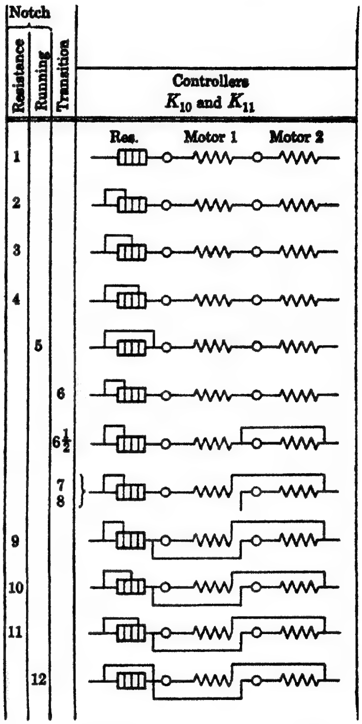


FIG. 53.—Successive stages of connections, K-10 controller.

Recent designs for the P.C.C. (President's Conference Committee) type of car embody, in addition to series-parallel control and the use of an increased number of resistance steps to secure smoothness of acceleration, such features as dynamic braking (Art. 25) and field control of speed. Field control is obtained by shunting a part of the series field winding. A schematic diagram of the main connections is shown in Fig. 56. One form of the master controller and limit relay is shown in Fig. 57, and the accelerator, with its incorporated resistors, is shown in Fig. 58.

22. Mechanics of Acceleration and Retardation of Motors.— It is a fundamental law of mechanics that a force of Δf dynes

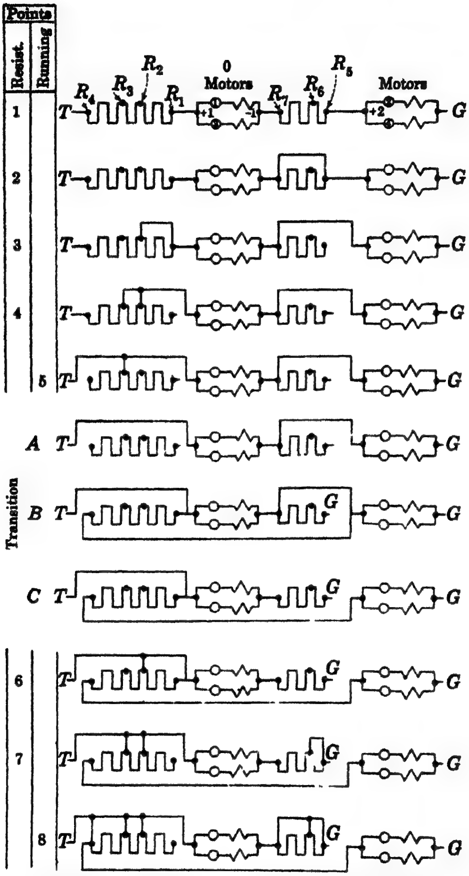


FIG. 54.—Bridge control, series-parallel system.

acting upon an unrestrained elementary mass of Δm g. will produce an acceleration of a cm. per sec. per sec. such that

$$\Delta f = \Delta m \cdot a = \Delta m \cdot \frac{dv}{dt} \tag{40}$$

where v is the velocity in centimeters per second. If the mass Δm is an elementary part of a rotating body and if it is distant

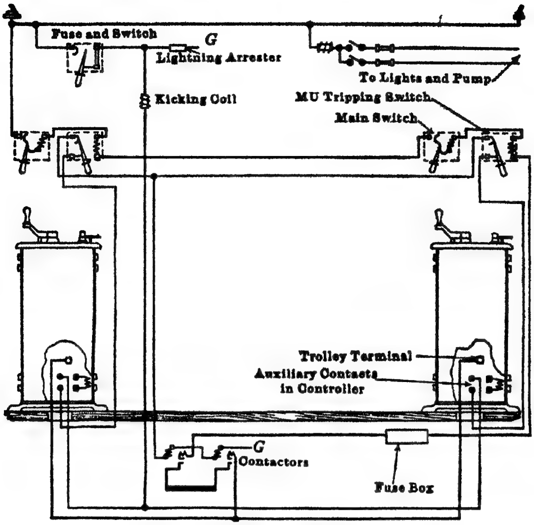


FIG. 55.—Railway motor controller with auxiliary circuits.

SEQUENCE CHART

	LS	M1	M2	R1	R2	F1	F2	F3	F4	B1	B2	Acc.
Switch	O	O		O				O	O			1 to 4
Rates	O	O		O	O			O	O			
	O	O		O	O				O			5 to 6
	O	O		O	O							7 to 55
	O	O	O	O	O							56 to 57
Field control	O	O	O	O	O	O						58
	O	O	O	O	O	O	O					59 to 60
	O	O	O	O	O	O	O	O				61
	O	O	O	O	O	O	O	O	O			62 to 98
Coast						O	O			O	O	98 to 1
Brake										O	O	

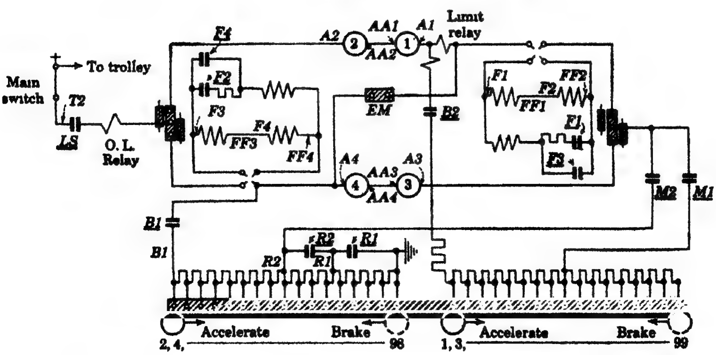


FIG. 56.—Main circuit connection diagram, P.C.C. type. (*Westinghouse Electric and Manufacturing Company.*)

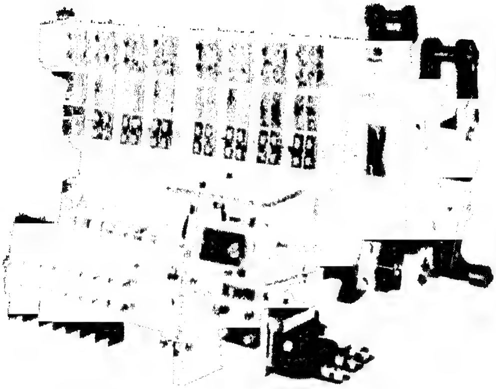


FIG. 57.—Master controller and limit relay. (*Westinghouse Electric and Manufacturing Company.*)

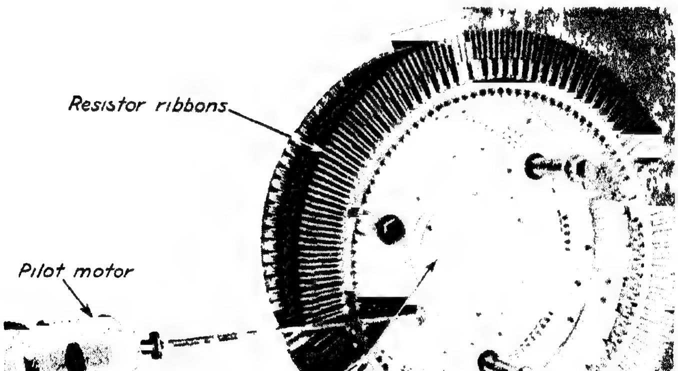


FIG. 58.—Accelerator for P C C control. (*Westinghouse Electric and Manufacturing Company.*)

r cm. from the axis of rotation, the accelerating moment or torque of the force is

$$\Delta T_a = \Delta f \cdot r = \Delta mra = \Delta mr\frac{dv}{dt} = \Delta mr^2\frac{d\omega}{dt} \tag{41}$$

where ω, the angular velocity of rotation, is related to the instantaneous velocity of translation by the equation

$$v = r\omega \tag{42}$$

The angular velocity is the angle, in radians, through which the body rotates in a unit of time (1 sec.), so that if the speed of rotation is n r.p.m., or $n/60$ r.p.s., the angular velocity is

$$\omega = 2\pi\frac{n}{60} \tag{43}$$

In Eq. (41), $d\omega/dt$ is the time rate of change of angular velocity, which is equal to the angular acceleration α. Accordingly, Eq. (41) can be written

$$\Delta T_a = \Delta mr^2 \cdot \alpha \tag{44}$$

If the rotating body is made up of a large number of elementary masses, each of which is distant from the axis of rotation by an amount r cm., the summation of all such products as Δmr^2 is the moment of inertia J of the rotating body, and Eq. (44) becomes

$$T_a = J\alpha \tag{45}$$

which is a fundamental equation of mechanics for determining the angular acceleration produced by a given torque. The same formula serves to determine the angular retardation due to a braking torque. Upon substituting in Eq. (45) the relation

$$\alpha = \frac{d\omega}{dt} = \frac{2\pi}{60}\frac{dn}{dt}$$

there results

$$T_a = \frac{2\pi}{60}J\frac{dn}{dt} \quad \text{dyne-cm.}$$

provided that $J = \Sigma\Delta mr^2$ is computed on the basis of mass in grams and dimensions in centimeters. If mass is expressed in

kilograms and dimensions in meters, the accelerating torque becomes

$$T_a = \frac{\pi}{30 \times 9.81} J \frac{dn}{dt} \quad \text{kg.-m} \tag{46}$$

If it is desired to use English units (mass in pounds and dimensions in feet), this equation becomes

$$T_a = \frac{\pi}{30 \times 32.2} J \frac{dn}{dt} \quad \text{lb.-ft.} \tag{47}$$

The computation of the moment of inertia J of a rotating body is facilitated by analyzing the actual body into a series of simple geometrical forms and adding together the moments of inertia of the several parts. In d-c machines, the typical forms are:

1. Hollow cylinders, such as the armature core and its winding; and the commutator.
2. Solid cylinders, such as the shaft.
3. Radially disposed arms, such as the spokes of the armature and commutator spiders.

In case 1, if the internal and external radii are r_1 and r_2, the formula for J is $m(r_1^2 + r_2^2)/2$; in case 2, where the internal radius r_1 is zero, $J = mr_2^2/2$; and in case 3, $J = m(r_1^2 + r_1r_2 + r_2^2)/3$, where r_1 and r_2 are the inner and outer radii of the arms which are assumed to be like straight rods. In general, the value of J is determined by the relation $J = \int dm \cdot r^2$, the integration being extended between limits fixed by the shape and dimensions of the part under consideration.

Quite commonly a motor is connected, either directly or through gearing or nonslip belting, to other rotating parts that have moments of inertia represented, say by J_1, J_2, J_3, . . . ; if their speeds of rotation are respectively n_1, n_2, n_3, . . . , measured in terms of the same unit that expresses the speed n of the driving motor, the effect of these connected rotating loads is to increase the moment of inertia of the original motor from J to an equivalent value

$$\sum J = J + J_1\left(\frac{n_1}{n}\right)^2 + J_2\left(\frac{n_2}{n}\right)^2 + J_3\left(\frac{n_3}{n}\right)^2 + \cdots \tag{48}$$

If instead of, or in addition to, being mechanically geared to rotating parts, the motor is coupled to a moving mass m' lb

which has imparted to it a velocity of translation v' ft. per sec. when the motor is running at a speed of n r.p.m., the effect is the same as if the moment of inertia of the motor were increased by

$$J' = \frac{m'}{32.2}\left(\frac{30}{\pi}\frac{v'}{n}\right)^2 \tag{49}$$

This case occurs with railway motors, and it is also encountered with motor-driven traveling cranes.

23. Time Required for Acceleration of Motors.—The total torque T developed by a motor is given by Eq. (11), page 470. Part of this torque, T_0, is consumed by the friction and core losses of the motor itself; another part, T_l, is consumed in overcoming the resisting moment due to the load carried by the motor; and the remainder, if any, is the accelerating torque T_a. Consequently

$$T_a = T - T_0 - T_l \tag{50}$$

If the load on the motor is expressed in horsepower, the equivalent torque or moment T_l corresponding to a speed of n r.p.m. can be found from the relation

$$T_l = \frac{\text{hp.} \times 33{,}000}{2\pi n} \quad \text{lb.-ft.} \tag{51}$$

If, for the sake of simplicity, it is assumed that T_0 and T_l are constant regardless of the speed and if it is also assumed that the armature current is held constant during the acceleration period, it follows from Eqs. (11), (47), and (50) that

$$\frac{\pi}{30 \times 32.2} J \frac{dn}{dt} = 7.05\Phi Z' I_a - T_0 - T_l \tag{52}$$

where all the quantities entering into this expression are constant except n and t. Therefore, subject to these assumptions,

$$\int_0^n dn = A \int_0^t dt$$

or

$$n = At \tag{53}$$

where

$$A = \frac{30 \times 32.2}{\pi J}(7.05\Phi Z' I_a - T_0 - T_l)$$

Equation (53) indicates that under the assumed conditions the speed increases uniformly with time, or the angular acceleration is constant; it is then a simple matter to find the time required to attain a given speed.

In general, however, the resisting torque due to the load is not independent of the speed. In the case of railway loads, fans, blowers, etc., the power required is more than proportional to the speed; consequently, from Eq. (51), T_l increases with increasing speed and T_a decreases. Moreover, as the speed rises the counter e.m.f. increases also; and since the armature current is thereby reduced, the magnitude of T also becomes smaller, unless the starting rheostat is of such special character that the current can be kept constant notwithstanding the changing counter e.m.f. It is obvious, therefore, that with the usual types of starters, which change the starting resistance by a series of discrete steps, neither the torque developed by the motor nor the resisting torque due to the load will remain constant. The problem of determining the time required to accelerate the motor may then be analyzed in the following manner:

In Fig. 46 and the preceding drawings from which it was derived, it will be noted that the speed-torque characteristic of any type of motor is a function of its magnetization curve and of such constants as armature resistance and armature demagnetizing effect. The particular curves of Fig. 46 (obtained from Figs. 15 and 16) are based upon the condition that the motor is running freely on the main line, without any external resistance in the armature circuit. If the starting resistance, or any part of it, is in circuit, a different speed-torque characteristic will be obtained.

Consider the particular case of the series motor. Figure 59 shows in a single diagram the torque-current characteristic and several speed-current characteristics, all constructed in the manner already explained in connection with Figs. 8 and 9. Each speed characteristic corresponds to a particular value of resistance in series for starting purposes. Variation of the series resistance has no effect upon the torque-current characteristic since torque depends only upon the current and is independent of speed and counter e.m.f. By taking pairs of simultaneous values of torque and speed, the several speed characteristics being used in turn, there is obtained a group of speed-torque characteristics,

as in Fig. 60, each curve corresponding to a definite value of resistance in series with the motor. If it is assumed that the successive steps of the starting rheostat have been so proportioned that the current fluctuates between a maximum value I_1 and a minimum value I_2, the relation between total torque T and speed,

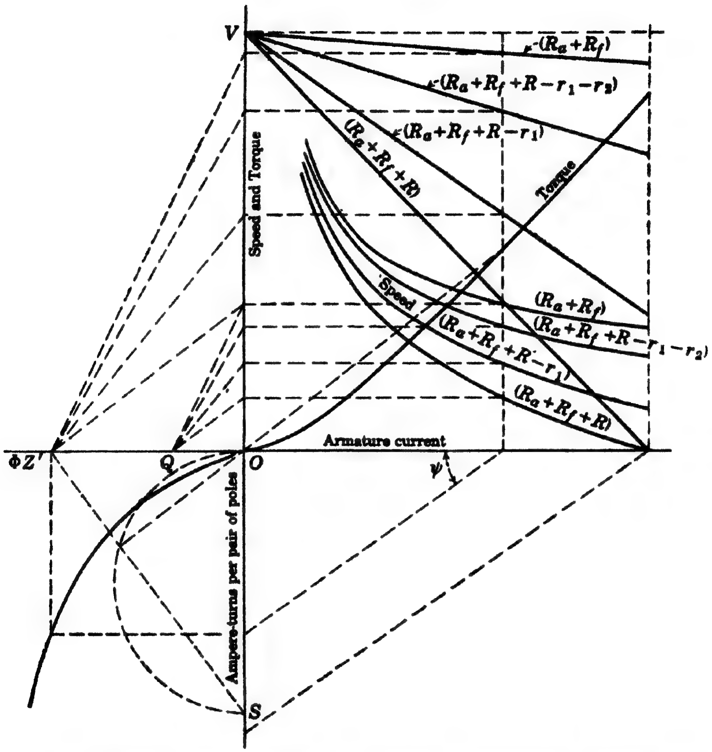


FIG. 59.—Effect of series resistance upon speed characteristic of series motor.

as the motor accelerates, is shown by the saw-tooth pattern indicated by the heavy line at the top of the diagram.

Let it be assumed that the combined resisting torques T_0 and T_l are represented by the curve marked $(T_0 + T_l)$ (Fig. 60); this is so drawn that it intersects the last torque-speed curve in the point Y, corresponding to the condition of all starting resistance out of the circuit and armature current equal to I_2. The torque developed by the motor is then just equal to the resist-

ing torques, acceleration will have ceased, and the speed will have attained the value represented by the abscissa OX.

If the ordinates of the curve $(T_0 + T_l)$ are subtracted from those of the heavy full line T, the remainder gives the relation between the accelerating torque T_a and speed, indicated in the

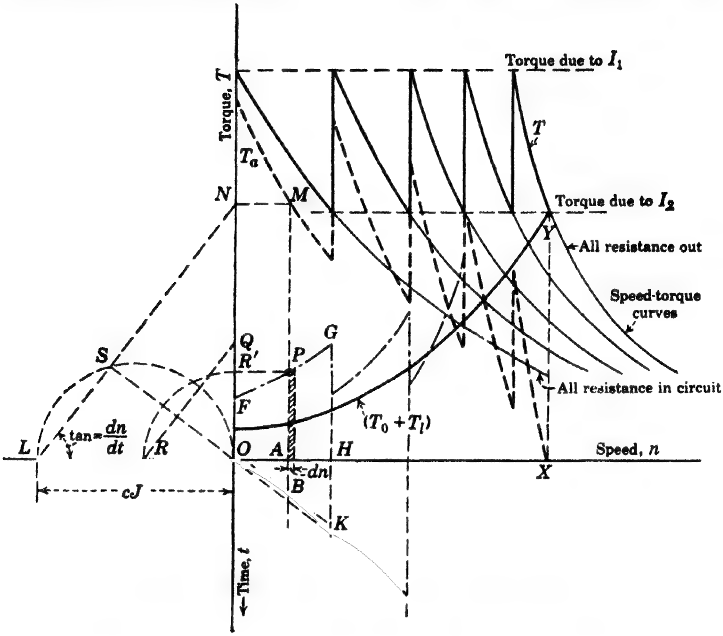


FIG. 60.—Acceleration and torque relationships.

diagram by the dashed line T_a. But from Eqs. (46) and (47) it is seen that T_a and n are related by the expression

$$T_a = cJ\frac{dn}{dt}$$

the magnitude of the constant c depending upon the system of units employed. This equation can be written

$$dt = \frac{cJ}{T_a}dn \tag{54}$$

whence

$$t = \int_0^t dt = \int_0^n \frac{cJ}{T_a}dn \tag{55}$$

In Fig. 60, the accelerating torque T_a corresponding to a speed OA is given by the ordinate of point M, that is, by ON. If, therefore, the constant length $OL = cJ$ is laid off to any arbitrary scale to the left of the origin, the slope of the line LN is $T_a/cJ = dn/dt$, which is equal, or proportional, to the slope of the speed-time curve, depending upon the choice of scale. Selecting a fixed point Q on the vertical axis of reference, draw QR parallel to NL, and lay off $OR' = OR$ by drawing the quadrant of a circle as indicated. The length $AP = OR'$ is then equal to or proportional to cJ/T_a and determines a point P on a curve of which all the ordinates, constructed in like manner, give cJ/T_a as a function of n. Only a portion of this curve, shown in Fig. 60 as a dot-and-dash line, has been drawn.

The area of the crosshatched element AP is evidently represented by $(cJ/T_a)dn$, and the integrated value of all such elements between $n = 0$ and $n = n$ is, by Eq. (55), the time required to attain the speed n. Thus, the length AB represents to scale the area of the trapezoidal figure $OFPA$, and HK represents the area $OFGH$. By proceeding in this manner from point to point, the entire course of the speed-time curve OK can be mapped out.

It is of interest to note that if a semicircle is drawn on the fixed length OL as a diameter, it intersects LN in a point S such that OS is parallel to the tangent of the speed-time curve at B, provided that the time scale has been suitably chosen.

Though Figs. 59 and 60 have been constructed to represent the case of the series motor, it is obvious that similar methods are applicable to shunt and compound motors. The shape of the torque-speed curves will not be the same, as may be seen from Fig. 46, and the shape of the curve $(T_0 + T_l)$ (Fig. 60) will vary depending upon the nature of the load. Otherwise the principle of the method is the same for all types of motors.

It is clear from Eq. (52), as well as from the construction of Fig. 60, that a motor will continue to accelerate as long as its developed torque exceeds the resisting torques of the load and of its own friction and core losses. The condition for the eventual attainment of steady speed, equivalent to a state of stable equilibrium, is, therefore, equality between the driving torque and the sum of all opposing torques. Whenever the speed-torque characteristic of a motor lies above the corresponding characteristic of the load, there will therefore exist a condition of *instability*.

24. Retardation of Motor Speed.—When the supply of power is cut off from a motor, it will come to rest when the kinetic energy stored in its rotating parts, and in moving parts of the load to which it is connected, has been dissipated. The same equation (46 or 47) that determines the angular acceleration produced by an accelerating torque also gives the relation between retarding or braking torque and the resultant rate of decrease of speed. The derivative dn/dt is negative as the speed falls, and the corresponding negative sign of the torque indicates that the torque acts in a direction opposite to the rotation.

These relations can be utilized to measure the losses in a machine at any given speed.* The method consists in raising

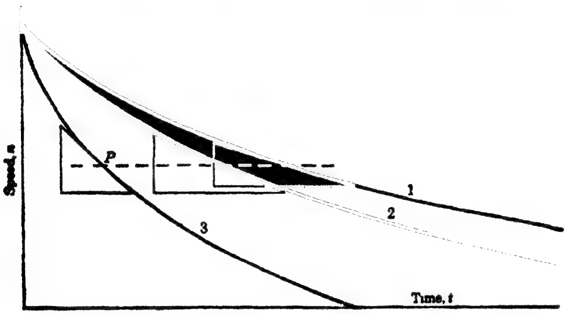


FIG. 61.—Retardation curves.

the speed to somewhat more than its rated value and then allowing the motor to "coast" under the retarding influence of the losses to be measured, readings of the speed being taken simultaneously with time as the motor slows down. Retardation curves obtained in this way are shown in Fig. 61. Curve 1 is taken with the brushes lifted and the field circuit disconnected, so that the only losses are those due to bearing friction and windage. Curve 2 is taken with the field disconnected but with the brushes in place, so that the losses are greater by the friction of the brushes on the commutator. Curve 3 is taken with the brushes lifted and the field excitation normal, the losses being then made up of bearing friction, windage, and core losses (hysteresis and eddy currents). In order to compute the losses

* See also O. E. CHARLTON, and W. D. KETCHUM, Determination of Generator Speed and Retardation during Loss Measurements, *Trans. A.I.E.E.*, **49**, 1095, 1930.

from these curves, it is to be noted that if T_a in Eq. (47) is multiplied by $(2\pi n \times 746)/33{,}000$ the product is the loss in watts. Hence, the loss is given by

$$P_l = \frac{2\pi^2 \times 746J}{33{,}000 \times 30 \times 32.2}n\frac{dn}{dt}$$

$$= 0.462 \times 10^{-3}Jn\frac{dn}{dt} \quad \text{watts} \tag{56}$$

provided that J is measured in English units (lb.-ft.^{2}) and n is in r.p.m. Thus, at a point P, Fig. 61, the value of dn/dt can be found by drawing a tangent to the curve, so that, if J is known, the losses at the corresponding speed n follow from Eq. (56). By taking the difference between losses determined from curves 1 and 2, the loss due to brush friction is determined; and by taking differences between the losses found from curves 2 and 3, the core losses are fixed.

The determination of the losses by this method depends upon an accurate knowledge of the moment of inertia. Computation of J by the formulas given in Art. 22 cannot be depended upon, since it is difficult to obtain all the necessary dimensions and masses. The determination of the moment of inertia can be made experimentally in the case of large machines by applying a known retarding torque by means of a brake and plotting a retardation curve in the same way as before. For example, if the slope of the retardation curve 1 is $(dn/dt)_1$ at some particular speed n, the resisting torque, due to bearing friction and windage, is $T_1 = cJ\ (dn/dt)_1$. If, with a known additional braking torque T_b applied to a pulley on the shaft, the rate of decrease of speed is found to be $(dn/dt)_2$ at the same speed n,

$$T_1 + T_b = cJ\left(\frac{dn}{dt}\right)_2$$

whence

$$J = \frac{T_b}{c\left[\left(\frac{dn}{dt}\right)_2 - \left(\frac{dn}{dt}\right)_1\right]} \tag{57}$$

where $c = \pi/(30 \times 32.2)$ if English units are used.

This method is difficult in the case of small machines because the machine comes to rest so quickly that accurate readings of speed and time cannot be made. Speed should be measured

by some form of stroboscope, since the use of a tachometer introduces additional friction that is not negligible, except in large machines. The difficulty with small machines can be overcome by placing on the shaft a heavy flywheel of known moment of inertia. The moment of inertia of the combined system can be found by means of Eq. (57), and that of the motor itself by subtracting the amount contributed by the flywheel.

25. Dynamic Braking. Regenerative Control.—When a motor is to be brought to rest quickly, as in elevator and rolling-mill service, the rate of retardation can be greatly increased by using the inertia of the load to drive the motor as a generator during the slowing-down period, the energy being consumed in a resistor connected across the armature. The controller must be so arranged that when this braking resistance is connected across the armature the field winding receives enough current to maintain the magnetic field at full, or even increased, strength. A shunt motor will act as a generator without any change in the connections of the field winding to the line. But in the case of a series motor it is necessary to change the field connections by placing a current-limiting resistor in series with the field winding and then to connect the resultant circuit across the line; or else, as in the electric locomotives of the Chicago, Milwaukee and St. Paul Railway, the series field must be excited by a special low-voltage generator. This method of obtaining a quick stop is called *dynamic braking*. It is commonly used in controlling elevators, rolling mills, printing presses, machine tools, etc. As the motor slows down, the generated e.m.f. likewise decreases, so that if the retarding torque is to be maintained the braking resistance must be decreased also. The calculation of the steps of the braking resistance can be carried out by methods similar to those indicated in Figs. 17, 18, and 19.

Figure 62 illustrates an application of dynamic braking to a crane hoist operated by a series motor. When the crane is used for lowering duty, the contactor L is closed and H (the hoisting contactor) is open, and the entire line current, limited to full-load value by the resistance between the contactor points R_2, passes through the solenoid of the magnetic brake, which is instantly released. It will be seen that the armature current is limited by the resistance E.

An application of dynamic braking that presents features of special interest is afforded by the case of crane bridge drives where series motors are used. Owing to the use of antifriction bearings, such a bridge will coast a considerable distance after the power has been shut off; with the series motor disconnected from the line, a sufficient time may elapse (before it becomes necessary to stop the bridge) to permit the magnetic field of the motor to

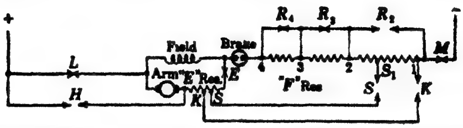


Fig. 62.—Wright dynamic lowering circuit. (*Electric Controller and Manufacturing Company.*)

collapse. On reconnecting the motor to act as a generator through a braking resistance it may fail to build up its field and therefore may fail to serve as a brake. To prevent this contingency, the connections may be arranged as in Fig. 63. The series field winding is permanently connected to the line through a current-limiting resistor in order to obviate the danger of a collapse of the magnetic field; in the application of this principle as worked out by the Electric Controller and Manufacturing

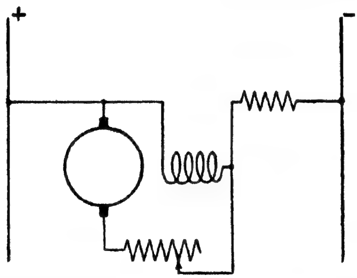


Fig. 63.—Teaser connection for dynamic braking.

Company, the arrangement is referred to as a "teaser" connection of the field winding.

In cases where a motor must be stopped quickly and then immediately reversed, as in rolling-mill service, the operation known as "plugging" is commonly used. In plugging, power is applied in the reverse direction before the motor has stopped.

When this is done the line voltage and the e.m.f. generated in the armature of the motor act in the same direction until the motor stops, so that, unless preventive measures are supplied, an enormous rush of current will result. This heavy current would, at the very least, trip the circuit breaker at the very time when uninterrupted excitation is essential. To forestall difficulty due to this cause, the plugging relay and contactor shown in Fig. 34 are provided. The plugging relay $S1P$ is shunt-wound and is permanently connected across the first step of the starting resistance. So long as the current through this resistance (and therefore also the drop of potential across it) exceeds a predetermined limit, the relay prevents the closure of the plugging contactor R_2, the usual setting being such that R_2 is held open until the motor comes to rest; an adjustment of the relay is provided whereby the plugging contactor may be made to close somewhat before the motor comes to rest, the reversal being thus hastened.

Regenerative braking is a modified form of dynamic braking in which the energy stored in the moving system is delivered back to the supply circuit instead of being consumed in a resistor. If, while one moving system is being slowed down in this manner, other similar loads are drawing power from the line, the effect is to relieve the powerhouse of part of the load. But if there are no other loads on the line, the effect would be to reverse the power flow, in which case special provision must be made to prevent damage to the prime movers.

26. Flywheel Load Equalizers.— The use of flywheels to equalize the loads on steam and gas engines, punch presses, and other machines subjected to fluctuating loads is an old and well-understood device. The same device is used to equalize the load on a motor-driven system where peak loads, of short duration, are large in comparison with the capacity of the supply system, as in mine hoists. The Ilgner-Ward-Leonard system, which is a modification of the Ward-Leonard system previously described, makes use of a heavy flywheel on the motor-generator set. The energy stored in the flywheel is drawn upon, when the load reaches a peak, by allowing the speed of the motor-generator set to fall; when the load falls off, the speed again increases, and energy is thus restored to the flywheel. The flywheel therefore acts as a buffer between the supply circuit and the load, the net

result being the same as though the peak loads were carried by a storage battery, but with the advantage of much lower cost.*

27. Division of Load between Motors.—Two or more shunt motors designed for the same voltage, when connected in parallel to the same supply circuit, with their shafts rigidly coupled, will divide the load in proportion to their capacities provided that their speed-current curves (Fig. 4) are identical, as discussed in connection with Fig. 26, Chap. X—that is, if the speed curves, plotted in terms of percentage of full-load current, are identical. Similar considerations are applicable to series motors operating in parallel.

Series-wound motors, when connected in series in a constant-current circuit, will develop approximately constant torque if the brushes are kept in a fixed position; but if the torque were constant the speed would have to vary in direct proportion to the load. To obviate this variation of speed and in particular to keep the speed constant, the series motors in the Thury constant-current system (see Chap. X) are provided with regulators which change the position of the brushes, the effect being to change the torque instead of the speed.

An interesting example of unequal division of load between series-wound motors is afforded by the case of a car starting on an upgrade on slippery rails. Assume, for instance, that the rear end of the car is more heavily loaded than the forward end; on turning the controller handle to the first notch, the same current will flow through both motors (or both pairs of motors) since they are in series with each other and therefore each will develop the same torque. If the weight on the forward trucks is fairly light, the adhesion between the wheels and the rail may not be sufficient to prevent slipping, in which case the forward motor will speed up and spin the wheels. The counter e.m.f. of the forward motor will increase as its speed rises, so that its impressed voltage must also increase; but any increase of the voltage on the forward motor will be at the expense of that impressed on the already overworked rear motor, and the result will be to stall the car unless the front wheels can be prevented from slipping.

* W. N. MOTTER, and L. L. TATUM, Flywheel Load Equalizer, *Trans. A.I.E.E.*, **30**, Part I, 729, 1911.

CHAPTER XII

COMMUTATION

1. Elementary Physical Theory.—The elementary discussion of commutation given in Art. 7, Chap. VIII, indicated that it is the function of the commutator and brushes to switch the individual elements of the armature winding from a circuit in which the current has one direction to an adjoining circuit in which the current has an opposite direction. During this transition period, called the *period of commutation*, the current in an element must be reduced from its original value to zero and then built up again to an equal value in the opposite direction. The period of commutation is necessarily very brief, of the order of 0.0005 to 0.002 sec., since it must be limited to the time required for a point on the commutator to pass under a brush. It may therefore easily happen that the reversal of current is either retarded or unduly accelerated; in either case, the current at the end of the period will tend to have a value which differs from that of the circuit to which the commutated coil is about to be connected, and the final equalization may in that case result in a spark, or arc, between the brush and the commutator segment, with results destructive to both. Study of the commutation process therefore has as its object the determination of the conditions that will result in satisfactory commutation, by which is meant not only the prevention of injurious sparking but more generally the preservation of a smooth sliding contact between commutator and brushes.

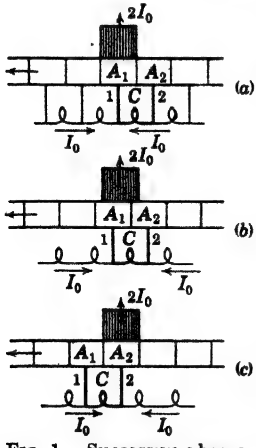


FIG. 1.—Successive phases of short circuit of coil.

For simplicity consider the case of a simple ring winding (Fig. 1), in which the brush width b is equal to the width β of a commutator segment. Under this condition only one winding

element will be short-circuited at a time, and the effect of the mutual induction of other elements is eliminated. Parts (*a*), (*b*), and (*c*) of Fig. 1 represent, respectively, the initial, intermediate, and final stages of the commutation of coil C. In (*a*), the coil is a part of the right-hand branch of the winding and carries the current $I_0 = I_a/a$. All of the current reaching the brush from the two adjoining paths must therefore pass through lead 1. Similarly, in the final position (*c*), coil C has become an integral part of the left-hand branch, its current has been completely reversed, and the combined current of the two adjoining paths must reach the brush by way of lead 2. The (*b*) position, in which coil C is short-circuited, shows that immediately after segment A_2 has reached the brush the current from the right-hand branch may reach the brush by way of both leads 1 and 2, and coil C therefore carries less current than before; as the contact area of segment A_1 diminishes and that of A_2 increases, the original current through C is diverted more and more from lead 1 to lead 2. At the same time that the right-hand branch current is being thus throttled out of coil C, the left-hand branch current finds its way more and more readily through C and the increasing contact area of segment A_2 and less and less readily through the diminishing contact area of A_1.

If the transfer of the brush current $2I_0$ from lead 1 to lead 2 occurs uniformly during the commutation period, *linear commutation* results. In that case the current in coil C will have zero value when the insulation between segments A_1 and A_2 is directly under the middle of the brush; and leads 1 and 2 will at that moment each be carrying current I_0. But if the axis of commutation (determined by the position of the brush) is too near the leading pole tip (in the case of a generator) the e.m.f. induced in the short-circuited coil by its rotation through the flux will accelerate the transfer of current from lead 1 to lead 2, and thus tend to produce excessive current density at contact A_2, a condition that is characteristic of *overcommutation*. On the other hand, a commutating field that is too weak will delay transfer of current from lead 1 to lead 2, so that the current density may then become excessive at segment A_1, this condition being characteristic of *undercommutation*.

The variation of current in a winding element during the period of commutation may occur in an infinite variety of ways. A few

of these *short-circuit current curves* are indicated in idealized form in Fig. 2, but in every case the current must change from $+I_0$ to $-I_0$ in a time T.

Curves a and b represent the case of overcommutation, the initial reversal having occurred too rapidly and implying excessive current density at the leading edge of the brush.

Curve c represents linear commutation, which is a desirable condition, for it corresponds to a condition of uniform current density and minimum loss of power at the brush contact surface (Art. 5).

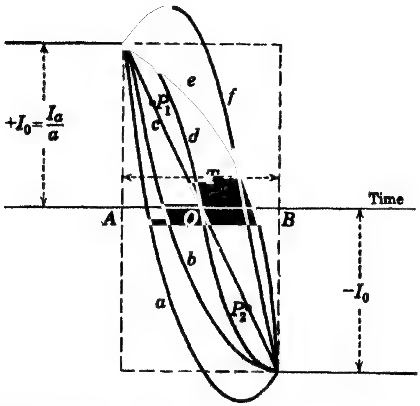


FIG. 2.—Types of short-circuit current curves.

Curve d, mainly of theoretical interest, represents so-called sinusoidal commutation; the curve (one-half of a sine curve), if actually realized, would result in satisfactory commutation.

Curves e and f represent cases in which reversal has been delayed; in case f the initial volume of current in the coil has even been increased. These cases are typical of undercommutation, and they involve excessive current density at the trailing tip of the brush.

Short-circuit current curves as actually observed by means of oscillograph records show marked departures from the smooth curves illustrated in Fig. 2. Their form is affected by vibration of the brushes, unevenness of the commutator surface, and other factors to be discussed later; in general, these curves have a saw-tooth shape, indicating the presence of high-frequency ripples in the short-circuit current.

The fact that over- and undercommutation may give rise to abnormal current density at the brush contact indicates the possibility of excessive localized heating under such circumstances. The heating may promote a disintegration of the brush material as well as chemical changes in the copper segments, even though there is no visible sparking or arcing It may be concluded, therefore, that sparkless commutation is not in itself a sufficient criterion of successful commutation.

2. Elementary Mathematical Relations.—Ordinarily the brush width b is greater than the width β of a commutator segment, so that several winding elements are simultaneously short-circuited; in Fig. 3, $b = 2.5\beta$, and hence three coils undergo commutation at the same time. The winding illustrated in Fig. 3 is the same as that of Fig. 23, Chap. VII. If it is arbitrarily assumed that the direction of current flow to the left of slot 1 is downward, as indicated by arrow A, then in successive belts of the armature surface the direction of current will be shown by arrows B and C. The currents in slots 1 and 17 are in process of change from one direction to the other, but it is assumed for convenience that at the instant shown in the diagram all of them have the original direction and that their magnitudes are i_1, i_3, and i_5. Applying Kirchhoff's first law to the junction points of the elements, it is seen that the currents in the risers connected to the segments touched by the brush are $(I_0 - i_1)$, $(i_1 - i_3)$, $(i_3 - i_5)$, $(i_5 + I_0)$; their sum is equal to $2I_0$, where $I_0 = I_a/a$ is the current per path in the winding.

Let it now be assumed that the instantaneous contact resistances between the brush and each of the four segments which it touches are R_1, R_3, R_5, and R_7, as in Fig. 3. Assume also that

R_c = resistance per winding element.
R_l = resistance of each commutator riser.
L = inductance of a winding element.
ΣM = mutual inductance between an element and all other elements that are simultaneously short-circuited.
E_c = commutating e.m.f. developed in an element by its rotation through a suitable reversing field.

Kirchhoff's second law may be applied to the closed circuit that includes the winding element (say number 3 in Fig. 3) and the brush contacts, in which circuit it is justifiable to assume that the commutator segments, as well as the body of the brush

itself, have negligible resistance. In tracing through this circuit, say in the clockwise direction, any active e.m.f. is to be taken as positive if it acts in that direction and as negative if it acts in the opposite direction; a drop of potential due to a current in the clockwise direction is to be considered negative, and vice versa. The commutating e.m.f. E_c must therefore be taken as positive, since it must tend to produce an upward flow of current on the left side of the element, as indicated by arrow B. In the same manner, the e.m.f. of self-induction $(-L\,di/dt)$

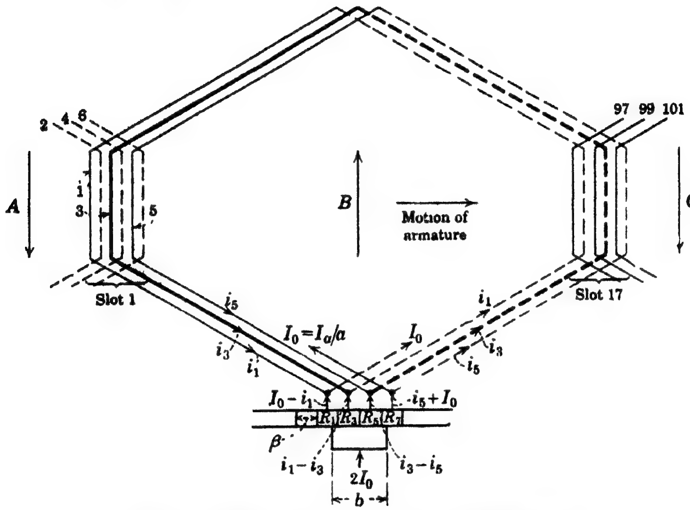


FIG. 3.—Short-circuit conditions with wide brush.

must be taken with a negative sign since it always opposes the change of current. The conditions with respect to mutual induction are somewhat ambiguous, for Fig. 3 shows that element 3 is subjected not only to the inductive effects of elements 1 and 5, but also to the effects of coil sides 2, 4, 6, and 97, 99, 101, which are simultaneously short-circuited by neighboring brushes. These adjacent elements which give rise to a mutually inductive effect are not all in the same phase of current reversal at the same moment, for their angular displacement in the slots, and the displacement of the commutator segments to which they connect, make it plain that there is a greater or less time interval between their several commutation periods. It is impossible to take all

these complications into account in any reasonably simple manner; it is sufficient for practical purposes to assume that all elements simultaneously short-circuited are at a given instant in the same phase of their variation, so that the total effect is to set up in element 3 an e.m.f. of mutual induction represented by $(-\Sigma M\, di/dt)$; and this, like $(-L\, di/dt)$, must be taken with a negative sign.

It is therefore possible to write

$$E_c - (-L\, di_3/dt) - (-\Sigma M\, di_3/dt) + i_3 R_c + (i_3 - i_5)(R_5 + R_l) - (i_1 - i_3)(R_3 + R_l) = 0$$

or, after rearranging terms,

$$E_c + i_3(R_c + 2R_l + R_3 + R_5) - i_5(R_5 + R_l) - i_1(R_3 + R_l) + (L + \Sigma M)di_3/dt = 0 \quad (1)$$

Figure 3 has been drawn with element 3 at or near its mid-position, so that element 1 is in the early stages of commutation, and element 5 is near the end. Currents i_1 and i_5 are therefore approximately equal in magnitude and opposite in sign, as indicated by points P_1 and P_2 in Fig. 2. The terms involving i_1 and i_5 in Eq. (1) therefore tend to cancel each other; and since $(R_c + 2R_l)$ is generally very small in comparison with $(R_3 + R_5)$ at least in large machines, Eq. (1) leads to the approximate relation

$$E_c + i(R_3 + R_5) = -(L + \Sigma M)\frac{di}{dt} \quad (2)$$

where $i(=i_3)$ is the instantaneous current in any element undergoing commutation.

The relation expressed by Eq. (2) was derived from Eq. (1), which was in turn based upon the assumption that the entire surface of the brush is in contact with the commutator. As a matter of fact, contact between brush and commutator is limited to a relatively small part of the brush area, and thus it is impossible to assign precise meaning to the resistances R_1, R_3, ... Moreover, even if the contact areas were definite, the contact resistances cannot be treated as ordinary ohmic resistances, for they are themselves inverse functions of the current, with the result that the drop of potential at a brush contact tends to be

constant regardless of the magnitude of the brush current. These circumstances will be discussed later in more detail, but it is plain that the uncertainties they imply make it impossible to develop a rigorous mathematical theory of commutation. It is, however, reasonable to assume that the relations embodied in Eq. (2) are valid in a general way and that, expressed in words,

Commutating e.m.f. + contact drop = reactance voltage

or

Contact drop = reactance voltage − commutating e.m.f. (3)

The phrase reactance voltage, commonly used in a-c terminology, means simply the e.m.f. of self- and mutual-induction, or $-(L + \Sigma M)(di/dt)$.

3. Relations between Commutating E.M.F., Reactance Voltage, and Contact Drop.—The commutating e.m.f., E_c in Eqs. (1)

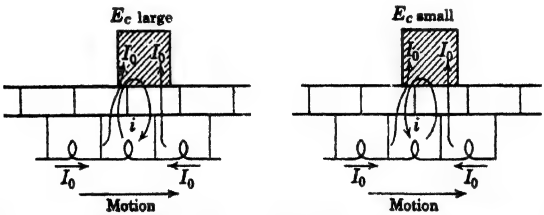


FIG. 4.—Direction of short-circuit current as affected by magnitude of commutating e.m.f.

and (2), is the net e.m.f. generated in the short-circuited coil while it is undergoing commutation. It is produced by rotation of the active sides of the winding element through the magnetic field in the vicinity of the neutral axis, and it is obvious that its magnitude varies from instant to instant during the commutation period. Its direction must be the same as the final direction of the current in the commutated coil, in order that it may sufficiently balance the reactance voltage to satisfy Eq. (3).

The commutating e.m.f. may be either greater or less than the reactance voltage. In the former case the current in the short-circuited coil will have one direction, in the latter case the opposite direction, as indicated in Fig. 4, in which it is also seen that if the reversing field is too strong the current density will be large at the leading tip of the brush. The total contact drop due to the circulating current in the short-circuited coil consists of two

parts iR_3 and iR_5 in Eq. (2), and the drop will tend to be greater at the segment where the current density is above normal than where it is below normal. Thus, in Fig. 5, suppose that one terminal of a low-reading voltmeter is connected to the brush holder and the other to the point of a lead pencil of medium hardness which can be touched successively to four or five equidistant points on the commutator between the heel and toe of the brush. With the machine running at normal speed and

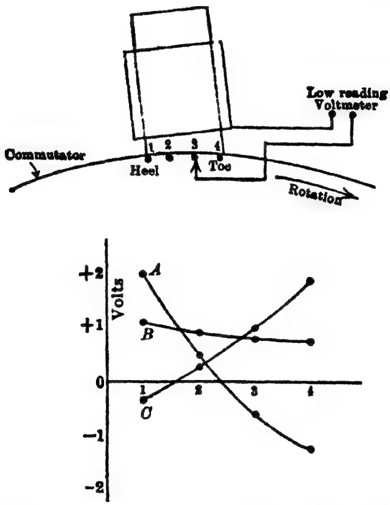


FIG. 5.—Measurement of brush contact drop.

with full load, the plotted voltmeter readings may give such typical curves as A, B, or C. Curve B indicates a nearly uniform drop all along the brush contact, a condition, as will be shown later, indicating nearly linear commutation. Curve A shows excessive drop and current density at the heel of the brush and reversed current at the toe, indicating a condition of overcommutation and a commutating field that is too strong. Curve C is typical of undercommutation, the current density being too great at the toe of the brush and the commutating field too weak.

The relation between contact drop and current density at the brush surface is shown in a general way in Fig. 6. The drop is affected by other variables than the current density, temperature

having a very marked effect. But with average carbon brushes under ordinary operating conditions the contact drop tends to remain nearly constant at about 1 volt at each contact; consequently, if the total contact drop at a brush, or $(iR_3 + iR_5)$, is appreciably more than 2 volts, the circulating current in the short-circuited element will be excessive, the loss of power at the brush contact will be unduly large, and both the brush and the commutator will be unduly heated, with results that may be injurious to both. The conclusion may therefore be drawn, in conformity with Eq. (3), that the difference between commutating e.m.f. and reactance voltage should not greatly exceed 2 volts if commutation is to be satisfactory.

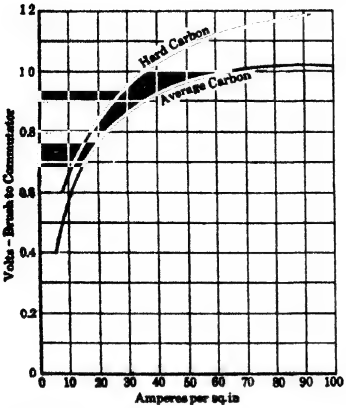


FIG. 6 Relation between brush contact drop and current density

In noninterpole machines the commutating e.m.f. is generated by rotation through the fringing field in the interpolar space, where the flux density from point to point is due to the resultant of the m.m.fs. of the armature winding and of the field excitation. If the effect of saturation of the pole tips and armature teeth is ignored, the commutating field may be thought of as the resultant of the individual fields produced by the field winding and by the armature winding acting independently. The curves of flux distribution due to the field and armature windings can be determined by the methods described in Chap. VIII. In the commutating zone they will have the forms indicated in Fig. 7, which represents the case of a shunt generator with a forward lead of the brushes. It will be seen that the strength of the commutat-

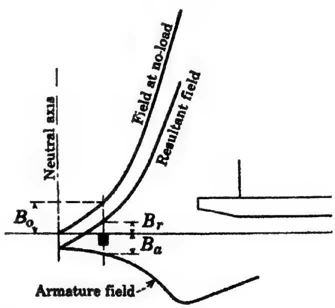


FIG. 7.—Distribution of commutating field.

ing field decreases with increasing load, even if the field excitation remains constant; this change is unfortunately in the wrong direction, inasmuch as the commutating e.m.f. should increase with increasing load and armature current. If the armature is magnetically too powerful, or if the brushes have insufficient lead, the commutating e.m.f. may even reverse under load, it being assumed as before that there are no commutating poles.

It follows, therefore, that in constant-speed generators and motors of the separately excited and shunt types, the commutating e.m.f. decreases gradually with increasing load. This fact is indicated in Fig. 8*a*. Since the difference between reactance

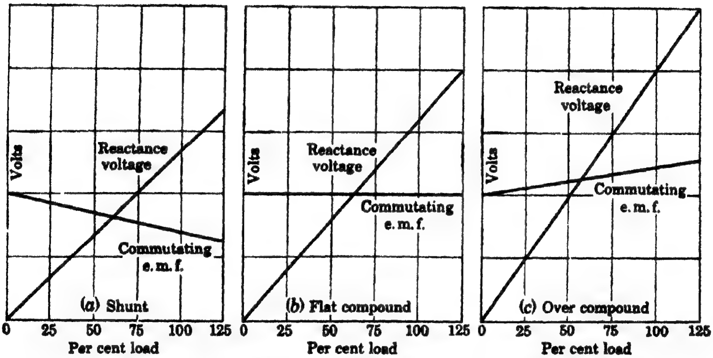


Fig. 8.—Unbalanced voltage in commutated coil.

voltage (which increases in direct proportion to the load) and the commutating e.m.f must be kept within a maximum limit of about 2 volts, the conclusion may be drawn that at maximum load (approximately 125 per cent of full load) the reactance voltage must be considerably less than double the 2-volt sparking limit. On the other hand, in over-compounded generators the commutating e.m.f. increases with increasing load, and so in such machines the reactance voltage at maximum load may be more than double the limiting value of contact drop. These considerations explain the superior commutating properties of over-compound and series machines as compared with shunt or separately excited machines of the noninterpole type.

4. Resistance and Voltage Commutation. Interpoles.—Prior to the general use of interpoles in all but small generators and

motors, successful operation necessarily depended upon the presence of a suitable fringing field in the interpolar space to provide the commutating e.m.f. That is, as may be seen in Fig. 7, the resultant distribution of flux density in the commutating zone must have a gradual slope, for if the slope is steep a very slight displacement of the brushes one way or the other will upset the balance between commutating e.m.f. and reactance voltage; such a machine would be entirely too sensitive to slight displacements of the brushes and to change of load. In many of the early machines the sensitivity to change of load was partly corrected by beveling and chamfering the pole tips to obtain a gradually fringing field and by making the field excitation "stiff" or powerful in comparison with the armature m.m.f., which of course added to the cost of construction.

Motors of the noninterpole type intended for reversible operation, as in railway and hoisting service, must have the brushes permanently set so that commutation takes place in the neutral axis. Under this condition the active conductors of the short-circuited coil cut through the cross field (B_a in Fig. 7) produced by the armature m.m.f., in the manner shown more in detail in Fig. 9. The commutating e.m.f. is $E_c = E_a$, the subscript a indicating that E_a is due to the armature flux; E_a is obviously in the wrong direction, for it tends to maintain the current in the original direction instead of helping to reverse it. In other words, this noncommutating e.m.f. E_a adds to the reactance voltage in Eq. (3) instead of neutralizing it, and the brush contact drop must then absorb the augmented reactance voltage. This type of commutation, in which the contact resistance of the brush is wholly relied upon, is called *resistance commutation.* It can be successful only if the reactance voltage is kept within the limit of contact drop attainable with hard carbon. In any case, resistance commutation imposes a decided limit upon the current that can be collected at a brush and so limits the maximum rating of machines of this type.

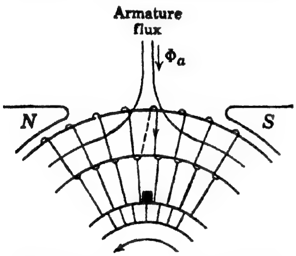


FIG. 9.—Production of armature flux (motor action).

Voltage commutation implies that the noncommutating e.m.f. E_a has been more than neutralized by means of a reversing field

(B_0 in Fig. 7) directed in opposition to the armature flux of Fig. 9. Such neutralization of the e.m.f. due to the armature cross field exists to varying degrees in shunt, series, and compound machines, as is evident from Fig. 8; but the difference between commutating e.m.f. and reactance voltage changes sign at about five-eighths of full load, at which point resistance commutation must be relied upon.

In the case of linear commutation the current in the commutated coil changes uniformly from $+I_0$ to $-I_0$ in the time T; counting time t from the instant commutation begins, we see from Fig. 2 that the current in the coil is

$$i = I_0 - \frac{2I_0}{T}t = I_0\left(1 - \frac{2t}{T}\right) \tag{4}$$

and thus the reactance voltage has the constant value

$$E_r = -(L + \Sigma M)\frac{di}{dt} = \frac{2I_0}{T}(L + \Sigma M) \tag{5}$$

From Eq. (2) it follows that when $t = 0$,

$$(E_c)_{t=0} + I_0(R_3 + R_5) = \frac{2I_0}{T}(L + \Sigma M)$$

and when $t = T$,

$$(E_c)_{t=T} - I_0(R_3 + R_5) = \frac{2I_0}{T}(L + \Sigma M)$$

whence

$$(E_c)_{t=0} = I_0\left[\frac{2(L + \Sigma M)}{T} - 2R\right] \tag{6}$$

$$(E_c)_{t=T} = I_0\left[\frac{2(L + \Sigma M)}{T} + 2R\right] \tag{7}$$

where $R = R_3 + R_5$.

Equations (6) and (7) show that the commutating e.m.f. should increase from the beginning to the end of the commutation period by $4I_0R$, the gradient of the commutating field being thus approximately determined; the average value of E_c should be $2I_0(L + \Sigma M)/T$, which is equal to the reactance voltage and proportional to the load current. The conclusion follows that, if the average commutating e.m.f. is equal and

opposite to the average reactance voltage and if both vary in direct proportion to the armature current, linear commutation will be closely approximated.

In order that the average commutating e.m.f. may be proportional to armature current, the flux in the axis of commutation must likewise be proportional to the armature current, and it must be directed in opposition to the flux Φ_a indicated in Fig. 9. If small auxiliary pole pieces, called *interpoles* or *commutating poles*, placed midway between the main poles, as in Fig. 10, are magnetized by the armature current, the resultant flux in the commutating axis can be made proportional to the load current except in so far as saturation of the interpoles causes a departure from proportionality. The interpole winding is usually directly in series with the armature, though in some cases it is shunted in the same manner as the series field winding of a compound-wound machine.

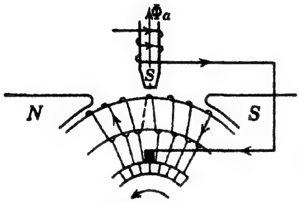


FIG. 10.—Relative polarity of interpole (motor)

The compelling reason for the general use of commutating poles is that present-day requirements overstep the limits which can be met by such expedients as high tooth saturation, saturated pole tips, and increased airgap. The use of higher speeds than were originally possible calls for armatures of limited diameter; and in order to obtain sufficient radiating surface to get rid of the heat developed in machines of large capacity, the axial length must be increased. These proportions necessarily involve a considerable increase in the armature m.m.f. relative to that of the field winding; and the inductance of the winding elements also increases with increasing length of core, which increases the reactance voltage, also. Hence, commutating poles are essential for satisfactory commutation when the design is pushed to the limit of effective utilization of material. Commutating poles make it possible to adjust the voltage of generators and the speed of motors between much wider limits than are attainable without them.

The presence of commutating poles increases the magnetic leakage of the main magnetic circuit because of the decreased length of the leakage paths adjacent to the main poles. The

coefficient of dispersion (ν) may reach values of 1.8 and higher. To reduce the leakage to a minimum, both the breadth and the length of the interpoles should be kept as small as possible, and the span of the main poles must be made smaller than in ordinary machines; the ratio of pole arc to pole pitch is usually between 0.60 and 0.65, instead of 0.70. The circumferential span of the interpole should be about equal to, or slightly larger than, the distance moved over by a slot while the coils in it are being commutated. The axial length of the interpole can be made less than that of the main poles, for it is immaterial in what portion of the short-circuited coil the commutating e.m.f. is generated; but if the interpole is shortened, the field intensity at its tip must be correspondingly greater than in an interpole of full length.

Since the commutating field should be proportional to the armature current, saturation of the commutating poles should be avoided, except beyond the limit of load to which the guarantee of satisfactory commutation applies. By tapering the tip of the interpole in the manner indicated in Fig. 10, it is possible to make a satisfactory compromise between the conflicting requirements of an unsaturated magnetic circuit and a fairly intense field in the commutating zone

In multipolar machines having a two-circuit armature winding and two brush sets, as in street-railway motors, only two interpoles are needed. For in this type of winding each brush short-circuits $p/2$ elements in series; and if the correct value of commutating e.m.f is generated in one element, it will be just as effective as smaller e.m.fs. in each of them. Railway motors of the four-pole type are usually built with two interpoles.

5. Linear Commutation.—Let it be supposed that commutation of the short-circuited coil of Fig. 3 takes place linearly and that the brush is in uniform contact with all the commutator segments within its span. These assumptions predicate ideal conditions which are not completely attained in actual machines, but it is nevertheless useful to examine what would happen if conditions were perfect.

The conditions with respect to the currents crossing the contacts at the several commutator segments are shown in detail in Fig. 11. Commutation in coil 1 begins at the moment when segment R_1 first touches the brush, and at that instant the current

in coil 1 is $+I_0$. Thereafter, the current i_1 falls off linearly, until finally it becomes $-I_0$; but at the instant represented in the diagram i_1 has the magnitude shown by the ordinate of the short-circuit current curve. At the same instant the current in coil 3, the short circuit of which began earlier than that of coil 1, has fallen to i_3; and the current in coil 5 is i_5 (which, being negative, indicates that i_5 has already reversed).

In the upper part of Fig. 11 the riser currents $(I_0 - i_1)$, $(i_1 - i_3)$, . . . are shown to scale. It is seen that the current

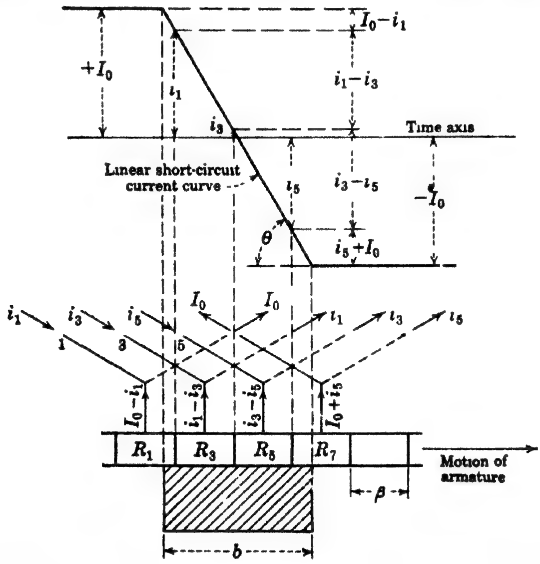


FIG. 11.—Brush currents, linear commutation.

density at segment R_1 is proportional to $(I_0 - i_1)$ divided by the width of the segment covered by the brush; according to the diagram this ratio is equal to tan θ, which is constant. Similarly, the diagram shows that the current density has the same value at all the segments which are simultaneously in contact with the brush.

Under the assumed ideal conditions, linear commutation is therefore characterized by uniform current density under the entire brush surface, and this in turn means that the contact drop is the same at all points of the brush: in that case the curve B in Fig. 5 would be a horizontal line.

The ohmic loss of power at a contact is smaller, and the heating is therefore less, when the current density is uniform than with any other possible distribution of current. Consider a conductor in which there is flowing a current that is uniformly distributed over the entire cross-section. Let a small amount of current Δi be removed from an area a in which the current was originally i amp., so that the resultant current is $(i - \Delta i)$; and let Δi be transferred to another equal area a in which the current then becomes $(i + \Delta i)$. The ohmic loss at these two areas is

$$(i - \Delta i)^2 r + (i + \Delta i)^2 r = 2[i^2 + (\Delta i)^2]r$$

which is greater than the original loss, $2i^2r$, at these two areas. At all other parts of the cross-section the loss remains unaltered, and so the effect of this deviation from strict uniformity of current distribution, however small it may be, is to increase the loss.

6. Simultaneous Commutation of Adjacent Coils.—Inasmuch as the process of commutation in a coil is affected by the mutual induction of neighboring short-circuited coils, it is important to be able to predetermine the number and relative positions of those coils in the same neutral zone which are simultaneously short-circuited. In a simple ring winding where b is the brush width and β the width of a commutator segment, the ratio b/β fixes the number of coils short-circuited at the same time, subject to the assumption that the whole of the brush surface is in contact with the commutator. This ratio is generally a mixed number, and the actual number of coils short-circuited will vary alternately between the two integers lying on either side of it. In lap and wave windings, however, the conditions are as a rule not so simple, since in a given neutral zone some of the conductors are short-circuited by a brush of one polarity and others by a neighboring brush of opposite polarity, as illustrated in Fig. 12. The diagram represents a duplex lap winding having the following constants:

$$Z = 122 \qquad S = 61 \qquad p = 6 \qquad a = 12$$
$$y = m = a/p = 2 \qquad y_1 = 23 \qquad y_2 = -19 \qquad b/\beta = 2.5$$

It is clear from the figure that, in the position shown, conductors 1 and 4 are simultaneously short-circuited; a moment earlier

conductors 1, 3, and 4 were short-circuited. The successive combinations of short-circuited coils can be conveniently studied by means of the following graphical method, due to Professor Arnold.*

1. *Lap Windings.*—It will be observed that in the winding here selected brushes B_1 and B_2 are not identically situated with respect to the segments of the commutator in contact with them. This is a consequence of the fact that S/p is not an integer.

Coil sides 1, 3, 5, drawn in full lines to indicate that they occupy the tops of the slots, are connected to commutator segments which are correspondingly numbered in the top row of

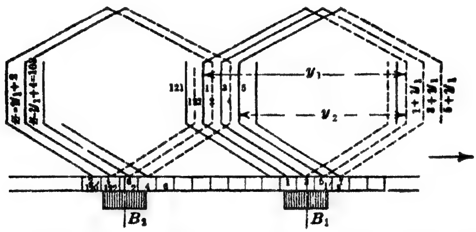


FIG. 12.—Simultaneous short circuit of elements of lap winding.

figures. The other sides of the same coils, the numbers of which are $1 + y_1$, $3 + y_1$, $5 + y_1$, are connected to segments which are numbered 1′, 3′, 5′ (that is, by dropping the term y_1 and priming the numeral) in the bottom row of figures. A coil will be short-circuited when the brush B_1 is in contact with any two segments that bear the same numbers. A similar arrangement is indicated in the case of coil sides 2, 4, 6 . . .

Now, coil side 2 is connected to one on the left that is separated by a pitch y_1 from 2 and by $y_1 - 1$ from 1. Segment 2 is therefore separated from segment 1 by $\frac{1}{2}(y_1 - 1)$ segments. But brushes B_2 and B_1 are separated by S/p segments; hence, the relative shift of segments in the vicinity of B_2 with respect to those at B_1 is

$$\Delta = \frac{S}{p} - \frac{1}{2}(y_1 - 1) \qquad (8)$$

and is toward the left when Δ is negative and toward the right when Δ is positive. In the case considered in Fig. 12, $\Delta = -5/6$.

* "Die Gleichstrommaschine," 2d ed., Vol I, p. 354.

The simultaneous action of the two brushes can now be studied by means of a diagram like Fig. 13. Take a strip of paper cut to the width of the hatched area to represent the brush, and slide it between the two commutators; when it touches segments similarly numbered, the corresponding coils will be simultaneously short-circuited.

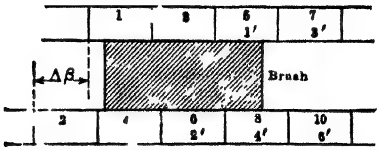


FIG. 13.—Diagram showing coils simultaneously short-circuited—lap winding

2. *Wave Windings.*—Figure 14 represents a portion of a duplex wave winding having the following constants:

$$Z = 122 \qquad S = 61 \qquad p = 6 \qquad a = 4$$
$$y = y_1 = y_2 = 21 \qquad b/\beta = 2.5$$

In the position shown, coil sides 122, 1, 2, 3, and 4 are short circuited.

Number the segments connected to coil sides 1, 3, 5, . . . with corresponding numbers, and the segments connected to the other sides of the same coils 1′, 3′, 5′, . . . ; similarly with respect to the other coils occupying the same neutral zone, as 2, 4, 6, . . .

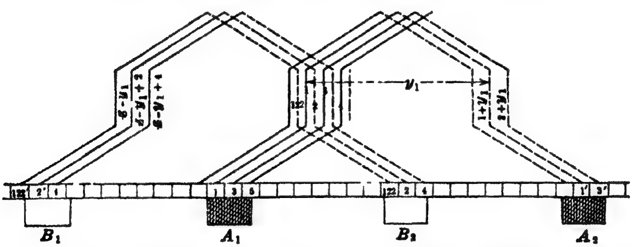


FIG. 14.—Simultaneous short-circuiting of elements of wave winding.

Brushes A_1 and A_2, which are of the same polarity, are not similarly placed with respect to the segments in contact with them. The brushes are separated by $2S/p$ segments, and the ends of an element are separated by $y = 2S/p \pm m$ segments, where $m = a/p$; segments 1′, 3′, 5′ are therefore shifted with respect to A_2 by an amount $m\beta$, as compared with the relative positions of segments 1, 3, 5 with respect to brush A_1.

The shift is to the right if m is positive (as in the case illustrated), to the left if m is negative. The short-circuiting of these elements can then be shown by drawing two commutators one above the other, as in the upper part of Fig. 15.

Obviously, brushes B_1 and B_2 and the segments in contact with them are related to each other in the same way as are A_1 and A_2. Now B_1 is separated from A_1 by S/p segments, and segment 2′ is separated from segment 1 by $\frac{1}{2}(y_1 - 1)$ segments. The displacement of segments 2′, 4′, 6′, with respect to B_1, as compared with that of segments 1, 3, 5 relative to A_1, is

$$\Delta = \frac{S}{p} - \frac{1}{2}(y_1 - 1)$$

and is to the right if Δ is positive (in the winding considered $\Delta = +\frac{1}{6}$), to the left if it is negative. The complete relations are shown in Fig. 15.

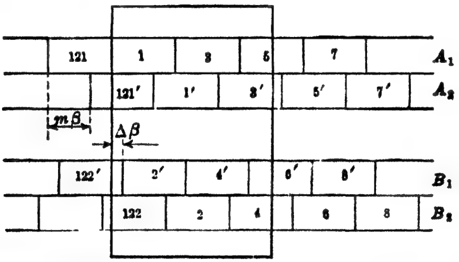


FIG. 15. Diagram showing coils simultaneously short-circuited—wave winding.

If a strip of paper of which the width is equal to that of the brush is moved across the fictitious commutators represented by A_1, A_2, B_1, and B_2, it will touch a series of similarly numbered segments and the corresponding coils will be simultaneously short-circuited.

7. Successive Phases of Short Circuit in Coils of a Slot.—The method described above may be used to investigate the order in which the coils occupying a given slot undergo commutation. Two distinct cases may be distinguished:

1. Coil sides lying in the same radial plane (one above the other) enter and leave short circuit simultaneously.

2. Coil sides lying in the same radial plane enter and leave short circuit at different times.

Case 1.—If the coil sides are numbered in accordance with the system described in Art. 13, Chap. VII, and illustrated in Fig. 16 of that chapter, coil sides 1 and 2 of a two-layer winding will occupy the same radial plane, and so also will 3 and 4, 5 and 6, etc. Reference to Figs. 13 and 15 shows, therefore, that if coil sides 1 and 2, 3 and 4, or, in general, any two in the same radial plane, are to enter and leave short circuit simultaneously, there must be no displacement between the correspondingly numbered commutator segments; the condition to be satisfied is that

$$\Delta = \frac{S}{p} - \frac{1}{2}(y_1 - 1) = 0$$

or

$$y_1 = \frac{2S}{p} + 1 \tag{9}$$

For example, consider the case of a simplex lap winding having six coil sides per slot, a brush width of $2\frac{1}{2}$ segments, and $\Delta = 0$. With the help of a diagram like Fig. 13, but with Δ made equal

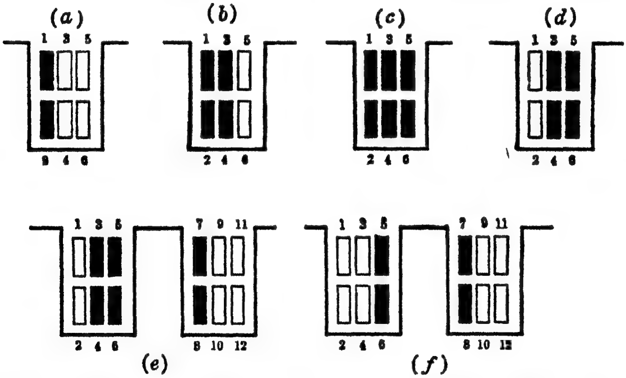


Fig. 16.—Successive phases of short circuit in adjacent coils, $\Delta = 0$.

to zero, it is readily shown that the successive phases of the short-circuiting of neighboring coils will follow the order shown in parts (*a*), (*b*), (*c*), . . . of Fig. 16, where the shaded coils indicate short-circuit conditions. During a brief interval the condition shown in diagram (*c*) will exist; that is, all the coil sides in a slot will be simultaneously short-circuited. A little later,

as in diagram (*e*), six coil sides are again short-circuited, but four are in one slot and two in the next slot.

A study of Fig. 16 shows that when coil sides 1 and 2 leave short circuit they are subject to the effect of mutual induction from the simultaneously short-circuited coils 3, 4, 5, and 6, all of which occupy the same slot. When coils 3 and 4 leave short circuit, they are subject to the mutual induction of coils 5 and 6, which are in the same slot, and of coils 7 and 8, which are in the next slot; because of this separation of the short-circuited group of coils, the inductive effect upon coils 3 and 4 will be smaller than upon coils 1 and 2. Similarly, when coils 5 and 6 leave short circuit, they are acted upon by the mutual induction due to the simultaneously short-circuited coils 7, 8, 9, and 10, all of which are in the slot adjacent to that occupied by 5 and 6; hence, the inductive effect upon these two coils is still less than upon coils 3 and 4. The commutating conditions are, therefore, not the same in all the winding elements, and their short-circuit current curves will have different forms.

An additional disturbing feature arises from the fact that when the successive coils of a slot, as 1–2, 3–4, 5–6, of Fig. 16, break contact with the brush, they are not identically situated with respect to the adjacent pole tip; consequently, the e.m.fs. generated in each of them during the final stage of the short circuit (by cutting through the fringing field) will be different. This difference is due to the fact that the successive commutator segments are evenly spaced, whereas the coils, being grouped in slots, are not. Thus, in Fig. 16, coils 1 and 2 are ahead of 3 and 4, etc., with respect to the direction of rotation, and their short circuit terminates when they are in a weaker field than that which acts upon 3 and 4 when the latter leave short circuit. Similarly, coils 5 and 6 leave short circuit when they are subjected to the action of a still stronger field than that which acts upon 3 and 4. If, therefore, the commutating e.m.f. acting upon coils 1 and 2 is just sufficient to overcome the e.m.f. of self- and mutual induction therein, it will be more than sufficient to balance the smaller inductive e.m.f. in 3 and 4, and considerably too great in coils 5 and 6. In the latter coils, there will be a condition of overcommutation, and under these circumstances every third commutator segment may become blackened because of the possible excessive current density. In order that there may be no marked

difference between the field intensities acting upon the various coils of a slot while they are undergoing commutation, the angle subtended by a slot should be small. For this reason the number of slots per pole should not be less than 12, and preferably greater than 12, and the angle subtended between the edges of a brush should not exceed one-twelfth of the angle from center to center of the poles.*

The order of commutation illustrated in Fig. 16 can occur only in full-pitch windings, since it is in such windings that the back pitch y_1 is made nearly equal to $2S/p$. Chord windings (fractional pitch) are, therefore, characterized by the condition $\Delta \lessgtr 0$.

Case 2.—It follows from the foregoing analysis that the second cases arises when $\Delta \lessgtr 0$. An interesting variation of this case occurs when $\Delta = 1$, that is, when

$$y_1 = \frac{2S}{p} - 1 \tag{10}$$

Thus, if a winding has pitches that satisfy Eq. (10) and is arranged so that each slot contains six coil sides, the brush

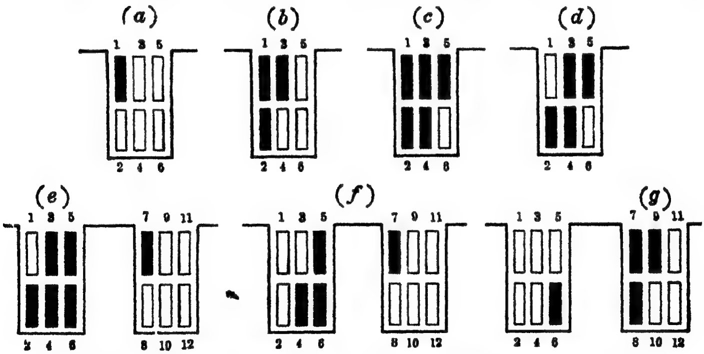


Fig. 17.—Successive phases of short circuit in adjacent coils, $\Delta = 1$.

width being $2\frac{1}{2}$ commutator segments, the order of commutation of adjacent coils will be as shown in Fig. 17. In this particular case, pairs of coils like 2 and 3, 4 and 5 enter and leave short circuit simultaneously.

8. Selective Commutation in Wave Windings.—A study of the simplex wave winding shown in Fig. 10, page 327, shows that

* A. Gray, "Electrical Machine Design."

the several brushes of one polarity are connected to each other not only by an external conductor but also through the winding by way of the coils that they short circuit. The figure also shows that the resistances of these internal paths are not equal because of the varying areas of brush contact; further, the short-circuited coils are not at any instant identically located with respect to the fringing fields through which they are moving, and hence the e.m.fs generated in them by rotation through the field, though small, are not the same in any two of them. Both these facts are responsible for an unequal division of the total armature current between the several brushes. The unequal components of the total current shift from brush to brush in cyclical order, in such a way that Kirchhoff's laws are continuously satisfied. This shifting of the current values at the brushes in the case of wave windings is called "selective commutation."

9. Theoretical Duration of Short Circuit.—In the case of a simple ring winding the theoretical duration of short circuit is simply the time required for a given point on the commutator to move through an arc equal to the width of a brush. But it will be seen from Figs. 13 and 15 that this simple relation does not hold in lap and wave windings, since a coil is short-circuited only when similarly numbered segments in those diagrams are simultaneously touched by the brush. These segments being displaced with respect to each other, the time of short circuit may be either greater or less than in a ring winding, on the assumption that the whole of the brush surface is in contact with the commutator.

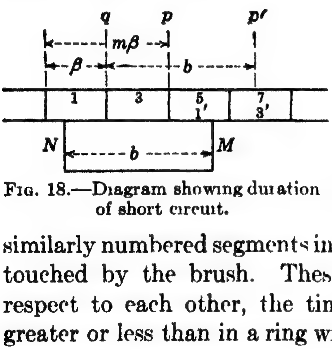


FIG. 18.—Diagram showing duration of short circuit.

Consider the case of a multiplex lap winding (Fig. 18) (drawn to represent a duplex winding); the distance between corresponding edges of similarly numbered segments is $m\beta = (a/p)\beta$, where β is the width of a segment. Short circuit of coil 1 endures while the edge M of the brush moves from position p until edge N reaches q. When N is at q, M will be at p'; hence, the short circuit exists over the distance from p to p', which equals

$$b - (m\beta - \beta) = b + \beta\left(1 - \frac{a}{p}\right)$$

The time of short circuit is, therefore,

$$T = \frac{b + \beta\left(1 - \dfrac{a}{p}\right)}{v_c} \tag{11}$$

where v_c is the peripheral velocity of the commutator.

An exactly similar result is obtained in the case of wave windings where the number of brush sets equals the number of poles. If one or more pairs of brush sets are omitted, the necessary correction can be applied by remembering that $m\beta$, Fig. 15, is the displacement corresponding to the distance between a given brush and the next brush of the same polarity. Therefore, if some of the brushes are removed, the term $m\beta = (a/p)\beta$ in the foregoing equation for T must be multiplied by the number of double pole pitches in the region from which the brushes have been omitted.

In simplex lap windings, $a/p = 1$, and hence $T = b/v_c$, or the same as in a ring winding. T is less than this in multiplex lap windings. In wave windings, on the other hand, T is greater than in a ring winding, other things being equal, since $a/p < 1$.

The assumption that the duration of short circuit is determined by the circumferential width of the brush gives results that are too large. For reasons described in following articles the actual width of contact is less than that of the brush, and the short-circuit period is therefore reduced. But if the current density at the brush is high, as under overload conditions, arcing at the brush edge may prolong the time of effective contact.

10. Mechanical Characteristics of Brush Contact.*—The permanent preservation of uniform contact between the commutator and the running surface of the brushes is impossible for both mechanical and electrical reasons. When new machines are made ready for service, and periodically thereafter, a running fit between the commutator and brushes is prepared by inserting a strip of sandpaper between the brush and the commutator and drawing it by hand in the direction of rotation, the paper being kept tightly stretched to fit the curvature of the commutator; on the back stroke the brush must be lifted so that cutting will

* "Carbon Brushes," by Dr. J. Neukirchen (translated by E. I. Shobert II), published by the Stackpole Carbon Company, has been used as source material for Arts. 10 to 12.

not occur. Sometimes the whole surface of a large commutator is tightly covered with thin sandpaper, the joints between successive sheets being made by beveling or tapering the overlapping portions so that the thickness of the sandpaper is uniform *all around the periphery;* the brushes are then placed in position, and the armature is driven mechanically at slow speed until the brushes are ground to the proper curvature. Thereafter the machine is operated under moderate load to allow the brushes to be "run in" under working conditions. Emery should never be used for the sanding-in operation because its fine conducting particles would lodge between the commutator segments and short-circuit elements of the armature winding.

The degree of mechanical perfection essential to complete and permanent contact of the entire surface of a brush is unattainable. The brush holder must be made with sufficient clearance to permit the brush to slide freely in the direction of its own length, so that there is the possibility of small but appreciable to-and-fro rotation of the brush about its longitudinal axis; the axis of the brush may also tip in planes passing through the commutator shaft and perpendicular thereto. In addition to these movements, the brush may move longitudinally in and out of its holder because of slight eccentricity of the commutator, or because of local irregularities—such as high or low spots—on the commutator surface; a commutator that is truly concentric when stationary may become slightly eccentric under load owing to unequal expansion of its parts. It is also possible that there may be small angles between the axis of rotation and the axes of the commutator spider and of the brush-holder ring, which will initiate some or all of these components of brush motion. The longitudinal motion of the brushes is restrained by the springs that hold the brush against the commutator; but unless auxiliary damping springs or pads are present, the chatter of the brushes will be greater the more nearly their natural period of vibration approaches that of the impressed disturbances. Chatter will of course occur if the brush becomes so tilted that its leading edge dips into the slots of an undercut commutator.

Experience has shown* that a slight tilt of the brush, in the manner indicated in Fig. 19, will produce either a partial vacuum

* W. E. STINE, Brush Friction Greatly Affected by Contact Air Pressure, *Elec. World,* **88** (No. 2), 67, 1926.

or an increase of pressure under the brush, depending upon the direction of tilt with respect to the direction of rotation. In one case the vacuum accounted for an increased brush pressure of 0.72 lb. per sq. in. of total brush surface, the friction loss being thus considerably increased.

Another factor that limits the contact to a portion of a wide brush is the plastic deformation of both the carbon of the brush and the copper of the segments when they are pressed together. Hertz derived the following formula for the circumferential

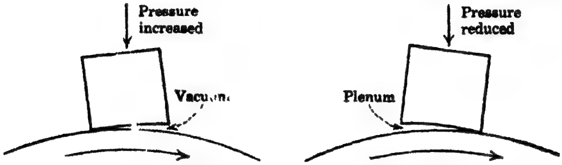


Fig. 19.—Effect of tilt upon brush pressure.

width b of the contact between a solid cylinder, of radius r_1 in., and an external hollow cylinder, of radius r_2 in., both l in. long in the axial direction, when pressed together by a total force of P lb., the moduli of elasticity being respectively M_1 and M_2:

$$\left(\frac{b}{4}\right)^2 = \frac{0.29P}{l}\,\frac{(1/M_1) + (1/M_2)}{(1/r_1) - (1/r_2)} \tag{12}$$

Since r_1 and r_2 are nearly the same, this can be written in the approximate form

$$b = 4r_1\sqrt{\frac{0.29P\left(\dfrac{1}{M_1} + \dfrac{1}{M_2}\right)}{l(r_2 - r_1)}} \tag{13}$$

and, on substituting $M_1 = 18{,}500{,}000$ lb. per sq. in. for copper and $M_2 = 1{,}430{,}000$ for carbon, it will be found that if $(r_2 - r_1)$ is as small as 0.001 in. the contact arc will be limited to about 1 in. Larger area of contact is occasionally attempted by slitting the brushes, as indicated in Fig. 20.

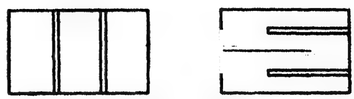


Fig. 20.—Contact surface of slit brushes.

These considerations are concerned with the macroscopic features of the contact between commutator and brushes. Microscopically, the actual contact takes place between minute

grains or crystals of the two materials where they penetrate through the thin layer of air (or other gases), moisture, dust, and oxide which tend to keep the true conducting surfaces apart. The actual current density at these minute contacts must be very high in comparison with the average current density computed on the basis of uniform contact over the entire brush surface, so that this average current density is really only a polite fiction. The heat developed at the actual contact points may therefore raise the particles to red or white heat, or even vaporize them, or, in the case of carbon particles, may cause them to explode by the sudden liberation of occluded gas. In this way there may be created between parts of the brush and the commutator a thin layer of ionized gas, which gives to the transition layer the properties of gaseous conduction; the particles of carbon dust thus broken off remain in part under the brush, in part adhering to the commutator surface, but in both cases the effect is to increase the contact resistance. It is therefore easy to see how bits of both the copper and the carbon may be torn away by a combination of mechanical tearing and electrolytic conduction, accounting for the wear that actually occurs under load conditions.

11. Types of Sparking at the Carbon-copper Contact.—When commutation takes place satisfactorily, the brushes ride quietly without appreciable sparking and the contact surfaces of both the brushes and the commutator take on a smooth polish. Cleaning of the commutator by means of a slightly oily cloth is, however, necessary from time to time to remove dirt and dust, including particles of carbon dust that come from the brush; for without such periodic cleaning there accumulate on the commutator high-resistance films of oil, carbon dust, and oxides and other copper salts that hinder commutation and eventually lead to sparking which begins gradually and, unless the cause is removed, grows progressively worse. Chattering of the brushes caused by dirt particles or by tipping of the brushes also produces sparking.

The type of sparking initiated by high-resistance films is called *pin fire*, which occurs close to the edge of the brush. The visible light, of bluish tint, is due to small particles heated to incandescence, or to small paths of illuminated gas, but is not the same as true arcing. Pin fire may occur at either the leading

or the trailing edge of the brush but is more common and more injurious at the trailing edge where it is due to retarded commutation. It always begins at the brush that is the *anode* (Fig. 21*b*).

At the other extreme from pin fire, which is the mildest form of commutation sparking, is the destructive *arcing* which occurs when there is complete rupture of the carbon-copper contact, so that the current must flow through a greater or less length of highly heated gas. The *anode* and *cathode* brushes (Fig. 21) exhibit marked differences in their arcing properties; for though the arrows in Fig. 21 indicate the conventional direction of current flow, the transfer of electrons takes place in the opposite direction. The high-temperature *cathode spot* from which the electrons

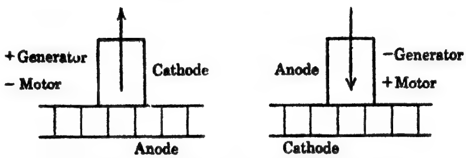


FIG. 21.- Brush as cathode and anode.

issue is therefore on the copper in the case of the anode brush and on the carbon in the case of the cathode brush. The result is that a greenish arc (copper vapor) at the trailing edge of the anode brush is drawn out by the motion of the commutator, being maintained so long as the ionized gas dragged along by the commutator remains sufficiently conducting. Copper dust will then be deposited on the brush. This arcing will be accentuated if the machine is part of a highly inductive circuit. At the cathode brush, however, the hot cathode spot is on the running surface of the brush itself, and so the arc is short and is visible only close to or under the trailing edge of the brush. The effect of such an arc is to create a burned or pitted strip on the brush surface, in general parallel to the commutator segments, and thus carbon dust is deposited on the commutator.

Next in severity to pin fire is *spit fire*, which is due to the explosive action of gases suddenly released from relatively large particles by high concentration of current at the points of contact. The heated particles are generally expelled from under the brush at the trailing edge, but sometimes from the sides and the leading edge.

Streamers are bands of reddish light that are drawn out from under the trailing edge of the cathode brush if a high-temperature cathode spot forms on the brush running surface. The streamers consist of highly heated particles of carbon which are drawn out by the layer of air that clings to the commutator.

Ring fire, not directly related to commutator sparking, is a symptom of a dirty commutator. Particles of dirt or dust, frequently carbon dust broken away from the brushes, become impregnated with oil and cling to the mica insulation between segments or lodge in the slots of undercut commutators. These particles may be heated to incandescence on passing under the brushes and so become sufficiently conducting to permit the flow of considerable leakage current between the segments thus bridged together; this current maintains the heat, and the incandescent spots, whirled around by the commutator, appear to the eye as continuous rings. The trouble is easily remedied by cleaning the commutator; if it is neglected, it may lead to a *flash-over,* which means complete short-circuiting of the machine from brush to brush, around the commutator.

12. Electrical Characteristics of Brushes and the Contact Surface.—Carbon brushes are made in a large number of grades, ranging from hard retort carbon having abrasive properties, to very soft graphite possessing lubricating qualities. The abrasive quality of hard carbon is due, in part, to mineral particles in the coke ash, such as quartz, mica, and iron oxide. In the manufacture of brushes, finely ground carbon flours of several degrees of hardness are blended in the desired proportions, organic binding material is added, and the molded mixture is baked to convert the binder into coke. Subsequent electric heat-treatment changes the components to electrographitized forms the hardness of which depends upon the temperature used.

Metal-graphite brushes, which consist usually of mixtures of copper dust and graphite, are available for cases where low contact resistance is required.

The relatively small part of the brush area (called the Hertz surface) in which there is intimate contact between the copper of the commutator and conducting projections of the granular brush may be anywhere on the brush surface, depending upon the position of the brush in its holder; adjoining this area are regions

in which the contact is loosely bridged by carbon or other dust particles clinging to the commutator; and beyond are regions in which the brush and commutator are separated by a thin layer of air or other gas and across which current can pass only as an arc. Radial movements of the brush may so reduce the Hertz surface that conduction can take place only through the dust zone or entirely through an arcing zone.

A layer of moisture exists normally on all relatively cool surfaces, but particularly on the porous brushes and the slightly hygroscopic copper oxide of the commutator; it becomes thicker as the relative humidity of the air increases. This layer must be traversed by the current when there is any loosening of the brush contact caused by brush movement, provided that it has not previously been evaporated by arcing or other heating effects. Electrolysis of the water then takes place, oxygen appearing at the anode and hydrogen at the cathode. Thus, in Fig. **21*b*** oxygen is liberated at the carbon surface, where it attacks and erodes the brush, forming carbon dioxide and oxidizing the metal in metal-graphite brushes; and the hydrogen liberated at the commutator (cathode) surface reduces the oxide layer to metallic copper. At the cathode brush (Fig. 21*a*) oxide of copper is formed by the action of the liberated oxygen at the metallic surface of the commutator, and in metal-graphite brushes any surface oxide is reduced to metallic form. The reducing action under the anode brush and the oxidizing action under the cathode brush become evident by the different colors that appear on the commutator if the brushes of opposite polarity are staggered so that each runs on a separate path.

Under normal conditions of operation the local heating at actual contact points may raise the temperature to red heat, especially at the carbon surface because of its low heat conductivity. This high temperature causes the emission of positively charged ions of occluded gas, and even of positive atoms of the body itself if the temperature approaches its vaporizing point. These positively charged ions can move only in the direction of the main current, and the velocity they can acquire is determined by the potential gradient in the transition layer. If the acquired velocity is sufficiently high, the impact energy will release electrons from the target, and a cathode spot and arcing will result. Arcing, when once established, tends to

become a self-supporting reaction; for the electrons produce additional ionization by collision with gas molecules, and these ions, falling upon the cathode, liberate more electrons. Clearly, though, this cannot occur unless the positive ions accelerate through a sufficient distance to attain the critical velocity—which implies a definite interruption of the physical contact between brush and commutator. In any case, the movement of the positive ions with the current indicates that there are characteristic differences between the cathode and anode brushes, some of which have already been mentioned.

At the cathode brush (Fig. 21*a*), partial loosening of the original contact will cause diffused burning (oxidation) of the brush surface by incipient arcs; but the burned spots will be removed by mechanical wear of the remaining surface if the brush is not too hard, and so the surface remains smooth. New particles of the harder constituents that make the principal contact are then exposed and are in turn burned away. It is for this reason that the voltage drop at the cathode brush is not appreciably affected by changes in brush pressure. However, if the contact is impaired by commutator roughness or brush movements so that arcing takes place, the resultant burned strips on the brush surface are characteristic of the cathode brush alone. This is particularly noticeable on the positive brushes of high-speed turbo-generators that supply the field excitation of large machines and also on the negative brushes of high-speed series-wound motors; in both cases these brushes are cathodes (see Fig. 21*a*), and the high inductance of the circuit in series with the armature may give rise to high brush voltages under arcing conditions.

At the anode brush (Fig. 21*b*), the permissible values of average current density are materially affected by the hardness of the brush. With soft brushes there are relatively few hard particles to serve as conducting material, and the localized current is thus very high; loosening of these conducting bridges because of brush motion will therefore quickly lead to incipient arcing, electrolysis, and high rate of brush wear. Medium-hard brushes contain more nonconducting mineral-ash particles which do not burn away and which by their abrasive action help to maintain the contact. Hard brushes contain enough hard conducting particles to carry the current unless they are torn out by brush movement or are burned away by excessive current density

In general, therefore, the higher the current density, the harder the brush must be; otherwise stated, the higher the load, the more the brushes vibrate, and the more graphite there is in the brush composition, the greater the tendency of the anodic brush surface to become too graphitic.

The most important effect at the anode brush is the transfer of graphite to the commutator surface, which occurs the more readily the softer the brush. This effect increases the resistance of the transition layer and promotes incipient arcing, which changes the crystalline graphite into a black, sooty form that sticks tightly to the commutator. The voltage drop at the brush then increases rapidly with slight reduction of pressure.

The erratic movements of a brush which change the nature and the resistance of the contact from time to time are responsible for the fact that the several parallel-connected brushes on a single brush stud frequently divide the total current unequally. This condition is called *selective action* of the brushes. It is probably started if any one brush has more hard conducting particles making contact with the commutator than the others have; thereafter the negative temperature coefficient of the carbon brushes accentuates the inequality of current distribution. The remedy is to clean the commutator, increase the brush pressure, and prevent the formation of a too smooth graphitic surface on the brush

13. Arcing and Nonarcing Relations.—The discussion in Art. 12 indicates that arcing, once initiated, tends to maintain itself. This is in accord with the experimental fact that the voltage across the arc decreases in an inverse manner with increasing current, as shown in Fig. 22. There is, however, a minimum voltage below which an arc cannot be maintained no matter how large the interrupted current may be. With voltages higher than this minimum, such as OV, arcing can occur only if the current equals or exceeds the abscissa VP. In other words, the hyperbolic curve divides the entire region into two parts, arcing being possible for the voltage-current relations above

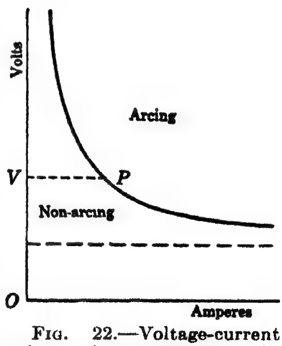


FIG. 22.—Voltage-current relation for nonarcing condition.

it and impossible for those below it. There is a definite curve of this kind for each set of two electrodes, one of which is the cathode, the other the anode. For example, when both electrodes (in rod form) are carbon, the minimum arcing voltage is about 25 volts; with a carbon-copper pair of rods, the minimum is 16 volts when carbon is the cathode and 25 volts when copper is the cathode. In general, when the electrodes are made of different materials, the minimum arcing voltage will be lower if the cathode has the lower heat conductivity; for heat is then less readily dissipated from the cathode, and the temperature will more quickly rise to the point at which electrons are emitted.

In the case of the carbon-copper pair represented by the brushes and commutator of a dynamo, the larger heat capacity and better cooling of the moving commutator introduce an asymmetry, so that it is possible to interrupt larger currents and voltages without arcing than the above figures would indicate; measurements* have shown voltages across the brush contact of 20 volts and, in small universal motors, of more than 100 volts.

14. Reaction of Short-circuit Current upon Main Field.—Let *a* and *b*, Fig. 23, represent the initial and final positions of a coil undergoing commutation in a generator. In the *a* position the coil exerts a demagnetizing m m f. upon the main magnetic circuit, whereas in the *b* position the action is a magnetizing one. If commutation takes place, on the average, in the geometrical neutral and if the short-circuit current curve is symmetrical with respect to the point *O*, Fig. 2, the demagnetizing and magnetizing effects annul each other. But if the mid-point of the commutation period occurs when the coil is in the geometrical neutral, and the short-circuit current curve is not symmetrical, one or the other of the two effects will preponderate. Thus, in the case of overcommutation, the current will have the direction of the *b* position during the greater part of the time of short circuit, and hence there will be a resultant magnetizing action; in case of undercommutation the

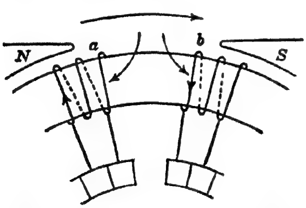


Fig. 23—Magnetizing action of short-circuited elements (generator).

* F. Punga and A Schliephake, *Z E u.M.*, **45**, No. 11, 201, 1927.

resultant action will be demagnetizing. Obviously, these statements are to be reversed in the case of a motor.

When a generator is running without load, currents will be set up in the coils short-circuited by the brushes. The direction of the current flow will depend upon the direction of the field in which the coils are moving and therefore upon the direction of displacement of the brushes. With a forward lead of the brushes the current in the short-circuited coils will have the direction shown in the *b* position (Fig. 23) and will then exert a magnetizing effect; a backward lead of the brushes will result in a demagnetizing action. The magnitude of these no-load short-circuit currents may be sufficiently great to materially influence the field flux, and hence also the experimentally determined open-circuit characteristic, it is for this reason that the experimental determination of the open-circuit characteristic should be made at reduced speed, as mentioned in Art. 3, Chap. VIII.

If a generator is suddenly short-circuited at its main terminals when running with full excitation, the current in the coils undergoing commutation may rise to enormous values, and the inductance of the coils will tend to delay the reversal of this current, thus producing a powerful demagnetizing effect. This effect is particularly prominent in commutating-pole machines. The natural effect of this demagnetizing action is to weaken the main flux of the machine and therefore relieve the severity of the short circuit; but the main flux of the machine represents a very considerable amount of stored energy, and this energy cannot instantly be changed. Consequently the main flux decreases very slowly in spite of the large demagnetizing action of the coils undergoing commutation, and it follows that this demagnetizing action must be neutralized by a correspondingly large increased current in the main field winding. Looked at in another way, the coils short-circuited by the brushes are inductively related to the field winding in much the same manner as are the primary and secondary windings of a transformer or induction coil. The sudden increase of current in the coils under the brushes caused by the principal short circuit induces a reflected current impulse in the field coils. Since the resistance of the field winding has not been changed, the sudden increase in the exciting current results in a marked rise of voltage at the terminals of the field winding.

15. Flashing at the Commutator.—If the voltage between adjacent segments of the commutator is too high, an arc may be established between them. When this condition exists from segment to segment between brushes of opposite polarity, it is equivalent to a short circuit of the entire machine and is called a *flash-over*. Flashing may occasionally be caused by a dirty commutator but is due chiefly to electromagnetic conditions in the machine itself.

The increase of armature current due to a sudden short circuit or to a transient overload will be accompanied by a nearly proportional increase in the cross flux set up by the armature current (Fig. 24). In the process of building up, this flux reacts inductively upon the armature winding, with which it links, and induces in the winding elements e.m.fs. proportional to the time rate of change of the current. The more sudden the increase of armature current, the greater will be the induced e.m.f. If the average value of the potential difference between adjacent segments is fairly high under normal load conditions, the additional e.m.f. may cause a spark discharge between the tips of adjacent commutator segments and so provide an ionized or conducting path for the main current. This induced e.m.f. will be greatest in the elements located midway between the brushes. In machines that are normally subject to heavy rushes of current, as in motors used to drive reversing rolling mills, this tendency to flash-over may be very serious, and in such cases special means must be employed to eliminate the danger. This is accomplished by the use of compensating windings which neutralize the armature cross flux and which are described in Art. 20.

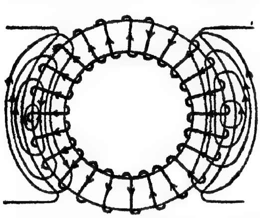


FIG. 24.—Linkage of cross flux with armature winding elements.

16. Pulsations of Commutating Field.—During the period of commutation the rotation of the armature periodically changes the positions of the teeth and slots with respect to the pole shoes, thereby giving rise to peripheral oscillations of the armature flux in the interpolar space. The changing current in the short-circuited coils produces a further pulsation of the flux in this region. There may also be pulsations of the flux as a whole, due to periodic changes in the reluctance of the magnetic circuit

if the surface of the teeth presented to the poles does not remain constant. All these effects are of high frequency and induce in the short-circuited coils e.m.fs. of rapidly changing direction which are superposed upon the main e.m.fs. considered in the preceding sections. They give rise to saw-tooth notches in the short-circuit current curves. These pulsations can be reduced by using numerous small slots with few coil sides per slot.

17. Commutation without Special Auxiliary Devices.—In machines not equipped with interpoles, satisfactory commutation is to a large extent dependent upon the resistance effect contributed by the brush contact and to a lesser extent upon such commutating e.m.f. as may be derived from the fringing field without the necessity of shifting the brushes with each change of load. These conditions call for a main field that is powerful, or stiff, in the manner analyzed in Art. 16, Chap. VIII; and at the same time the reactance voltage of the short-circuited coils must be kept within limits.

FIG. 25.—Chamfered and eccentric pole faces.

FIG. 26.—Construction of laminated pole shoe.

Prevention of the distortion, or skewing, of the main field by the cross-magnetizing m.m.f. of the armature is accomplished to some extent by saturation of the armature teeth. Additional reluctance in the path of the cross flux may be obtained by using chamfered or eccentric pole faces, as in Fig. 25; the same effect results from the use of pole shoes made of laminations that have only one projecting tip, assembled alternately in opposite directions, as in Fig. 26. A few machines, now obsolete, were designed with longitudinal slots in the main pole cores (Fig. 27) for the purpose of increasing the reluctance in the path of the cross flux; but the armature flux tends to pass around behind the

slots, rather than to cross them, and so they are not effective. One design, the Lundell generator (Fig. 28), incorporated several of these features.

The limitation on reactance voltage, which has an average value of $\frac{2I_0(L + \Sigma M)}{T}$, requires that both L and ΣM be kept small unless $I_0 = I_a/a$ is itself so small, as in fractional-horsepower motors, as to relieve this restriction. The inductance L is proportional to the square of the number of turns of the com-

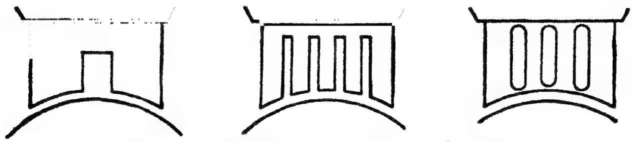


FIG. 27.—Longitudinal slotting of pole cores.

mutated coil and to the permeance of the leakage paths associated with it; hence, it is minimized by using one turn per element in machines of large current rating (which implies the use of a considerable number of commutator segments) and by designing the armature with a short axial length of core. Both these considerations lead to design proportions in which the ratio of armature diameter to length of core is relatively large, these proportions being characteristic of slow- or moderate-speed machines. The selection of a large diameter carries with it the possibility of using a large number of slots each containing but few coil sides, thereby reducing the mutual inductance. The presence of numerous armature slots means further that there will be a sufficient number of them between pole tips to cut down oscillations of the commutating flux and minimize pulsations of the flux as a whole. A large number of commutator segments involves a moderate average value of volts per segment, $V \div S/p = pV/S$, and this value in simplex lap windings should not exceed 20 volts, and should preferably be less than 15 volts. The maximum difference of potential between adjacent segments in machines of the series type should not exceed 40 volts; otherwise, there is the possibility of flashing over the commutator.

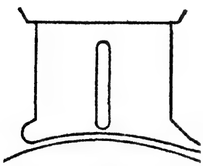


FIG. 28.—Pole of Lundell generator.

It would at first appear that the average reactance voltage can be reduced by increasing the duration of the commutation period T, which is theoretically proportional to the width of the brush. The practical difficulty of maintaining nonarcing conditions over a wide brush interferes with this procedure; but even if the contact were adequate in a wide brush, the additional winding elements that would be simultaneously short-circuited would increase ΣM at the same time that T increases, any decrease in reactance voltage being thus largely annulled.

18. Early Commutating Devices.—Quite early in the development of d-c generators and motors, commutation difficulties led to the development of numerous devices intended to ensure

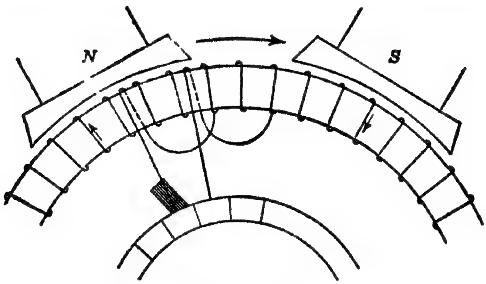


FIG. 29.—Sayers' winding.

sparkless operation, or at least to minimize the cost of maintaining commutators in good condition. Some of these devices were based upon the principle of introducing into the coil, during the period of short circuit by the brush, an e.m.f. large enough to neutralize the reactance voltage; others were designed to introduce into the circuit of the coil a resistance sufficient to limit the short-circuit current to safe values.

Thus, in the Sayers' winding (Fig. 29) the reversing e.m.f. is introduced into the short-circuited coil by an auxiliary winding which is so placed as to cut a part of the main flux during the commutation period and which, during that interval, is in series with the coil undergoing commutation. At all other times the auxiliary coil is not in circuit. The auxiliary coils are in reality merely extensions of the commutator leads that have been wound around the armature. The main coils are connected to auxiliary coils which lie *behind* them with respect to the direction of rotation so that commutation is not dependent upon the flux

density at the leading pole tip as in the ordinary machine, but upon that at the trailing pole tip; and since the field intensity at the trailing tip increases with increasing current, the commutating e.m.f. increases with it. A limit is set to this automatic adjustment of commutating conditions by the saturation of the trailing pole tip. The Sayers' device is of historical rather than practical interest, as is also that of Swinburne.* In the latter,

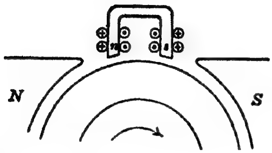


FIG. 30.—Swinburne's commutating device.

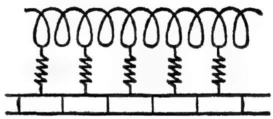


FIG. 31.—High-resistance leads to commutator.

small U-shaped electromagnets (Fig. 30), excited by the main current, were placed in the neutral zone.

The simplest method that suggests itself for introducing resistance into the circuit of the short-circuited coil is indicated in Fig. 31, the commutator leads being made of an alloy of high resistivity; this arrangement was used in early types of a-c commutator motors. It is open to the serious objection that the main current must flow through a set of these extra resistors with consequent heating and loss of efficiency.

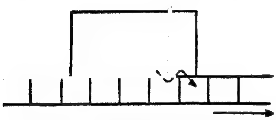


FIG. 32.—Special brush with auxiliary insulated section.

Instead of inserting resistance in the main circuit in the manner described above, it was at one time proposed to construct the brushes of alternate longitudinal layers of carbon and copper. The copper was intended to provide a low resistance path for the main current, whereas the short-circuit current had to traverse the higher resistance of the carbon and copper layers in series. This arrangement has no practical value to offset its obvious disadvantages. A better design, due to Young and Dunn,† provided for the final rupture of the short circuit at an auxiliary carbon brush insulated from the main brush in the manner indicated in Fig. 32. This tandem arrangement of

* *Jour. Inst. Elec. Eng. (London)*, **19**, 106, 1890.

† *Elec. World*, **45**, 481, 1905.

brushes has recently been reintroduced in some European and American designs, but with the difference that each part of the brush is mechanically independent of the other, each having its own holder and pressure finger (see Fig. 25, Chap. VI).

At the present time, however, the device that is used to the practical exclusion of all others, wherever commutating conditions present special difficulties, is the interpole or commutating pole; and this is supplemented, in cases where the operating conditions are particularly severe, by the use of *compensating windings*. The principal features of these auxiliary devices are discussed in the following articles.

19. Principle of Compensation.—The cross- or transverse-magnetizing action of the armature current is the primary cause of the field distortion, which in turn necessitates the shifting of the brushes and thereby brings into existence the demagnetizing action of the armature. Clearly, if the transverse m.m.f. of the armature were balanced by an equal and opposite m.m.f. *having the same distribution in space*, the distortion of the field would be completely eliminated and appreciable brush displacement would be unnecessary. Moreover, if the armature m.m.f. is somewhat overcompensated, there will exist in the neutral zone a component of flux having the proper direction to reverse the current in the short-circuited coils, and the brushes could be permanently fixed in the geometrical neutral axis.

If the ratio of compensating ampere-turns to armature ampere-turns is unity, nearly complete neutralization of the armature flux will result; if the ratio is slightly greater than unity, there will exist in the commutating zone a reversing flux that increases proportionally with the armature current (unless saturation of the iron of the magnetic circuit sets in), which is precisely the condition necessary to secure good commutation at all loads. In either case, whether the above ratio is unity or greater than unity, the compensating winding must be traversed by the main armature current or a fixed fractional part thereof; consequently the compensating winding is connected in series with the armature and may or may not be provided with a diverting shunt, as in the case of the series winding of a compound machine.

The problem of compensating armature reaction then consists of two parts: (1) the prevention of field distortion in order to minimize the danger of flash-over at the commutator, as dis-

cussed in Art. 15; (2) the production of a commutating e.m.f. for the purpose of neutralizing the reactance voltage of the short-circuited coils and reversing the current in them. Of these two the latter is usually the more important.

20. Compensating Windings.—In a German patent granted to Menges in 1884 there is the first exposition of the principle of compensating armature reaction. The patent specifications call for the use of a stationary compensating winding wound around the armature at the sides of the poles and traversed by the armature current, or a part of it, in such a direction as to oppose the magnetizing action of the armature. Later, in 1892, H. J. Ryan and M. E. Thompson experimented extensively along similar lines and developed the method shown diagrammatically in Fig. 33 for the case of a bipolar generator. The compensating winding is embedded in slots in the pole faces. The space distribution of the compensating winding being very nearly the same as that of the armature winding, the neutralization of armature reaction is practically complete. Since in the case shown in Fig. 33 the current in the armature conductors is half the total current, the number of turns in the compensating winding should be half the effective number of cross-magnetizing armature turns. In larger multipolar machines the number of compensating turns per pole should be $1/a$ times the armature turns per pole, where a is the number of current paths through the armature.

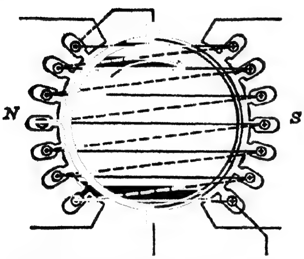


FIG. 33.—Diagram of compensating winding.

Figure 34 illustrates the construction of the field frame of a Ridgway generator of this type built by the Elliott Company. The entire magnetic circuit is built up of sheet-steel stampings, the main ring or yoke being clamped between cast-iron frames; the main pole pieces that carry the coils of the shunt winding are bolted to the yoke, and the cores that carry the compensating winding are held by the bolts which pass through the main poles and by the wedges which hold the commutating lugs in position. These wedges also serve to reduce the cross-section of the magnetic path from pole to pole and so keep down magnetic leakage.

The object of laminating the entire magnetic circuit is to prevent the setting up of eddy currents that would be induced in a solid mass of metal by rapid changes in the load current. Such eddy currents, if permitted to exist, would by Lenz's law oppose the action that produces them and hence would retard

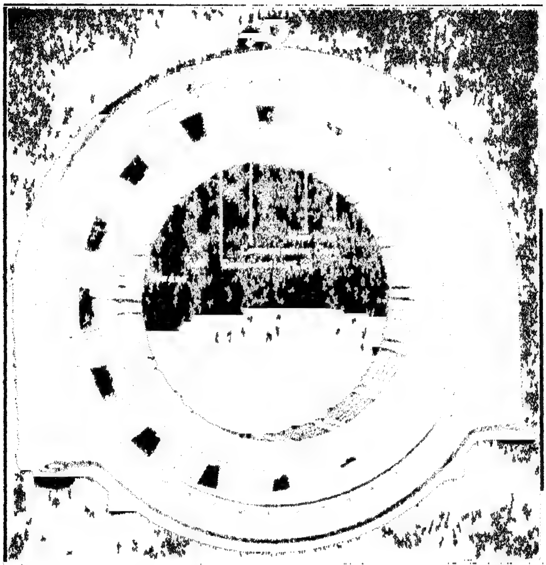


FIG. 34.—Frame of Ridgway generator, showing slotted pole face.

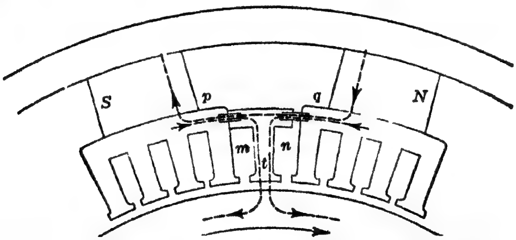


FIG. 35.—Flux paths in poles of compensated machine.

adjustments of the flux at the very time when the flux in the commutating zone should be most responsive to changes in the load current.

Figure 35 illustrates diagrammatically a portion of the magnetic circuit of a machine embodying the above-mentioned device. Under load conditions the m.m.f. of the compensating winding

acts in the directions shown by the dotted lines, generator action and clockwise rotation being assumed. Section q of the bridge is acted upon by two m.m.fs. having the same direction, but the flux is not materially increased on account of the initial saturation of the iron; and section p is acted upon by two m.m.fs. of opposite direction. The result is that the central tooth t

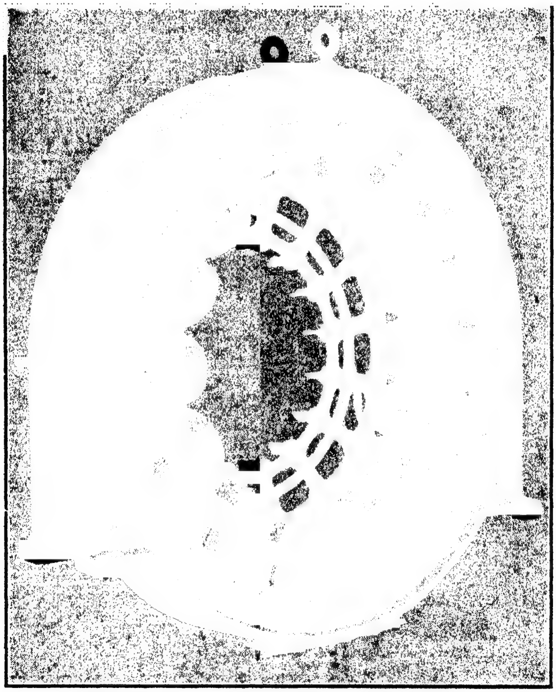


FIG. 36.—Frame of Ridgway generator, coils in place.

is acted upon by a resultant m.m.f. which makes it a north pole under the assumed conditions; hence, it produces a local field of the proper direction to assist the reversal of the current in the short-circuited coils. This effect is accentuated by making slots m and n somewhat larger than the others (see also Fig. 34) and winding in them more than the normal number of conductors. The arrangement of the compensating winding and main field winding is shown in Fig. 36.

Closely akin to the Thompson-Ryan device is an arrangement due to Deri. Instead of using a field frame of the salient-pole type with the addition of a slotted ring, the field structure consists of a slotted cylinder wound with two independent sets of coils and concentrically surrounding the armature, as indicated in Fig. 37. The main winding *M* produces poles the axes of which are indicated by the arrows; the compensating winding *C* sets up m.m.f. acting along axes midway between the poles and in opposition to the armature m.m.f. The field structure closely resembles the stator of an induction motor; and since the reluctance of the magnetic path is the same along all diametral paths, the compensation can be made complete.

21. Distribution of Airgap Flux with Compensating Windings and Commutating Poles.—Machines that are called upon to

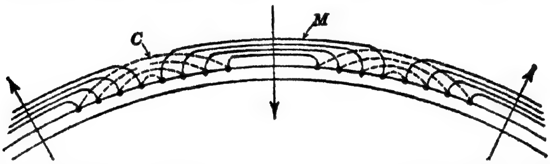


FIG. 37 —Deri's arrangement of main and compensating windings.

carry loads that fluctuate rapidly between wide extremes, for example, reversing rolling mills, require both compensating windings and commutating poles. The compensating winding, spread over what amounts to the main pole arc, serves to annul most of the transverse flux due to armature m.m.fs. within that space, with beneficial results in suppressing any tendency to flash-over; but in the commutating zone it is desirable to secure the benefit of the independently adjustable excitation of commutating poles so that the reversing field will have the proper space distribution and intensity to commutate any current, in either direction, from no load to heavy overload.

Figure 38 represents in diagrammatic form a compound-wound machine that is provided with commutating poles and compensating winding. In conformity with the system that has been adopted as standard throughout this book (with the exception of a few illustrations taken from catalogues), this diagram shows not only the order of the electrical connections but also the relations between the magnetic axes of the several windings.

The arrows in the diagram show the relative directions of the various m.m.fs.; the compensating winding and the commutating-pole winding act in direct opposition to the armature m.m.f., and along the brush axis; and the m.m.fs. of the main windings, shunt and series, act at right angles to the brush axis.

Complete neutralization of the armature flux is not possible through the use of interpoles alone for the reason that the space distributions of the m.m.fs. of armature and interpoles are different. This difference is not objectionable, however, except where complete neutralization is required to prevent flash-over, since in most cases the essential feature is the production, in the commutating zone only, of a reversing flux of proper

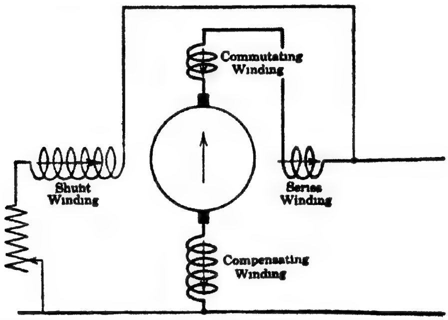


Fig. 38.—Compound machine with compensating winding and commutating poles

intensity; the flux distribution outside of this zone, so far as commutation is concerned, is of minor importance. Since the ratio of interpole ampere-turns to armature ampere-turns is constant, the difference between them will always be proportional to the armature current, and hence the commutating flux would satisfy the requirement of being proportional to the armature current at all loads, provided that the reluctance of the interpole circuit were constant; unfortunately, however, the interpoles tend to become saturated under heavy loads, with the result that the commutating flux does not increase in strict proportion to the armature current.

When a machine is provided with commutating poles, but not with a compensating winding, the resultant flux may, as a first approximation, be considered to be made up of two components: (1) a field due to the main winding considered as acting

alone; (2) a field due to the armature and commutating-pole windings in series, assumed to be acting together, but independently of the main field winding. The error involved in this superposition of the two separate fields has been pointed out in Chap. VIII, but for present purposes it may be neglected.

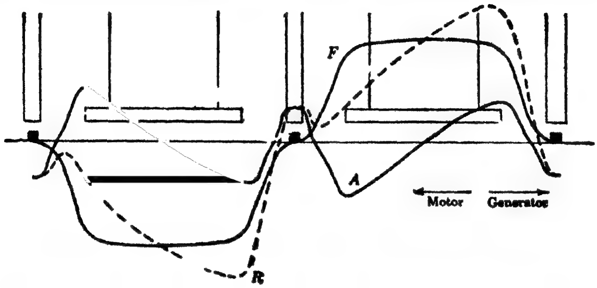


FIG. 39.—Flux distribution, commutating-pole machine.

In Fig. 39, curve F represents the flux distribution due to the main field winding, and curve A represents the field due to the combined magnetizing actions of the armature and commutating poles. Curve R, obtained by adding the ordinates of the two curves, shows the resultant flux distribution. It will be noted that curve R shows a pronounced dip in the vicinity of the brush

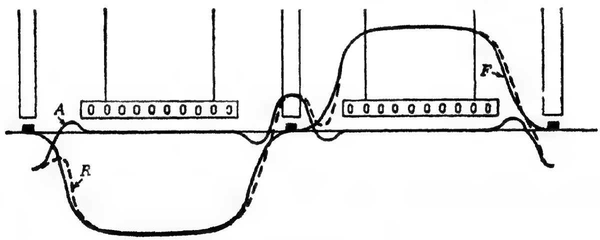


FIG. 40.—Flux distribution, commutating-pole machine with compensating windings.

axis. If the main field excitation is weak and the armature is magnetically powerful, this dip may be so accentuated as to produce a region of reversed polarity under each of the main pole tips. The effect of this reversal is to increase the voltage between adjacent commutator segments, and if this effect becomes extreme it may lead to flash-over. In such cases it is necessary to use compensating windings in the pole faces to neutralize the armature reaction. The curves showing flux

distribution in a fully compensated machine are indicated in Fig. 40.

22. Winding of Commutating Poles.—The calculation of the winding to be placed on the commutating poles presents no special difficulties. It is necessary to provide a sufficient number of ampere-turns to balance those of the armature and to supply the m.m.f. required to drive the commutating flux through the transverse path n, s (Fig. 41), there being taken into account the m.m.f. supplied by the main poles N, S in those parts of the path of the lines of induction which are common to both magnetic circuits. The figure represents a bipolar generator revolving in

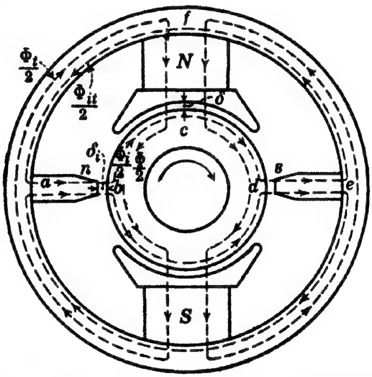


FIG. 41.—Magnetic circuits in commutating-pole machine.

the clockwise direction; if the machine were a motor revolving in the same direction, the polarity of the interpoles would have to be reversed, a condition that would be met automatically upon reversing the current through the armature, since the two windings are in series.

Figure 41 reveals the fact that the m.m.fs. of the main and commutating poles act in the same direction in two of the four quadrants of the armature core and of the yoke, and in opposite directions in the other two quadrants. The flux in the armature core is $(\Phi + \Phi_i)/2$ in two of the quadrants and $(\Phi - \Phi_i)/2$ in the other two, where Φ_i is the working flux produced by the interpole. Similarly, in two of the quadrants of the yoke the flux is $(\Phi_t + \Phi_{it})/2$, and $(\Phi_t - \Phi_{it})/2$ in the other two; here $\Phi_t = \nu\Phi$ where ν is the coefficient of dispersion of the main poles, and

$\Phi_{it} = \nu_i \Phi_i$ where ν_i is the coefficient of dispersion of the interpoles. The magnitude of the commutating flux Φ_i is given by

$$\Phi_i = B_{ig} b'_i l'_i$$

where B_{ig}, the flux density in the gap under the interpole, is determined by the value of the commutating e.m.f. to be generated and b'_i and l'_i are, respectively, the corrected breadth and length of the interpole; these corrected lengths are greater than the actual lengths, by three to four times the airgap δ_i under the interpole. Corresponding to this value of Φ_i and to that of the main flux Φ there will be definite flux densities in each part of the closed magnetic circuit *abcdefa*, and to each flux density there will correspond a definite number of ampere-turns which may be determined from the appropriate *B-H* curves. Thus, let

AT_{ic} = ampere-turns required by the two interpole cores and shoes.
AT_{ig} = ampere-turns for the two interpole airgaps δ_i.
AT_{it} = ampere-turns for the two sets of teeth opposite the interpoles.
AT'_a = ampere-turns for the armature core, *b* to *c*.
AT''_a = ampere-turns for the armature core, *c* to *d*.
AT'_y = ampere-turns for the yoke, *e* to *f*.
AT''_y = ampere-turns for the yoke, *f* to *a*.

Then, in the closed magnetic circuit *abcdefa*, the algebraic sum of all the m.m.fs. must be zero, in accordance with Kirchhoff's law. In this circuit there act, in addition to the m.m.fs. listed above, the m.m.f. due to the two interpole windings AT_i and that due to the armature $AT_{arm.}$, where

$$AT_{arm.} = \frac{2}{p}\frac{Z}{2}I_a = \frac{ZI_a}{\pi d}\cdot\frac{\pi d}{p} = q\tau$$

$$\therefore AT_i = q\tau + AT_{ic} + AT_{ig} + AT_{it} - AT'_a + AT''_a + AT'_y - AT''_y$$

It follows that the number of turns to be wound on each interpole is $\frac{1}{2}(AT_i/I_a)$, provided that the interpole coils are not shunted. A larger number of turns may be used if a diverting shunt is placed across the interpole winding.

The discussion above applies directly to the bipolar machine of Fig. 41, and with obvious modifications to multipolar machines, also.

23. Compounding Effect of Commutating Poles.—In Chap. VIII it is shown that a forward lead of the brushes in a generator produces a demagnetizing effect and consequently reduces the generated e.m.f., whereas a backward lead causes a compounding action. If the generator is provided with commutating poles, these effects of brush displacement are accentuated, as

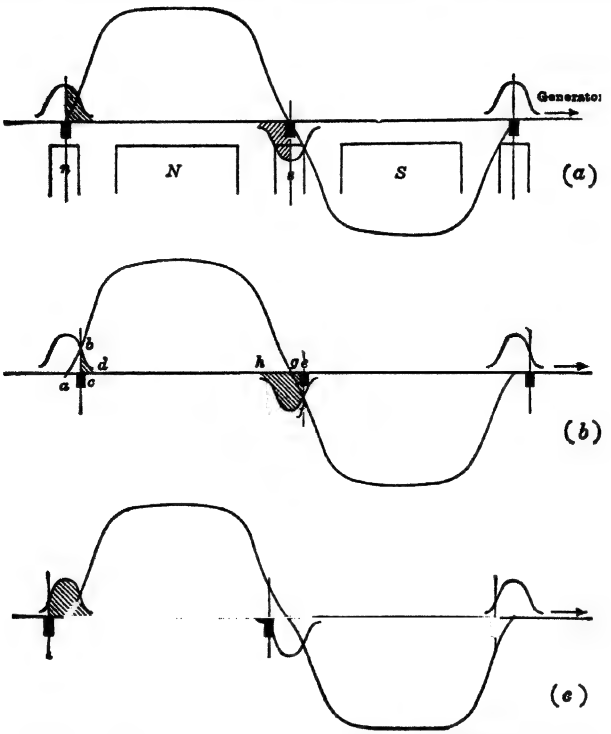


FIG. 42. · Compounding effect of interpoles.

may be seen from Fig. 42. Thus, Fig. 42*a* represents the conditions when the brushes are in the geometrical neutral axis; the armature winding between adjacent brushes of opposite polarity is then acted upon by the flux due to a main pole only, the effect of the oppositely directed fluxes due to the interpoles (shown by the hatched areas) being to annul each other. If the brushes are displaced in the direction of rotation (Fig. 42*b*), the total flux is reduced by the difference between the hatched

areas *efh* and *bcd* and, in addition, by the sum of areas *abc* and *efg*. In the same manner a backward displacement of the brushes (Fig. 42*c*) results in an increase of the flux linked with the armature winding.

Similar considerations show that in commutating-pole motors a forward lead of the brushes will increase the flux and therefore reduce the speed, whereas a backward lead will decrease the flux and raise the speed. In Chap. XI it was shown that a considerable backward displacement of the brushes of a commutating-pole motor may result in a continuous succession of reversals of rotation. If actual reversal of direction does not occur, pulsation of speed may result; for since the effect of a backward displacement is to weaken the active field, there will be a corresponding decrease of counter e.m.f. and an increase of armature current to produce the necessary acceleration of the armature. The increased current further strengthens the interpoles and so still further weakens the field and accelerates the armature. But the counter e.m.f. is proportional to the active flux and to the speed. The tendency to accelerate the armature will then continue until the decrease of counter e.m.f., due to reduced flux, is offset by the increase due to greater speed. The decrease of field strength cannot, however, go on indefinitely because the interpoles eventually become saturated; but up to the time that saturation of the interpoles sets in, the speed has been continuously increasing, and the momentum of the armature tends to maintain the speed even after the flux has reached a practically constant value—especially if the rotating parts have large moment of inertia. The result will be a rapid increase of counter e.m.f., possibly to a value greater than the line voltage, in which case the machine becomes a generator drawing upon its kinetic energy of rotation to send current back to the supply line. The speed under these conditions rapidly falls, and the counter e.m.f. is so reduced that the armature current again rises, the speed also increases, and the above-described cycle of changes is repeated.

The compounding effect of commutating poles, due to slight displacement of the brushes from the geometrical neutral, is equally prominent in the case of generators. Here, however, it affects the e.m.f. generated in the armature, so that if several generators are operating in parallel the proper division of the

load may be seriously disturbed if the brushes are not accurately adjusted. Very slight displacement of the brushes may produce disproportionately large unbalancing of the load distribution.

24. Leakage Flux in Commutating-pole Machines.—The presence of commutating poles between the main poles obviously increases the leakage of the main flux and therefore increases

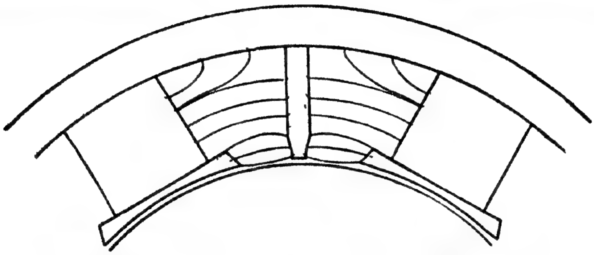


FIG. 43.—Leakage of main flux, as influenced by presence of interpoles.

the coefficient of dispersion ν. The calculation of the coefficient ν can be carried out in the manner explained in Chap. IX, modified however to suit the conditions sketched in Fig. 43. This diagram maps the leakage paths as they are affected by the presence of the interpoles, upon the assumption that the main excitation alone is operative.

The coefficient of dispersion of the interpoles themselves, designated ν_i, can be computed similarly by mapping the leakage

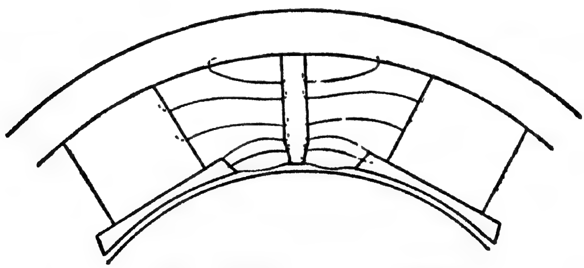


FIG. 44.—Leakage flux from interpoles.

paths as in Fig. 44 on the assumption that the excitation of the interpoles is operative whereas that of the main poles is not. Since the commutating poles are usually very narrow and have an axial length less than that of the main poles, it is hardly necessary to compute separately the leakage flux that emanates from the end flanks; this can be included in the leakage flux between

the sides of the commutating and main poles by increasing the actual axial length of the commutating pole by an amount (at each end) equal to half the breadth of the commutating pole. A similar simplification can be made in the case of the flank leakage from the shoe of the commutating pole.

In computing the leakage flux due to the commutating pole, it is necessary to take into account, just as in Chap. IX, the varying difference of magnetic potential that acts upon the several tubes of force. At the yoke, and above the interpole winding, the magnetic potential may be taken as zero, whereas at other points along the pole core it increases linearly to a maximum of $\frac{1}{2}AT_i$ amp.-turns at the pole shoe. If the computed value of the leakage flux associated with a commutating pole is φ_i and the useful flux is Φ_i, the coefficient of dispersion is

$$\nu_i = \frac{\Phi_i + \varphi_i}{\Phi_i} = 1 + \frac{AT_i}{\Phi_i} \times \text{function of dimensions}$$

Since the useful flux Φ_i contributed by the commutating poles is relatively small, being just enough to produce in the short-circuited coils the necessary reversing e.m.f., the leakage flux φ_i will be relatively large, so that ν_i may range in value from 2 to 5.

25. Estimate of Average Reactance Voltage.—An estimate of the reactance voltage, given by Eq. (5),

$$E_r = \frac{2I_0(L + \Sigma M)}{T}$$

is dependent upon the magnitude of the equivalent inductance $(L + \Sigma M)$, since $I_0 = I_a/a$ is known and T can be determined (approximately) by means of Eq. (11). It is possible to evaluate $(L + \Sigma M)$ by a method due to Parshall and Hobart; thus, in Fig. 45a, assume that there are two coil sides per slot, that the elements of the simplex lap winding there represented are of full pitch, and that the brush width b is equal to the width of a segment β. Each element has $z = Z/2S$ turns. The coil sides belonging to simultaneously short-circuited elements then lie one above the other. Let

φ_s = number of lines of induction that link with each inch of length of the "embedded" or slot part of coil C, per

ampere-conductor of the group of conductors simultaneously short-circuited.

φ_f = number of lines of induction that link with each inch of "free" length of the end connections of coil C, per ampere-conductor of the group of conductors simultaneously short-circuited.

In each slot of length l' there are $2z$ conductors each carrying a current I_0, so that the flux linking one side of coil C is $\varphi_s \times l' \times 2z \times I_0$. In each group of end connections of length $\frac{1}{2}l_f$ there are z conductors, so that the flux linking one group of end connections is $\varphi_f \times \frac{1}{2}l_f \times z \times I_0$; the total flux linking both sides and both end connections of coil C is double the sum of these fluxes, or

$$\varphi = 2zI_0(2\varphi_s l' + \tfrac{1}{2}\varphi_f l_f)$$

The equivalent inductance $(L + \Sigma M)$ is the number of flux linkages per ampere divided by 10^8, or

$$L + \Sigma M = \frac{z\varphi}{I_0} \times 10^{-8} = 2z^2\left(2\varphi_s l' + \frac{1}{2}\varphi_f l_f\right) \times 10^{-8} \quad \text{henry} \tag{14}$$

Experiments made by Parshall and Hobart* show that with average slot dimensions φ_s may be taken as 10 lines per amp.-conductor per in. of core (or 4 lines per amp.-conductor per cm. of core) and φ_f as 2 lines per amp.-conductor per in. of free length (or 0.8 line per cm. of free length). Accordingly, if inch units are used,

$$L + \Sigma M = 2z^2(20l' + l_f) \times 10^{-8} \tag{15}$$

Now let the brush width be increased until the coil sides in three adjoining slots are simultaneously short-circuited, as indicated in Fig. 45*b*, on the assumption that the wide brush makes good contact over its entire span. The path of the leakage flux surrounding the entire group crosses three slots instead of only one as in Fig. 45*a*, and hence the reluctance is very nearly three times as great since the reluctance of the iron part of the path is negligible in comparison with that of the nonmagnetic content of the slots; but the total m.m.f. acting around this leakage path is also three times as great as before, and hence the

* "Electric Generators," 1900.

actual flux linking an element is very nearly the same as before and $(L + \Sigma M)$ will have the same value as Eq. (15). On the other hand, the duration of the short circuit is likewise three times as great as before, and so the average reactance voltage is reduced to one-third of the value corresponding to the winding of Fig. 45*a*.

Consider windings like those indicated in Fig. 45*c* and *d*; in case (*c*), where the brush again covers only one segment, the leakage flux is only one-third as great as in case (*a*), and the period of commutation is the same; hence, the average reactance voltage of (*c*) is one-third that of (*a*). In case (*d*), the leakage flux is

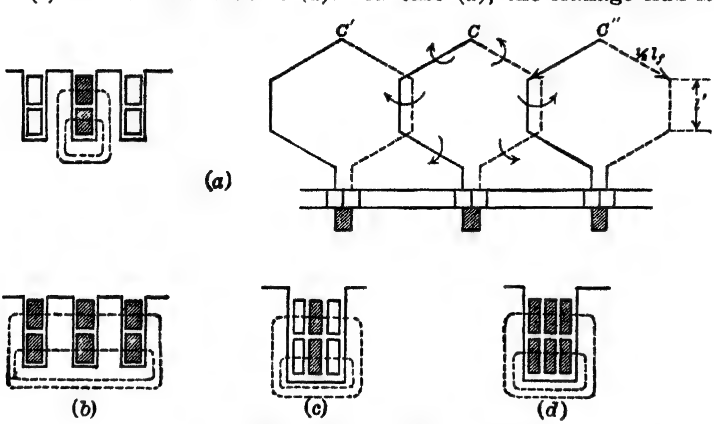


FIG. 45.—Estimation of reactance voltage.

the same as in (*a*) because both m.m.f. and reluctance have been increased in the ratio of 3:1, while the period of commutation is three times as great; hence, the average reactance voltage is again only one-third that of (*a*).

The above conclusions may be summarized in the formula

$$E_r = \frac{2I_0(L + \Sigma M)}{T} = \frac{2I_a}{a}\frac{L + \Sigma M}{T} \tag{16}$$

where

$$L + \Sigma M = 2z^2(20l' + l_f) \times 10^{-8}$$

and

$$T = \frac{b + \beta\left(1 - \dfrac{a}{p}\right)}{v_c}$$

In the case of simplex lap windings, $a = p$, so that

$$T = \frac{b}{v_c} = \frac{60}{nS} \times \textit{segments covered by brush}$$

where n = r.p.m. and S = total number of commutator segments; for this special case, and upon the assumption of full-pitch elements,

$$E_r = 1.33 \times 10^{-8} \frac{z^2 nSI_a(l' + 0.05l_f)}{a \times \text{number of segments covered by brush}} \tag{17}$$

l' and l_f being in *inches*. In this formula, if the number of segments covered by the brush is a mixed number (such as $2\frac{1}{2}$), the number to be substituted is the next higher integer.

If the shape of the slots departs materially from the average types used by Parshall and Hobart in their experiments, the empirical constants used in the foregoing computation of $L + \Sigma M$ must be modified. More refined methods for calculating L and M are given in Art. 26. But aside from this limitation it should be noted that in extending Eq. (17) to the case of wide brushes short-circuiting more than one element at a time, the assumption was tacitly made that the currents in all of the coils simultaneously undergoing commutation are at a given instant in the same phase of their variation and that their time rates of change are also equal at all times. This is not the case in reality, and to the extent to which the actual facts depart from these assumed conditions the formula will be in error; as a matter of fact the changing currents in the neighboring short-circuited coils have a more or less differential effect upon one another, and hence the results of Eq. (17) are more or less pessimistic and err on the side of conservatism.

In the case of *fractional-pitch* lap windings, some of the coil sides in the simultaneously short-circuited group occupy positions in the manner shown in Fig. 17, parts (*e*), (*f*), and (*g*). The leakage flux crossing the slot portions of a given coil is therefore somewhat less than in the corresponding full-pitch winding, and hence the average reactance voltage is also less, other conditions being the same. It is difficult to summarize all the possibilities into a single formula; the best procedure is to make such a diagram as Fig. 17, selecting that particular grouping which gives the highest value of leakage flux, and estimating the leak-

age flux in the general manner outlined for a full-pitch winding, or by using the more accurate methods of Arts. 26 and 27.

In simplex *wave windings* where only *two brushes* are used, each brush short-circuits $p/2$ elements in series, as may be seen from Figs. 7 and 10, Chap. VII. Accordingly, formula (17) must be modified by multiplying it by $p/2$, or

$$E_r = 1.33 \times 10^{-8} \frac{z^2 nSI_a(l' + 0.05l_f)}{a \times \text{no segments covered by brush}} \times \frac{p}{2} \quad (18)$$

In wave windings in which all the p possible brushes are used, each element is separately short-circuited during the period of commutation, and the $p/2$ elements comprising the group short-circuited by brushes of the same polarity are in parallel instead of in series. If the current divided equally between these $p/2$ elements, the average reactance voltage of each of them might be calculated from Eq. (17); but because of the so-called selective commutation in such windings, described in Art. 8, the average reactance voltage is higher than is indicated by that equation; on the other hand, Eq. (17) involves a period of commutation, T, that is somewhat too small, Eq. (11) showing that in wave windings T is somewhat greater than in simplex lap windings, other things being equal. These two factors therefore tend to counterbalance each other.

Summarizing the conclusions drawn from diagrams (*a*), (*b*), (*c*), and (*d*) (Fig. 45), it will be seen that the average reactance voltage is the same in cases (*b*), (*c*), and (*d*) and less in each of them than in case (*a*). When the number of coil sides per slot is greater than two, as in cases (*c*) and (*d*), an increase in brush width has no effect upon the average reactance voltage so long as the maximum number of simultaneously commutated coil sides does not exceed the number per slot. But if the brush width is increased until the number of simultaneously commutated coil sides exceeds the number per slot [compare cases (*b*) and (*a*)], there is a decrease in the average reactance voltage.

26. Calculation of Self-inductance of Embedded Coils.—The self-inductance of a coil has been shown to be equal to the number of flux linkages* per ampere, divided by 10^8. In the

* In using the m.k.s. system, the flux is expressed in webers, this unit being so selected that the factor 10^8 is included in it; hence the inductance in henrys is the number of flux linkages, in webers, per ampere.

case of an armature coil embedded in a slot, the self-excited flux linking with the coil may be separated into three parts:

1. The flux crossing the slot from wall to wall of the teeth and completing its path through the core, as indicated by φ_1 (Fig. 46).

2. The flux passing from tip to tip of the teeth within the space between pole tips, as indicated by φ_2.

3. The flux φ_3 linking with the end connections beyond the edges of the core.

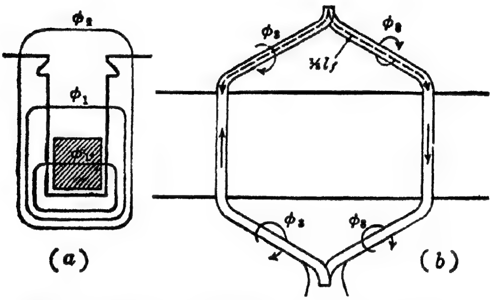


FIG. 46.—Paths of leakage flux surrounding coil.

The number of linkages due to each of these fluxes will now be computed separately.

1. *Slot Leakage Flux.*—Practically without exception the windings of all d-c machines are arranged in two layers, one side of each coil being in the top layer, the other side in the bottom layer. The magnitude and distribution of the flux are therefore not the same on the two sides of a coil.

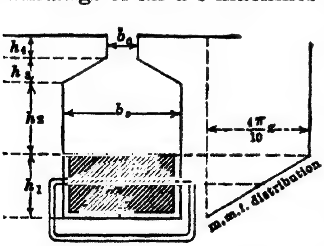


FIG. 47.—Slot leakage flux, coil occupying bottom of slot.

a. Coil Side Occupying Bottom of Slot (Fig. 47).—The coil side contains $z = Z/2S$ conductors, so that their total m.m.f. per unit current is $(4\pi/10)z$ gilberts. The m.m.f. acting upon an elementary tube dx is $(4\pi/10)(x/h_1)z$, on the assumption that the lines of force pass straight across the slot. The flux produced in this elementary path is

$$d\varphi_1' = \frac{4\pi}{10}\frac{x}{h_1}z\frac{\mu_0 l'\,dx}{b_s}$$

where l' is the corrected length of the armature core, all dimensions being in centimeters. The expression $b_s/\mu_0 l' dx$ represents the reluctance of the air part of the path, that of the iron part being negligible in comparison. This flux links with $(x/h_1)z$ conductors; hence, the number of linkages due to it is

$$\frac{4\pi}{10}\left(\frac{x}{h_1}z\right)^2 \mu_0 \frac{l'\,dx}{b_s}$$

and this may be reduced to henrys by dividing by 10^8.

$$\therefore dL'_{1b} = \frac{4\pi}{10}\left(\frac{x}{h_1}z\right)^2 \mu_0 \frac{l'\,dx}{b_s} \times 10^{-8}$$

The total number of linkages through the entire depth of the coil is found by integrating in x from 0 to h_1 or

$$L'_{1b} = \frac{4\pi\mu_0}{10^9}\frac{z^2 l'}{h_1^2 b_s}\int_0^{h_1} x^2\,dx = \frac{4\pi\mu_0}{10^9}z^2 l'\frac{h_1}{3b_s} \tag{19}$$

Above the coil the m.m.f. has the constant value $(4\pi/10)z$. Within the region h_2 the flux is uniformly distributed and has the magnitude

$$\varphi_1'' = \frac{4\pi}{10}z \cdot \frac{\mu_0 h_2 l'}{b_s}$$

and since it links with all of the conductors,

$$L''_{1b} = \frac{4\pi\mu_0}{10^9}z^2 l'\frac{h_2}{b_s} \tag{20}$$

In the same way the inductances due to the flux in the regions h_3 and h_4 are

$$L'''_{1b} = \frac{4\pi\mu_0}{10^9}z^2 l'\frac{2h_3}{b_0 + b_s} \tag{21}$$

and

$$L^{\mathrm{IV}}_{1b} = \frac{4\pi\mu_0}{10^9}z^2 l'\frac{h_4}{b_0} \tag{22}$$

The total inductance due to slot leakage is

$$L_{1b} = L'_{1b} + L''_{1b} + L'''_{1b} + L^{\mathrm{IV}}_{1b} =$$

$$\frac{4\pi\mu_0}{10^9}z^2 l'\left(\frac{h_1}{3b_s} + \frac{h_2}{b_s} + \frac{2h_3}{b_0 + b_s} + \frac{h_4}{b_0}\right) \tag{23}$$

In the case of straight slots (Fig. 48*a*), this reduces to

$$L_{1b} = \frac{4\pi\mu_0}{10^9}z^2l'\left(\frac{h_1}{3b_s} + \frac{h_s}{b_s}\right) \tag{24}$$

and with the shape of slot Fig. 48*b* it becomes

$$L_{1b} = \frac{4\pi\mu_0}{10^9}z^2l'\left(\frac{h_1}{3b_s} + \frac{h_2}{b_s} + \frac{2h_3}{b_0 + b_s} + \frac{h_4}{b_s}\right) \tag{25}$$

b. Coil Side Occupying Top of Slot (Fig. 49).—If the same methods are used as in case *a*, the resulting expressions for L_{1t} are identical with those for L_{1b} except that h_2 is replaced by h_2'. The total inductance of the coil due to slot leakage is

$$L_1 = L_{1b} + L_{1t}, \tag{26}$$

or less than twice L_{1b}.

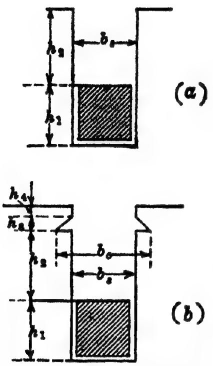


Fig. 48.—Coil occupying straight slots.

2. *Tooth-tip Leakage Flux.*—Assume that the lines of force from tip to tip of teeth are made up of a straight portion b_0 and of two quadrants of circles, as in Fig. 50. The flux in an elementary path dx is

$$d\varphi_2 = \frac{4\pi}{10^9}z\frac{\mu_0 l'\,dx}{b_0 + \pi x}$$

Hence, the total inductance for both sides of the coil is

$$L_2 = 2 \times \frac{4\pi\mu_0}{10^9}z^2l'\int_0^{\frac{1}{2}(\tau - b)}\frac{dx}{b_0 + \pi x} =$$

$$\frac{4\pi\mu_0}{10^9}z^2l' \times 1.46\log_{10}\left[1 + \frac{\pi(\tau - b)}{2b_0}\right] \tag{27}$$

The expression $(\tau - b)$ represents the distance between pole tips, and the superior limit of the integral is taken to be half this amount on the assumption that the coils undergoing commutation are approximately midway between pole tips. Values of L_2 calculated by the above equation are somewhat too large because no account has been taken of the effect of neighboring slot openings in reducing the flux.*

* The limit of integration used in the above equation has been checked by numerous tests and gives results that agree fairly well with actual meas-

3. *End-connection Leakage Flux.*—Various approximate formulas have been developed for calculating the inductance due to the end connections.

Niethammer* gives

$$L_3 = \mu_0 z^2 l_f \left[0.4 \log_{10}\left(\frac{l_f}{s}\right) - 0.1\right] \times 10^{-8} \qquad (28)$$

where s is the diagonal of the rectangular coil section (including insulation between turns) and l_f (Fig. 46*b*) is the total free length per element.

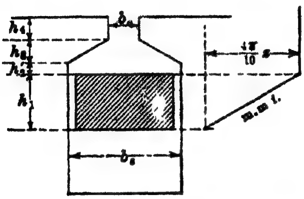


FIG. 49.—Coil occupying top of slot.

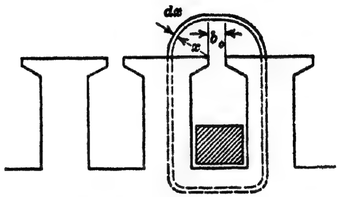


FIG. 50.—Tooth-tip leakage.

Arnold† gives

$$L_3 = \mu_0 z^2 l_f \left[0.46 \log_{10}\left(\frac{\frac{1}{2} l_f}{d_e}\right) - 0.092\right] \times 10^{-8}$$

$$= \mu_0 z^2 l_f \left[0.46 \log_{10}\left(\frac{l_f}{d_e}\right) - 0.23\right] \times 10^{-8} \qquad (29)$$

where d_e is the diameter of a circle whose circumference equals the perimeter of the coil section, including the insulation between turns (see Fig. 51); that is,

$$d_e = \frac{2(a + b)}{\pi}$$

The total inductance of a winding element is the sum of the inductances due to the several parts of the leakage, or

urements. Arnold ("Die Gleichstrommaschine") uses the entire pole pitch as the superior limit; Gray ("Electric Machine Design") uses only one tooth width.

* "Elektrische Maschinen Apparate u. Anlagen," Vol. I, p. 139, Stuttgart 1904.

† "Die Gleichstrommaschine," 2d ed., Vol. I, p. 376.

$$L = L_1 + L_2 + L_3 =$$
$$\frac{4\pi\mu_0}{10^9} z^2 l' \left[\left(\frac{2h_1}{3b_s} + \frac{h_2 + h_2'}{b_s} + \frac{4h_3}{b_0 + b_s} + \frac{2h_4}{b_0} \right) + \right.$$
$$\left. 1.46\mu_0 \log_{10} \left(1 + \frac{\pi}{2} \cdot \frac{\tau - b}{b_0} \right) \right] + \mu_0 \frac{z^2 l_f}{10^8} \left[0.4 \log_{10} \frac{l_f}{s} - 0.1 \right] \quad (30)$$

The values of leakage flux given by Parshall and Hobart (Art. 25) check fairly well with the results of the foregoing formulas when customary dimensions are inserted. Thus, consider a machine with straight open slots of which the ratio of depth to width is 5:1 (Fig. 52) and in which $(\tau - b)/b_0$ is

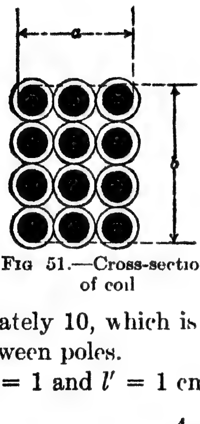


FIG 51.—Cross-section of coil

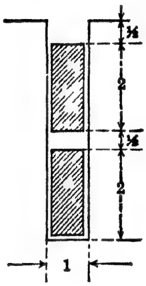


FIG 52 Relative slot dimensions, particular case.

approximately 10, which is equivalent to about five slots in the space between poles.

Taking $z = 1$ and $l' = 1$ cm.,

$$L_{1b} = \frac{4\pi\mu_0}{10}\left(\frac{2}{3} + 3\right) \times 10^{-8}$$
$$L_{1t} = \frac{4\pi\mu_0}{10}\left(\frac{2}{3} + \frac{1}{2}\right) \times 10^{-8}$$

or an average of $(4\pi\mu_0/10)(2/3 + 1\frac{3}{4}) \times 10^{-8} = 3.05 \times 10^{-8}$. The value of L_2 for one side of the coil, with $z = 1$ and $l' = 1$, is

$$\frac{4\pi\mu_0}{10} \times 0.73 \log_{10} [1 + 15.7] \times 10^{-8} = 1.11 \times 10^{-8}$$

or a total of 4.16×10^{-8} henry, corresponding to 4.16 lines per amp.-conductor per cm. of length. Parshall and Hobart's method is rapid and simple, but it is open to the objection that the designer must exercise great discretion in selecting the proper unit value of flux to fit the dimensions of his machine.

27. Calculation of Mutual Inductance of Embedded Coils.—The mutual inductance of two coils is equal to the number of flux linkages (in weber-turns) with one of them when a current of 1 amp. flows through the other.

The previous discussion of the simultaneous short-circuiting of several coils indicates that there are two cases to be considered: one in which the coils in question occupy the same slot, the other in which they lie in different slots.

1. *Coils Occupying Same Slot.*—In this case two distinct conditions are possible: (*a*) the coil sides lie side by side; (*b*) the coil sides lie one above the other, one in the top layer, the other in the bottom layer.

a. If the coil sides lie side by side in the slots, and therefore also throughout their entire lengths, there is no great error in writing

$$M = L$$

b. Since the coil sides are in different layers, the end connections run in opposite directions; hence, the mutual inductance is due only to the slot and tooth-tip fluxes.

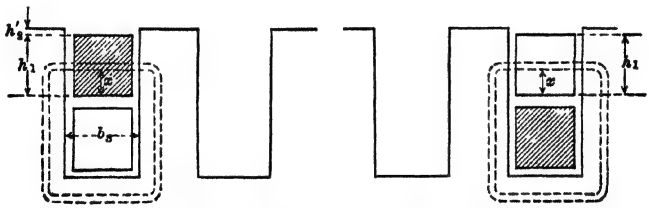


FIG. 53.—Flux leakage paths, mutual inductance.

In Fig. 53 the crosshatched areas represent the two sides of a coil. On the left-hand side the inducing coil is at the bottom of the slot, and the m.m.f. of $(4\pi/10)z$ gilberts per amp. acts in the elementary tube dx; the slot-flux linkages are

$$\int_0^{h_1} \frac{4\pi}{10} z \frac{\mu_0 l' \, dx}{b_s}\left(z\frac{x}{h_1}\right) + \frac{4\pi\mu_0}{10} z^2 l' \frac{h'_2}{b_s} = \frac{4\pi\mu_0}{10} z^2 l' \left[\frac{h_1}{2b_s} + \frac{h'_2}{b_s}\right]$$

and on the right-hand side of the coil the slot-flux linkages are given by the same expression.

The linkages due to the tooth-tip leakage flux are obviously the same as in the calculation of L. Finally, therefore,

$$M = \frac{4\pi\mu_0}{10^9} z^2 l' \left[\left(\frac{h_1}{b_s} + \frac{2h'_2}{b_s} \right) + 1.46 \log_{10} \left(1 + \frac{\pi}{2} \frac{\tau - b}{b_0} \right) \right] \qquad (31)$$

2. *Coils Occupying Different Slots. a. Both Coils in Same Layer.*—In this case the sides of the coils will be parallel to each other throughout their entire lengths, and the interlinked flux will consist of tooth-tip leakage flux along the embedded portion and end-connection flux along the free lengths. If we consider the tooth-tip leakage first, coil side 1 (Fig. 54) acts

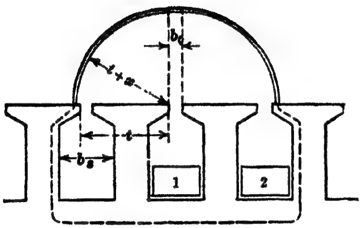


FIG. 54.—Mutual inductance, coils in adjacent slots.

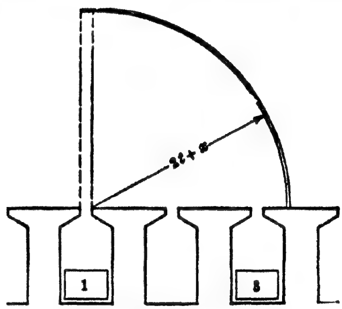


FIG. 55.—Mutual inductance, coils not in adjacent slots.

upon paths surrounding coil side 2 with m.m.f. $(4\pi/10)z$ gilberts per amp. In the tube dx the flux linkages are

$$\frac{4\pi}{10} z^2 \frac{\mu_0 l' \, dx}{b_0 + \pi(t + x)}$$

and the mutual inductance due to this flux on both sides of the coil is

$$M'_{12} = 2 \times \frac{4\pi\mu_0}{10^9} z^2 l' \int_0^{\frac{\tau - b}{2} - t} \frac{dx}{b_0 + \pi(t + x)} =$$

$$\frac{4\pi\mu_0}{10^9} z^2 l' \frac{4.6}{\pi} \log_{10} \frac{b_0 + \frac{\pi}{2}(\tau - b)}{b_0 + \pi t} \qquad (32)$$

This value is somewhat too large since in carrying out the integration continuously to the pole tips the effect of the slot openings is ignored.

The mutual inductance due to end-connection leakage is difficult to estimate. Arnold recommends taking it as one-half of the corresponding self-inductance of a single coil. On this basis,

$$M''_{12} = \mu_0 z^2 l_f\left[0.2 \log_{10}\left(\frac{l_f}{s}\right) - 0.05\right] \times 10^{-8} \qquad (33)$$

and

$$M_{12} = M'_{12} + M''_{12} \qquad (34)$$

When the coils considered are not in adjoining slots but are placed as in Fig. 55, the above equations become

$$M'_{13} = 2 \times \frac{4\pi\mu_0}{10^9}z^2 l' \int_0^{\frac{\tau-b}{2}-2t} \frac{dx}{b_0 + \pi(2t + x)} =$$

$$\frac{4\pi\mu_0}{10^9}z^2 l'\frac{4.6}{\pi} \log_{10} \frac{b_0 + \frac{\pi}{2}(\tau - b)}{b_0 + 2\pi t} \qquad (35)$$

and

$$M''_{13} = \frac{1}{4}L_s = \mu_0 z^2 l_f\left[0.1 \log_{10}\left(\frac{l_f}{s}\right) - 0.025\right] \times 10^{-8} \qquad (36)$$

$$M_{13} = M'_{13} + M''_{13} \qquad (37)$$

It is not necessary to carry the computation beyond the case shown in Fig. 55 for the reason that the numerical values become relatively small and the brushes are seldom so wide that coils are simultaneously short-circuited in more than three consecutive slots.

b. Coils Not in Same Layer.—In this case the mutual inductances M'_{12} and M'_{13}, due to tooth-tip leakage, remain the same as before; the end-connection leakage reduces to zero because the coils separate and run in opposite directions after leaving the slots. Then,

$$M_{12} = M'_{12} = \frac{4\pi\mu_0}{10^9}z^2 l'\frac{4.6}{\pi} \log_{10} \frac{b_0 + \frac{\pi}{2}(\tau - b)}{b_0 + \pi t} \qquad (38)$$

and

$$M_{13} = M'_{13} = \frac{4\pi\mu_0}{10^9}z^2 l'\frac{4.6}{\pi} \log_{10} \frac{b_0 + \frac{\pi}{2}(\tau - b)}{b_0 + 2\pi t} \qquad (39)$$

These values are somewhat too large inasmuch as the effect of the slot openings has been neglected.

28. Effect of Commutating Poles upon Coil Inductance.—The presence of commutating poles causes an increase in the inductance of the short-circuited coils under them. In the expression $L = L_1 + L_2 + L_3$ (Art. 26), the term L_2, due to tooth-tip leakage, is affected Its value may be computed as follows:

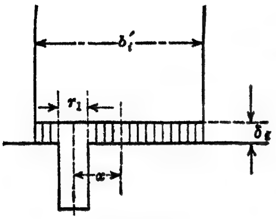


FIG. 56—Tooth-tip flux under interpole.

Suppose the center of the slot containing a coil side to be at a distance x cm from the center of the commutating pole of corrected breadth b'_t (Fig. 56). The tooth-tip flux within the limits of the pole is

$$\frac{(4\pi/10)z}{\dfrac{\delta'_t}{\mu_0 l'_t\left[\dfrac{b'_t}{2}+\left(x-\dfrac{r_1}{2}\right)\right]}+\dfrac{\delta'_t}{\mu_0 l'_t\left[\dfrac{b'_t}{2}-\left(x+\dfrac{r_1}{2}\right)\right]}}$$

$$= \frac{4\pi\mu_0}{10} z \frac{l'_t}{4\delta'_t}\left[\frac{(b'_t - r_1)^2 - 4x^2}{b'_t - r_1}\right]$$

and the average inductance, both sides of the coil being included, is

$$L_2 = 2 \times \frac{1}{\frac{1}{2}(b'_t - r_1)} \times \frac{4\pi\mu_0}{10^9} z^2 \frac{l'_t}{4\delta'_t}\int_0^{(b'_t - r_1)/2}\left[(b'_t - r_1) - \frac{4x^2}{b'_t - r_1}\right]dx$$

$$= \frac{4\pi\mu_0}{10^9} z^2 l'_t \frac{(b'_t - r_1)}{3\delta'_t}$$

In the above equations, δ'_t is the corrected gap length $\delta_t \dfrac{t}{t - \sigma r_1}$, all dimensions being expressed in centimeters.

If $l' > l'_t$, there must be added to the above expression, a term

$$\frac{4\pi\mu_0}{10^9} z^2 (l' - l'_t) \times 1.46 \log_{10}\left[1 + \frac{\pi(\tau - b)}{2r_1}\right]$$

CHAPTER XIII

EFFICIENCY, RATING, AND HEATING*

1. Conventional and Measured Efficiency.—The efficiency of a machine is defined as the ratio of the useful power delivered by the machine to the total power received by it. Naturally, both input and output must be expressed in terms of the same unit of power before computing their ratio; thus in the case of generators the input is mechanical and the output is electrical, whereas in motors the input is electrical and the output is mechanical. The usual units of power are the watt, the kilowatt, and the horsepower, where

$$1 \text{ hp.} \equiv 33{,}000 \text{ ft.-lb. per min.} \equiv 746 \text{ watts}$$
$$1 \text{ kw.} \equiv 1000 \text{ watts} \equiv 1.34 \text{ hp.}$$

In practice the rating of generators is given in terms of kilowatts available at the terminals at the specified speed and voltage of the machine. In the case of motors it is customary to express the rating in terms of the number of horsepower available at the shaft at the specified speed and voltage (except in the case of railway motors).

Two distinct efficiencies are recognized in engineering specifications, the *conventional* efficiency and the *directly measured* efficiency. The conventional efficiency, which is defined in detail in subsequent articles, is used unless otherwise specified. In either case, when the efficiency is referred to without specific reference to the load conditions, it is to be understood that it is the efficiency at the rated load of the machine and at a temperature of 75°C.

The conventional efficiency is computed from the fundamental relations

* See American Standards for Rotating Electrical Machinery, approved by American Standards Association, Jan. 6, 1936, published as Bulletin C50, A.I.E.E. Standards, from which the substance of Arts. 2, 3, 13, 14, and 15 has been taken. See also NEMA Motor and Generator Standards No. 38-49, May, 1938.

$$\text{Input} = \text{output} + \text{losses}$$
$$\text{Output} = \text{input} - \text{losses}$$

whence

$$\text{Efficiency} = \frac{\text{output}}{\text{input}} = \frac{\text{output}}{\text{output} + \text{losses}} = \frac{\text{input} - \text{losses}}{\text{input}} \tag{1}$$

If it were possible to measure all the losses corresponding to particular values of the load throughout the working range of the machine, the efficiency computed from the above equation would be the true efficiency. But for reasons to be explained later it is not possible to measure all the individual losses with entire accuracy, and in such cases conventional values are arbitrarily assigned to one or more of them; hence the origin of the term "conventional efficiency."

Special cases may arise when it is desirable to determine the efficiency of large machines by direct measurement. Two methods for making such measurements are recognized, namely:

1. Simultaneous measurements of input and output are made by using carefully calibrated electrical instruments to measure electrical input or output and a dynamometer or its equivalent to measure mechanical input or output.

2. The total losses are determined by suitably enclosing the machine and measuring the total quantities and temperature rises of the cooling media for the machine and its bearings. This is a calorimeter method.

Direct measurements of efficiency are relatively simple and inexpensive in the case of small machines and are to be preferred to conventional determinations because of their greater accuracy; the losses being relatively larger, the magnitudes of input and output are sufficiently different so that small percentage errors in each of them will not seriously affect their quotient. But in large machines the losses constitute a much smaller percentage of the total power, and small percentage errors in the measured values of input and output introduce much larger percentage errors in the computed efficiency. Moreover, direct measurements of input and output are impracticable in the case of large machines, because of the prohibitive cost of supplying the power.

Machines that are too large to be tested by means of dynamometers or brakes may under certain conditions be subjected

to direct measurement of efficiency by means of the *circulating-power* method, also called the *opposition* or *loading-back* method, provided that two machines of the same type and rating are available. Thus, in the Blondel opposition method (Fig. 1), the two identical machines are mechanically coupled together, preferably by direct connection, with a small auxiliary motor on the common shaft. Their armatures are connected electrically in opposition, but with a small booster, or other source of electrical power, in the circuit so that its e.m.f. acts in the same direction as that of the generator unit. The two field windings are connected to a supply circuit of suitable voltage. The set

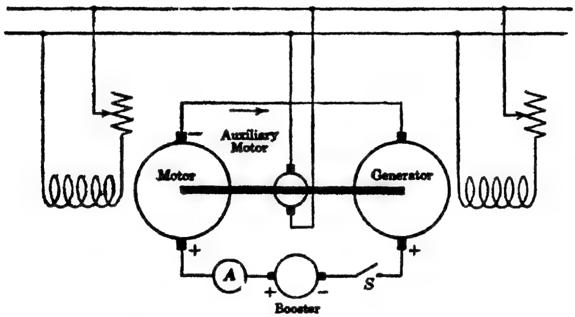


FIG. 1.—Blondel opposition method for measuring losses.

is started by means of the auxiliary motor and the speed adjusted to the rated value. With switch S open, the field windings are connected to the supply circuit and the exciting current is adjusted until both machines develop the rated terminal voltage. Under these circumstances the auxiliary motor supplies the friction and core losses of both machines, and these losses may be determined by measuring the power input to the auxiliary motor and subtracting the losses in the motor itself. On closing switch S, but with the booster not excited, no current will flow, since the generated e.m.fs. of the two main machines are equal and opposite; if then the voltage of the booster is increased, the balance is disturbed and any amount of current may be made to circulate between the two machines, one of which will act as a generator, the other as a motor. The auxiliary motor will continue to supply the friction and core losses, and the booster will supply the copper (ohmic) losses in the two main armatures.

It is obvious that the booster must have sufficient current-carrying capacity to operate in series with the main machines up to the highest value of current they are intended to carry.

This method has the advantage that both machines may be tested up to any desired rating without drawing from the supply circuit any more power than is represented by their combined losses plus the losses in the auxiliary motor and in the booster set. It gives results that are theoretically correct, but it is open to the objection that auxiliary machines are required.

The Kapp opposition method (Fig. 2) dispenses with the auxiliary machines but does not give precise results. The two machines are mechanically coupled, with their armatures

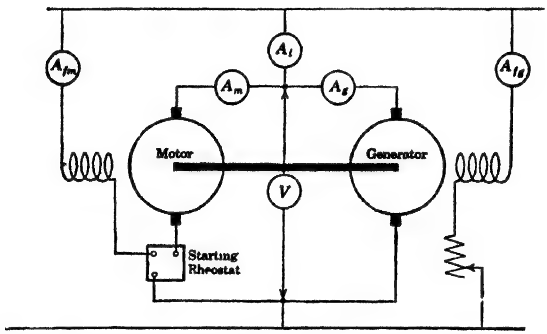


FIG. 2.—Kapp opposition method for measuring losses.

in electrical opposition, one of them being provided with a starting rheostat in order to bring the set up to speed. All the losses will then be supplied from the line. By increasing the excitation of machine G, it will act as a generator supplying electrical power to the motor M, and the latter will in turn drive the generator. The electrical input from the line supplies the combined losses of both G and M, but these are not equal since the excitation of G must be greater than that of M. It follows that this method cannot give precise results so far as efficiency measurements are concerned, but it is a very convenient way to load the two machines without drawing upon the supply line for more than the sum of the losses. It is useful for determining the rise of temperature of a machine under load conditions without incurring heavy expense for the energy consumed.

2. Measurement of Losses.—With the exception of the I^2R (copper) losses, the individual losses are measured by one of the following methods:

1. The *mechanical input method* consists in driving the machine under test with a motor whose output can be determined from the electrical input supplied to it. This method is sometimes called the *rated motor* method; in effect, the driving motor is calibrated, and its output is equal to the power consumed by the losses of the machine under test.

In using this method, the rating of the calibrated motor should not greatly exceed the magnitude of the loss to be measured. For if a large rated motor is used to measure a relatively small loss, variations in the indeterminable losses of the rated motor (such as its brush contact loss) may mask the loss intended to be measured.

2. The *retardation method* is used when the mechanical input method is inconvenient, as in the case of large machines after their final installation. It is particularly applicable to machines having a large moment of inertia. This method has been described in Art. 24, Chap. XI, and leads to the formula

$$\text{Loss} = 0.462 \times 10^{-3} Jn\frac{dn}{dt} \quad \text{watts}$$

where J is the polar moment of inertia in lb.-ft.^{2} and n is in r.p.m. If the retardation curve (Fig. 61, Chap. XI) is fairly flat, the value of dn/dt corresponding to the speed n can be obtained by observing the time t_1 (in seconds) when the speed is $(n - \Delta n)$ and again at time t_2 when the speed is $(n + \Delta n)$, where Δn is small in comparison with n. Then,

$$\frac{dn}{dt} = \frac{2\Delta n}{t_2 - t_1}$$

and the power loss is

$$P = 0.924 \times 10^{-3} \frac{Jn\,\Delta n}{t_2 - t_1} \quad \text{watts} \tag{2}$$

3. The *electrical input method* consists in supplying the machine from a suitable electrical source and measuring the input with electrical instruments.

4. The *calorimeter method,* suitable only for large, high-speed turbogenerators, requires that the machine under test be com-

pletely enclosed; the losses can be found from the temperature rises and the volumes of the cooling media or by comparing the temperature rise of the cooling medium due to the unknown loss with the rise caused by a known loss.

3. Schedule of Losses.—The following losses, some or all of which may occur in d-c generators and motors, must be included in computations of conventional efficiency, with the exceptions indicated:

1. The *shunt field loss* $I_s^2R_s$ is computed on the basis of the measured resistance R_s, corrected to a temperature of 75°C. The current I_s is that which corresponds to the rated voltage and speed; in the case of adjustable-speed motors the base speed is to be used unless otherwise specified.

2. The *rheostat loss* is charged against the plant of which the machine is a part, not against the machine itself. In the case of a single machine, constituting the entire plant, this is equivalent to charging the machine with its own rheostat loss.

3. *Exciter losses*, in plants where several machines are excited from separate generators or circuits, are charged against the plant as a whole, not against the individual machine.

4. *Friction and windage loss*, excluding brush friction, is the power required to drive the unexcited machine at its normal speed, with the brushes lifted.

The windage loss, if measured separately from bearing friction, may, if so specified, be corrected to standard atmospheric conditions by multiplying it by the factor $\frac{690(1 + 0.0040t)}{H}$, where H is the barometric pressure during the test (in millimeters of mercury) and t is the temperature of the ambient air, in degrees centigrade. This correction may also be made to allow for the decreased windage loss at high altitudes, on the basis of the following values of H for different altitudes:

Altitude, feet	0	2000	4000	6000	8000	10,000	12,000	14,000
H	760	706	655	608	564	520	483	445

In case machines are furnished with incomplete bearings, as when a generator is to be direct-connected to an engine made by a different manufacturer, the friction and windage losses are

determined by a factory test with temporary bearings of the size and type normal to the machine when operated alone. Increased loss caused by direct-connected flywheels or other direct-connected apparatus is charged against the plant as a whole, not against the individual machine.

5. *Brush friction loss*, because of its highly variable nature. is conventionalized by using the following average values:

	Watts per Square Inch of Brush Contact Surface per 1000 Ft. per Min. Peripheral Speed
Carbon and graphite brushes.......	8.0
Metal graphite brushes............	5.0

In measuring brush friction for research or laboratory purposes, the machine is driven by a rated motor whose output is determined (1) when the machine is driven with the brushes in contact with the commutator but with the field unexcited; (2) when the brushes are lifted and the field is unexcited. The difference between the two readings is the brush friction.

6. *Ventilating loss*, represented by the power required to circulate air through the machine (and through the ventilating system if one is provided) is charged against the machine, except under the following conditions:

a. If the power required to force the air through the ventilating system external to the machine is appreciable, this part of the total ventilating loss is charged to the plant.

b. If an external fan is provided to supplement the fanning effect built into the machine itself, for the purpose of driving air through long or restricted ducts, the extra loss is charged to the plant.

c. In cases where a single external fan serves several machines, the procedure is to be determined by special agreement.

When a machine is to be installed as part of a system involving elaborate ventilation, it is impracticable to make a factory test that will exactly duplicate actual operating conditions. In such cases the ventilating losses are measured without the ventilating duct and cooler but with the fan running as in service.

7. *Core loss* is determined by measuring the power required to drive the machine at normal speed when excited to produce a

terminal voltage equal to the rated terminal voltage corrected for *IR* drop. From this is subtracted the power required to drive the unexcited machine at the same speed.

8. *Armature I^2R loss* is the square of the armature current at rated voltage, multiplied by the armature resistance, as measured by direct current, and corrected to 75°C.

9. *Series field-winding I^2R loss* is the product of the square of the current in the series-connected field windings and the measured resistance corrected to 75°C. In case shunts are used around the windings, the multiple resistance is to be used.

10. *Brush contact loss*, because of its variable nature, is conventionalized by assuming that the voltage drop at the brushes of each polarity remains constant at all loads, as follows:

Carbon and graphite brushes, shunts attached .	1 volt
Carbon and graphite brushes, without shunts ..	1½ volts
Metal-graphite brushes, shunts attached	¼ volt

11. *Stray load losses* are caused by (1) eddy currents in the armature conductors and in general in any winding of large cross-section subjected to the inductive action of a flux whose density varies from point to point of the conductor cross-section; (2) additional core loss produced by distortion of the magnetic flux by load current; (3) the short-circuit currents in the coils undergoing commutation. All these losses are indeterminable, and their effect is allowed for by assuming that they aggregate 1 per cent of the output.

Inspection of the list of individual losses shows that some of them may be constant, or nearly constant, within the working range of the machine, whereas others are variable with the load, and that those losses which are substantially constant in one type of machine may be variable in another, depending upon its characteristics. For example, the friction and windage loss is constant in constant-speed generators and motors but is variable in such machines as series motors; the I^2R loss is practically constant in the field winding of a shunt motor and in the shunt winding of a long-shunt compound generator adjusted for constant terminal voltage, but is variable in all series field windings and in the armature; the core loss is nearly constant in constant-flux machines but variable in series motors and generators, where the flux changes with the load.

4. Efficiency and Losses in Constant-potential, Constant-speed Machines.—For the purpose of computing the conventional efficiency in constant-potential, constant-speed machines such as shunt generators and motors, the losses may be grouped into two classes, namely, (1) those which remain substantially constant at all loads, and (2) those which are variable with the load.

The constant loss includes iron or core losses, friction and windage, and I^2R or ohmic losses in the shunt winding. The ohmic loss in the shunt field winding will not change with the load if it is connected to the main terminals, if there is disregarded a possible change in the resistance of the winding due to change of temperature; but if the machine is a constant-potential, short-shunt compound generator, the difference of potential between the terminals of the shunt winding will rise slightly with increasing load and correspondingly increase the shunt field loss.

The variable loss includes ohmic losses in the armature and in field windings in series therewith, such as series-field coils and commutating and compensating windings.

The constant loss of a machine of the shunt type can be determined experimentally in a simple manner by running it as a motor, without load, at its rated voltage V and measuring the no-load current input to its armature $(I_a)_0$ and to the field winding (I_s). The total current input under this no-load condition is $(I_a)_0 + I_s$, and the total power input is $V[(I_a)_0 + I_s]$. The only loss in addition to the constant loss is the ohmic loss in the armature $(I_a)_0^2R_a$; and since the output is zero, the constant loss is

$$P_c = V[(I_a)_0 + I_s] - (I_a)_0^2R_a \tag{3}$$

This test should be made after the machine has been running under load for a sufficient time to raise its temperature to the normal operating value; and the value of the armature resistance R_a should also be measured at working temperature. The brushes should be placed so that commutation takes place in the neutral axis.

The value of the constant loss determined by Eq. (3) includes the core loss, friction and windage, and ohmic loss in the shunt winding. The latter is equal to VI_s, so that the combined value of core loss, friction, and windage is given by

$$P_c - VI_s = V(I_a)_0 - (I_a)_0^2R_a \tag{4}$$

The core loss of a machine of the constant-flux, constant-speed type does not remain absolutely constant under varying conditions of load, as is assumed in computing conventional efficiency. The hysteresis loss in any given element of volume of the core varies as the one and six-tenths power of the flux density, and the eddy-current loss varies as the square of the flux density (see Art. 10). The effect of armature reaction is to distort the flux, thus increasing the flux density in some portions of the core and decreasing it in others; the net result of this shift is to increase the total core loss, even though the total flux remains unaltered. This change in the core loss between no load and full load is a part of the stray load losses.

As an example of the method of computing conventional efficiency, let it be assumed that the following data have been determined from a 250-kw., 550-volt, flat-compound, long-shunt generator (current output at full load, 455 amp.).

Constant loss:

Core loss	3500 watts
Friction and windage	2000 watts
Shunt-field loss	2250 watts
Total	7750 watts
Resistance of armature and series field (hot)	0.03 ohm

For any given value of current output I, the armature current is

$$I_a = I + I_s = I + {}^{2250}\!/_{550} = I + 4.09$$

The output in watts is $550I$ and the losses are made up of the following components: constant loss, 7750 watts; copper loss in armature and series field, $I_a^2 \times 0.03$; brush contact loss, $2 \times I_a$; and the stray load loss, computed as 1 per cent of the output, or $0.01 \times 550I$, in accordance with A. I. E. E. Standards. Hence, for any output current I, the efficiency is

$$\eta = \frac{550I}{550I + 7750 + (I + 4.09)^2 \times 0.03 + 2(I + 4.09) + 5.5I}$$

Upon assuming values of I and substituting in this equation, data are obtained from which the curves of Fig. 3 are plotted.

It will be observed that the efficiency rises rapidly from zero value at no load and approaches a maximum value; if the calcu-

lations were carried beyond the limits shown in the figure, the efficiency would begin to fall for the reason that the total losses increase at a greater rate than the first power of the current output.

In the above problem, illustrative of the case of a generator, it will be noted that the output is proportional to the line current, on the assumption that the terminal voltage is constant. If the voltage is not constant, line current will not be proportional to

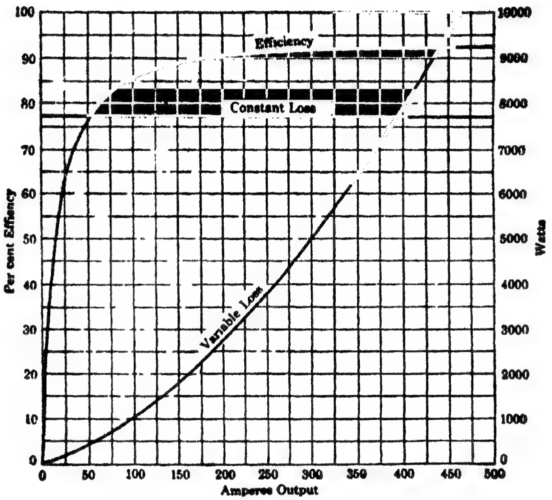


FIG. 3.—Losses and efficiency.

power output; and if it is desired to plot efficiency and losses in terms of power output instead of in terms of current, suitable corrections would have to be made. For example, suppose that a machine has a rising external characteristic, the terminal voltage rising linearly from 110 volts at no-load to 115 volts at full-load current; at full-load current output, the power output is $115I$ watts; at half of full-load current output, the power output is $112.5 \times \frac{1}{2}I = 56.75I$, which is 49.3 per cent of full-load output, instead of exactly 50 per cent.

In computing motor performance from the measured no-load losses, it must be remembered that the current input is not proportional to the power delivered by the motor. Thus, in a shunt motor, the power input is VI and the power output is

$$P = VI - P_c - I_a^2R_a$$

But

$$I = I_a + I_s$$

and

$$\begin{aligned} P_c &= \text{core loss} + \text{friction and windage loss} + \text{shunt-field loss} \\ &= P_{h+e} + P_f + VI_s \end{aligned}$$

where P_{h+e} = core loss.

P_f = friction and windage loss.

It follows, therefore, that

$$\begin{aligned} P &= V(I_a + I_s) - P_{h+e} - P_f - VI_s - I_a^2R_a \\ &= VI_a - (P_{h+e} + P_f) - I_a^2R_a \end{aligned}$$

and

$$I_a = \frac{V}{2R_a} \pm \sqrt{\left(\frac{V}{2R_a}\right)^2 - \frac{P + P_{h+e} + P_f}{R_a}}$$

In this equation the minus sign before the radical is the only one that has a physical meaning. As an example, consider a 110-volt, 10-hp. motor which has an armature resistance of 0.11 ohm and a field resistance of 40 ohms and which takes an armature current of 3.1 amp. at no load. The combined value of core loss and friction and windage is

$$P_{h+e} + P_f = 110 \times 3.1 - (3.1)^2 \times 0.11 = 340 \quad \text{watts}$$

The armature current intake at full load (P = 10 hp. = 7460 watts) is

$$I_a = \frac{110}{2 \times 0.11} - \sqrt{\left(\frac{110}{2 \times 0.11}\right)^2 - \frac{7460 + 340}{0.11}} = 76.8 \quad \text{amp.}$$

and the line current is

$$I = I_a + I_s = 76.8 + 110/40 = 79.55 \quad \text{amp.}$$

When the machine is delivering half its rated load, or 5 hp. = 3730 watts, the line current is 42.25 amp., or considerably more than half the line current at full load.

5. Efficiency of Conversion. Electrical and Mechanical Efficiency.—The efficiency discussed in the preceding articles may be thought of as the *over-all* efficiency of the machine, since it is the ratio of net output to gross input. But in analyzing the

operation of a generator or a motor, three additional subsidiary values of efficiency may be distinguished, as follows:

In a generator, the total mechanical power input is not converted into electrical power, but a portion is dissipated as mechanical loss in friction and windage and in core loss; the core loss in this case acts as a brake and is in fact equivalent to an increase in the frictional resistance. The difference between the total mechanical input and these losses is then converted into electrical form in the armature; a portion of the electrical power thus developed is lost in the ohmic resistance of the various windings of the machine and in the brush contact resistance, and the balance is available as net output. It follows, therefore, that

Electrical power developed = mechanical input − friction and windage − core loss

so that the *efficiency of conversion* is

$$\eta_c = \frac{\text{electrical power developed}}{\text{mechanical power input}} = \frac{\text{output} + \text{ohmic losses}}{\text{mechanical power input}} \tag{5}$$

Similarly, the ratio of net electrical power output to the total electrical power developed in the armature may be called the *electrical efficiency* η_e, and

$$\eta_e = \frac{\text{electric power output}}{\text{electrical power developed}} = \frac{\text{output}}{\text{output} + \text{ohmic losses}} \tag{6}$$

Comparison of equations (5) and (6) with the expression for over-all efficiency,

$$\eta = \frac{\text{output}}{\text{input}}$$

shows that

$$\eta = \eta_c \eta_e \tag{7}$$

In the case of motors, a part of the electrical power input is initially lost in the ohmic resistance of the windings and brush contacts, and the balance is converted into mechanical power; but of the total mechanical power developed, a part is lost in friction and windage and in core loss, the latter again acting as a

brake on the machine and being equivalent to an increase in the friction load. Consequently the following relations hold:

Mechanical power developed = electrical input − ohmic losses

and

Mechanical output = mechanical power developed − friction and windage − core loss

Hence, the *efficiency of conversion* is

$$\eta_c = \frac{\text{mechanical power developed}}{\text{electrical power input}}$$
$$= \frac{\text{electrical input} - \text{ohmic losses}}{\text{input}} \tag{8}$$

and the *mechanical efficiency* of the motor is

$$\eta_m = \frac{\text{mechanical power output}}{\text{mechanical power developed}}$$
$$= \frac{\text{output}}{\text{output} + \text{friction and windage} + \text{core loss}} \tag{9}$$

It follows that

$$\eta = \eta_c\eta_m \tag{10}$$

6. Condition for Maximum Efficiency. In any machine in which the total losses comprise a part that remains constant independently of the load and a part that varies as the square of the load, *maximum efficiency* will occur at that particular value of load at which the fixed and variable losses are equal. Thus, if the load (or output) is P, the constant loss is P_c, and the variable loss is kP^2, where k is a constant,

$$\eta = \frac{P}{P + P_c + kP^2} \tag{11}$$

Differentiating η with respect to the load P and equating the result to zero to find the condition for maximum value of η,

$$\frac{d\eta}{dP} = \frac{(P + P_c + kP^2) - P(1 + 2kP)}{(P + P_c + kP^2)^2} = 0$$

whence

$$P_c = kP^2 \tag{12}$$

and the maximum efficiency is

$$\eta_{max.} = \frac{P}{P + 2P_c} \tag{13}$$

The above relations are very nearly those which occur in machines of the constant-potential, constant-speed type, such as shunt generators and motors. The constant loss includes friction, windage, ohmic loss in the field winding, and core loss; the variable loss is equal to $I_a^2R_a$ plus the brush contact loss (conventional value $= 2I_a$ watts). If the brush contact loss is absorbed into the ohmic loss in the armature by assigning to R_a a suitably increased average value, the variable loss is seen to be nearly proportional to I_a^2, and I_a is in turn nearly proportional to the output. Upon examining the actual conditions somewhat more in detail, the facts with regard to the shunt *generator* may be summarized as follows:

Let V be the terminal voltage, and let I, I_a, and I_s represent, respectively, the line current, armature current, and shunt field current; then,

$$I_a = I + I_s$$

and

$$\eta = \frac{VI}{VI + P_c + I_a^2R_a} \tag{14}$$

where P_c is the constant loss and R_a is the value of armature resistance modified to include brush contact resistance. Since I_s is generally only a small percentage of I at values of the load approaching full-load rating, very little error will be introduced by writing

$$\eta = \frac{VI}{VI + P_c + I^2R_a}$$

whence

$$\frac{d\eta}{dI} = \frac{(VI + P_c + I^2R_a)V - VI(V + 2IR_a)}{(VI + P_c + I^2R_a)^2}$$

and if

$$\frac{d\eta}{di} = 0$$

$$P_c = I^2R_a \cong I_a^2R_a \tag{15}$$

or the efficiency is a maximum for that value of current output at which the constant and variable losses are equal.

Similarly, in a shunt *motor*, the input is VI, and the output is

$$VI - P_c - I_a^2R_a$$

whence the efficiency is

$$\eta = \frac{VI - P_c - I_a^2R_a}{VI} \cong \frac{VI - P_c - I^2R_a}{VI} \tag{16}$$

and

$$\frac{d\eta}{di} = \frac{VI(V - 2IR_a) - (VI - P_c - I^2R_a)V}{V^2I^2}$$

and since for maximum efficiency $d\eta/dI = 0$, maximum efficiency will occur when

$$P_c = I^2R_a$$

In this case the differentiation is made with respect to the input current I as independent variable, whereas in the case of the generator the output current was the independent variable.

The above conclusions are not entirely rigorous for either the generator or the motor, but they are sufficiently accurate for practical purposes. A somewhat more accurate analysis might be based on the fact that the variable loss is made up of parts which depend upon the first power of the current as well as upon the second power; thus, the core loss P_{h+e} (due to hysteresis and eddy currents), instead of remaining constant, may be assumed to vary linearly with the load current, or

$$P_{h+e} = (P_{h+e})_0 \pm cI \tag{17}$$

where $(P_{h+e})_0$ is the core loss at no load and c is a constant; the brush contact loss is nearly proportional to I_a, and therefore also to the line current I; the shunt field loss is V^2/R_s in plain shunt and in long-shunt compound machines and is $(V \pm IR_f)^2/R_s$ in short-shunt machines, the positive sign being used in the case of generators, the negative sign in the case of motors; the series field loss is $I_a^2R_f$ in long-shunt machines and I^2R_f in short-shunt machines. The summation of all the losses therefore includes a constant term, a term that varies directly with the line current,

and a term that varies as the square of the current; hence, the efficiency is given by an expression of the form

$$\eta = \frac{VI}{VI + P_c + C_1 I + C_2 I^2}$$
$$= \frac{V}{V + (P_c/I) + C_1 + C_2 I} \tag{18}$$

For maximum efficiency the denominator must be a minimum; hence, upon differentiating the denominator and equating to zero, the condition for maximum efficiency is found to be

$$-\frac{P_c}{I^2} + C_2 = 0$$

or

$$P_c = C_2 I^2 \tag{19}$$

from which it follows that for maximum efficiency the constant loss should be equal to that part of the variable loss which varies as the square of the line current.

7. Location of Point of Maximum Efficiency.—From the preceding article it is clear that by a proper choice of the relation between the fixed and the variable losses the point of maximum efficiency may be made to fall at any desired output. For example, assume that the total losses consist of a constant term P_c and a term variable with the square of the load. Let the rated full-load output be P, and let the constant loss be cP, where c is any fractional part of the output, and let the variable loss at full load be vP, where v is any fractional part of the output. Then the efficiency at full load is

$$\eta = \frac{1}{1 + c + v} \tag{20}$$

Let xP be the output at which it is required that the efficiency be a maximum. The variable loss will be $x^2(vP)$, and for maximum efficiency

$$x^2(vP) = cP$$

or

$$x = \sqrt{\frac{c}{v}} \tag{21}$$

For example, let it be required to divide the total losses in such a way that the maximum efficiency shall occur at three-fourths load and the efficiency at rated full load shall be 85 per cent. Then,

$$\frac{1}{1 + c + v} = 0.85$$

$$x = \sqrt{\frac{c}{v}} = 0.75$$

from which $c = 6.35$ per cent and $v = 11.3$ per cent.

It is seen that if the fixed losses, represented by c, are relatively large, and the variable copper losses, represented by v, are small, maximum efficiency will probably occur beyond full load. To make the maximum efficiency occur at a fractional part of full load, the copper losses should be large compared with the fixed losses. Thus, if it is known that a machine is to be operated for considerable periods at light loads and only occasionally at full load or overloads, it should be so designed as to have a relatively high armature resistance in order to make the efficiency a maximum at or near the point of average load. In other words, a high-efficiency machine is not necessarily the best machine for all conditions of service, for its high efficiency involves additional first cost which can be justified only if the saving in operating cost is sufficiently great.

8. All-day Efficiency.—*The all-day efficiency* of a machine is the ratio of the net *energy output* to the total *energy input* during a working day. Inasmuch as charges for electrical service are based largely on energy consumption (kilowatt-hours), it is important that the all-day efficiency of a motor that runs continuously should be as high as possible. The all-day efficiency of a machine is dependent to a large extent upon the shape of its *load curve A* (Fig. 4) and is also affected by the ratio of its fixed and variable losses. The ordinates of the load curve represent power output and the abscissas represent time, and so the area under the curve is proportional to the energy output. If this load is carried by a motor of which the fixed losses are 5 per cent and the variable losses are 10 per cent of its rated output, the power input will vary as shown by curve B; if the fixed and variable losses are, respectively, 10 and 5 per cent of the rated output, the power input will be given by curve C. In the former case the

all-day efficiency is 85.8 per cent, in the latter 81.7 per cent. The difference between the two becomes greater and greater, in favor of the machine with the lower fixed loss, as the period of light load increases; for example, if the machine runs for 9 hr. at 10 per cent load and 1 hr. at full load, the all-day efficiency

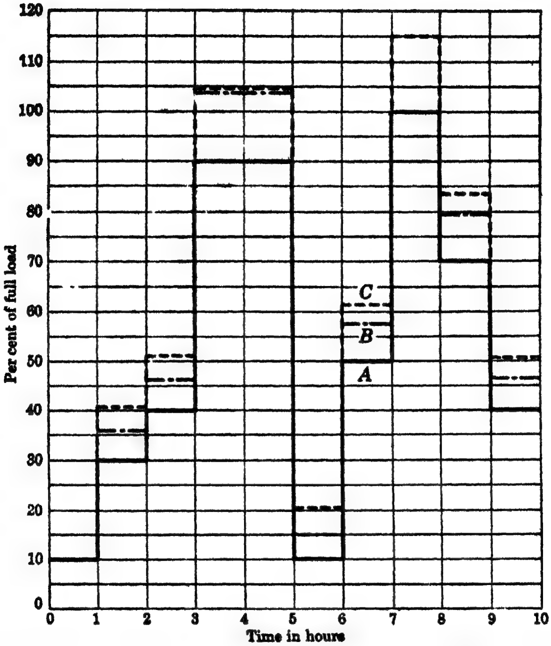


FIG. 4.—*A*, load curve; *B* and *C*, input curves.

of the first machine is 75.7 per cent as against 64.3 per cent for the second.

9. Efficiency and Losses in Variable-flux, Variable-speed Machines.—In machines in which the speed or the flux, or both of them, are inherently variable under operating conditions, as for example in series motors, the core loss and friction and windage loss are also variable. The combined value of core loss and friction and windage loss in a series motor can be found experimentally by the following method:

Separately excite the field winding from any suitable source, and start the motor without load by gradually increasing the voltage impressed upon the armature. After the motor starts,

increase the excitation until the field current has the highest value it will have under load conditions, and adjust the armature voltage to a value at which a reading is desired. Note the field current, armature current and voltage, and speed. Keeping the excitation constant, adjust the armature voltage to two or three different values, and for each setting take readings as before. The armature voltages ordinarily used in this test are 250, 400, and 550 volts for 550-volt motors and 300, 450, and 650 volts for 650-volt motors. Repeat this series of readings for a number of other values of field current, down to the lowest field current consistent with safe speed. For each setting the combined value of core loss and friction and windage loss will

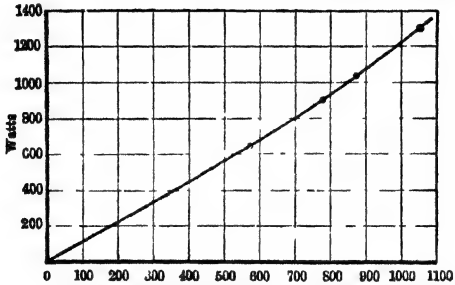


FIG. 5.—Friction and windage as a function of speed. Railway motor test.

be equal to the power input to the armature minus the ohmic loss in the armature winding and at the brush contact.

If it is desired to separate the total loss thus determined into core loss and friction and windage, the machine should be run as a series motor without load and at reduced voltage. By varying the impressed voltage through a sufficient range to cover the working range of speed and taking simultaneous readings of impressed voltage, current, and speed, the friction and windage corresponding to any speed may be taken as equal to the power supplied, less the ohmic losses in the armature and field windings, the core loss being negligible under these test conditions. The core loss at any given speed is then equal to the difference between the total loss, as determined by the first test, and the friction and windage loss at the same speed, as found in the second test. In this manner curves like Figs. 5 and 6 may be determined;*

* "Motor and Generator Testing," Sec. 8, pp. 8 and 11, Westinghouse Electric and Manufacturing Company, July, 1913.

data taken from these curves can be used to calculate the efficiency at any load.

10. Core Losses. 1. *Hysteresis Loss.* *a. In the Armature Core.*—The relative motion between the armature core and the magnetic field produces a periodic reversal of the magnetism of the core, thereby giving rise to a loss of power through molecular

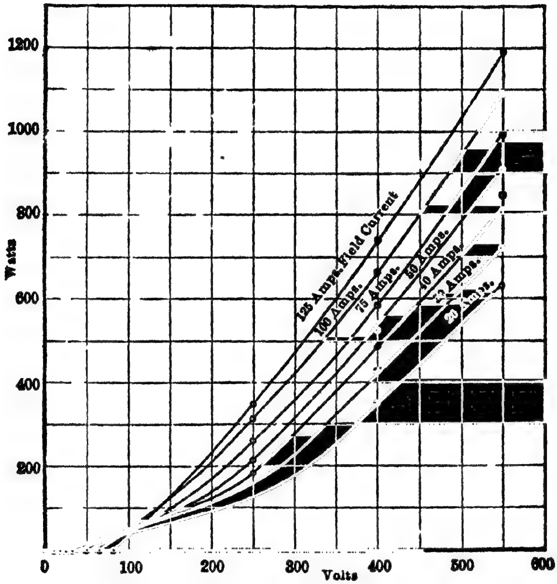


Fig. 6 — Core loss of series motor.

friction in the mass of the armature core. This *hysteresis* loss can be represented by an empirical equation due to Steinmetz,

$$P_{ha} = \eta f V B_a^{1.6} \quad \text{watts} \tag{22}$$

where η = a constant depending upon the material of the core.

$f = pn/120$ = number of magnetic cycles per second.

V = volume of the core.

B_a = maximum value of the flux density in the core.

If c.g.s. units are used in the above equation (volume in cubic centimeters and flux density in lines per square centimeter), $\eta = 0.0021 \times 10^{-7}$ for ordinary sheet steel; if volume is expressed in cubic inches and flux density in lines per square inch, η =

0.0017×10^{-7}. Since the weight W of the core is proportional to its volume, the equation for the hysteresis loss can also be written

$$P_{ha} = \eta f W B_a^{1.6} \quad \text{watts} \tag{23}$$

in which case $\eta = 0.0062 \times 10^{-7}$ if British units are used (weight in pounds, flux density in lines per square inch).

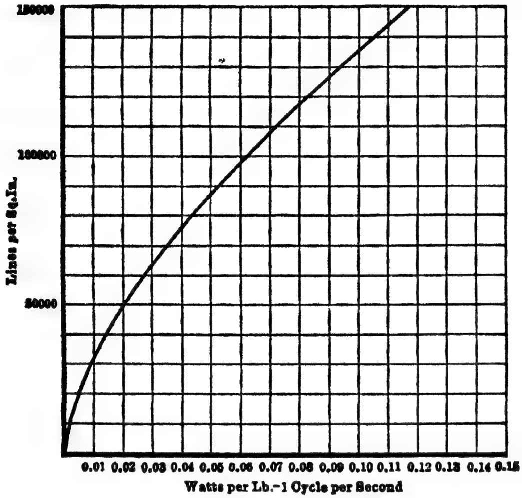


Fig. 7.—Curve of hysteresis loss.

The curve of Fig. 7 shows the variation of the hysteresis loss, expressed in watts per pound per cycle per second, as a function of the flux density expressed in lines per square inch, the foregoing value of η being used.

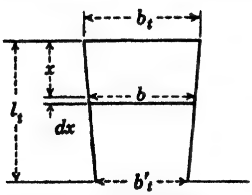


Fig. 8.—Computation of hysteresis loss in teeth.

b. In the Armature Teeth.—The flux density varies from section to section because of the taper of the teeth, and it is not correct to compute the hysteresis loss by substituting an average value of flux density in Eq. (23).

Consider an element dx (Fig. 8) at a distance x below the tip of the tooth; its volume is

$$dV = blk\,dx = \left(b_t - \frac{b_t - b'_t}{l_t}x\right)lk\,dx$$

where k is the lamination factor (0.85 to 0.90). On the assumption that the total flux is the same at all sections of the tooth, the flux density will vary inversely as the width of the section, or

$$B = \frac{b_t}{b}B_t$$

where B_t is the actual, or corrected, flux density at the tip of the tooth. The hysteresis loss in the element is

$$dP_{ht} = \eta f B^{1.6}\,dV = \eta fklb_t^{1.6}B_t^{1.6}\frac{dx}{b^{0.6}}$$

and the total loss per tooth is

$$P_{ht} = \eta kflb_t^{1.6}B_t^{1.6}\int_0^{l_t}\frac{dx}{\left(b_t - \frac{b_t - b_t'}{l_t}x\right)^{0.6}}$$

$$= 2.5\eta kflb_t^{1.6}B_t^{1.6}(b_t^{0.4} - b_t'^{0.4})\frac{l_t}{b_t - b_t'} \qquad (24)$$

Since the volume of a tooth is

$$V_t = \frac{b_t + b_t'}{2}kl_tl$$

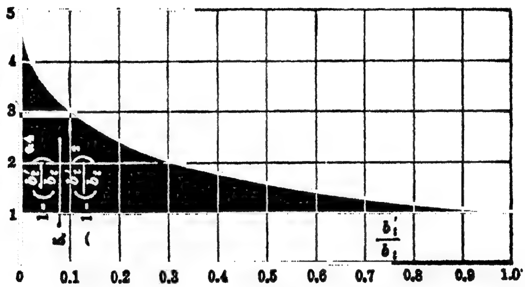


FIG. 9.—Correction factor, hysteresis loss in teeth.

the above expression can be written

$$P_{ht} = \eta fV_tB_t^{1.6} \times 5\frac{1 - (b_t'/b_t)^{0.4}}{1 - (b_t'/b_t)^2} \qquad (25)$$

The expression for the hysteresis loss in the teeth is similar to the general expression, but with the addition of the factor

$$5\,\frac{1 - (b_t'/b_t)^{0.4}}{1 - (b_t'/b_t)^2}$$

The ordinates of Fig. 9 give the value of this factor for various values of b'_t/b_t.

2. *Eddy-current Losses.*—That part of the core loss due to eddy, or Foucault, currents can be approximately calculated by the formula derived below, but it is more usual to determine the loss under known experimental conditions, for reasons that will appear later.

Consider a radial element Q, Fig. 10, of one of the armature-core stampings. Let the thickness of the stamping be t, and let ct be the radial depth of the core, where c is a numeric. When the element is in the vertical position OA the flux passing through its lateral walls is a maximum, and when it is in the horizontal axis OB the flux is zero. This change of flux occurs four times per revolution in a bipolar machine, or, in general, four times per magnetic cycle. The changing flux induces an alternating e.m.f. and sets up a corresponding alternating current which may be assumed to flow in paths like those indicated in the lower part of the figure; an elementary current path is bounded by similar rectangles of widths $2x$ and $2(x + dx)$, and lengths $2cx$ and $2c(x + dx)$. The change of flux through such an elementary circuit will be $4B_a \times 4cx^2$ lines per magnetic cycle, where B_a is the maximum flux density in the core, or $16B_acx^2f$ lines per sec., where f is the number of magnetic cycles per second. The average e.m.f. in the elementary circuit will be $16B_acfx^2 \times 10^{-8}$ volts, and the effective value will be

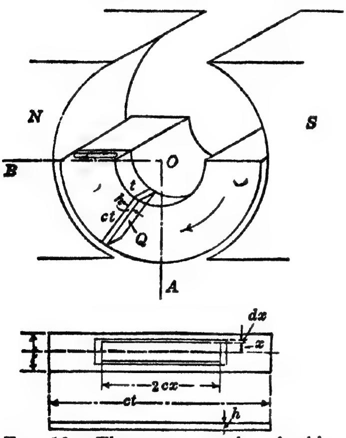


FIG. 10.—Elementary paths of eddy currents.

$$\frac{\pi}{2\sqrt{2}} \cdot 16B_acfx^2 \cdot 10^{-8}$$

The resistance of the path is

$$\gamma\left(\frac{4cx}{h\,dx} + \frac{4x}{hc\,dx}\right)$$

where γ is the resistivity of the material of the core. The loss in the elementary path is

$$i^2r = \frac{e^2}{r} = \frac{\pi^2}{8} \cdot \frac{(16B_acfx^2 \cdot 10^{-8})^2}{\frac{4\gamma x}{h\,dx}\left(c + \frac{1}{c}\right)} = \frac{8\pi^2hB_a^2f^2x^3\,dx}{\gamma} \cdot \frac{c^2}{c^2+1} \cdot 10^{-16}$$

and the total loss is

$$P_{ea} = \frac{8\pi^2hB_a^2f^2}{\gamma \times 10^{16}} \frac{c^2}{c^2+1} \int_0^{t/2} x^3\,dx = \frac{\pi^2}{8} \frac{hB_a^2f^2t^4}{\gamma \times 10^{16}} \frac{c^2}{c^2+1}$$

But hct^2 is the volume of the element; hence, the loss in watts is

$$P_{ea} = \frac{\pi^2}{8} \frac{B_a^2f^2t^2}{\gamma \times 10^{16}} \cdot \frac{c^2}{c^2+1} \times \text{volume of tooth} \tag{26}$$

This equation shows that the eddy-current loss varies as the square of the flux density, the square of the frequency of the magnetic reversals, and the square of the thickness of the laminations and inversely as the resistivity of the core material. The equation cannot, however, be relied upon for accurate results, because the actual distribution of the current may differ considerably from the assumed distribution, and the laminations are not perfectly insulated from each other, as has been tacitly assumed. Owing to these causes the actual measured loss will be 50 to 100 per cent greater than that computed from the formula. Thus, assuming

$$\begin{aligned} B_a &= 10{,}000 \text{ gausses} \\ f &= 60 \text{ cycles} \\ t &= 14 \text{ mils} = 0.0356 \text{ cm.} \\ \gamma &= 12 \times 10^{-6} \text{ ohm per cm.-cube} \\ \frac{c^2}{c^2+1} &= 1 \text{ (nearly)} \end{aligned}$$

the loss in watts per pound by the formula above is 0.27, whereas the observed value for these data in the case of annealed sheet steel is 0.44 watt per lb. Figure 11 shows the variation of eddy-current loss with flux density at frequencies of 25 and 60 cycles per sec. and for laminations 14 mils thick. The loss at other frequencies and thicknesses can be computed by observing that the loss varies as the squares of these quantities.

a. Eddy-current Loss in the Teeth.—With reference to Fig. 8, the eddy-current loss in an elementary section of a tooth is

$$
\begin{aligned}
dP_{et} &= \epsilon f^2 t^2 B^2 \times \text{volume} = \epsilon f^2 t^2 B^2 bkl\,dx \\
&= \epsilon f^2 t^2 b_t^2 B_t^2 kl \frac{dx}{b}
\end{aligned}
$$

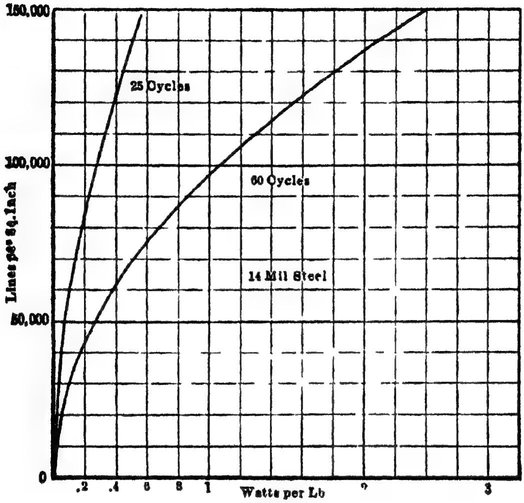


FIG. 11.—Curves of eddy-current loss.

where ϵ is the eddy-current constant. Integrating,

$$
\begin{aligned}
P_{et} &= \epsilon f^2 t^2 b_t^2 B_t^2 kl \int_0^{l_t} \frac{dx}{b_t - \dfrac{b_t - b_t'}{l_t}x} \\
&= \epsilon f^2 t^2 b_t^2 B_t^2 kl \frac{l_t}{b_t - b_t'} \log_e \frac{b_t}{b_t'} \\
&= \epsilon f^2 t^2 B_t^2 \times \text{volume of tooth} \times 2\frac{\log_e (b_t/b_t')}{1 - (b_t'/b_t)^2} \qquad (27)
\end{aligned}
$$

This equation differs from the original equation (26) in that it contains the additional factor

$$2\frac{\log_e (b_t/b_t')}{1 - (b_t'/b_t)^2}$$

the value of which is shown as a function of b_t'/b_t in Fig. 12

b. Eddy-current Loss in the Pole Faces.—Reference has been made in Chap. VI to the cause of the eddy-current loss in the pole faces. This loss is confined to a relatively thin layer at the face of the pole because the flux pulsations that produce

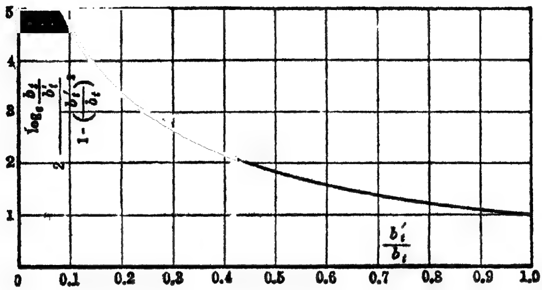


FIG. 12.—Correction factor, eddy-current loss in teeth.

the eddy currents are rapidly damped, in accordance with Lenz's law.

The flux pulsation at any given point in the pole face will pass through a complete cycle of changes in the time required for a point on the armature to move over a distance equal to the tooth pitch, that is, in a time $t' = t/(\pi\, dn/60)$ sec. This gives a frequency of $f_t = 1/t' = \pi\, dn/60t$ = number of teeth × r.p.s.

Figure 13 represents the variation of flux density at the pole face on the assumption that the curve of distribution is sinusoidal. The amplitude of the pulsation is $B' = (B_{max.} - B_{min.})/2$. If

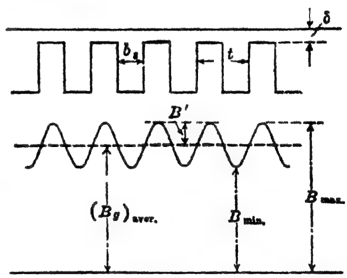


FIG. 13.—Variation of flux density opposite teeth and slots.

v = peripheral velocity of the armature in centimeters per second

μ = relative permeability of the material of pole face

ρ = resistivity of material of pole face in absolute e.m.u.

the pole-face loss in watts per square centimeter is*

* POTIER, "L'Industrie électrique," p. 35, 1905; RÜDENBERG, *Elektrotech. Z.*, **26**, 181, 1905.

$$P_p = \frac{B'^2}{8\pi}\sqrt{\frac{v^3t}{\mu\rho}} \times 10^{-7} = k^2\frac{B_g^2}{8\pi}\sqrt{\frac{v^3t}{\mu\rho}} \times 10^{-7} \tag{28}$$

where $k^2 = (B'/B_g)^2$ is a function of the relative dimensions of the airgap, teeth, and slots. Adams* has worked out the curve of Fig. 14 as giving fairly satisfactory values of k^2 in terms of the ratio b_s/δ. If British units are used (B_g in lines per square inch, v in feet per second, t in inches, and μ and ρ as above) the loss in watts per square inch of pole face is given as

$$P_p = 1.65 \times 10^{-7}\, k^2 B_g^2 \sqrt{\frac{v^3t}{\mu\rho}} \tag{29}$$

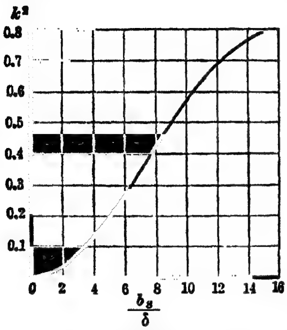


FIG. 14.—Constant for calculation of pole-face loss.

3. *Total Core Loss.*—Except in the case of special designs, it is more convenient to have access to the combined values of hysteresis and eddy-current losses than to compute each of these losses separately. Test methods lend themselves readily to the determination of the total core loss, and from the results of such tests curves like Fig. 15 can be prepared.†

11. Mechanical Losses. 1. *Bearing Friction and Windage.*—Though it is possible to compute the loss due to bearing friction, the loss due to windage involves so many complex variables that calculation of its magnitude is impossible. As it is likewise impossible to separate the combined value of the two losses as obtained by test measurements, they are always grouped as *friction and windage* loss. This loss varies from 1 to 3 per cent of the rated capacity in high-speed machines of moderate capacity and from 0.8 to 2 per cent in low-speed machines of moderate size. In large direct-connected machines the loss will be ½ to 1 per cent. In very high speed machines, such as turbo-generators, the loss due to windage will be increased.

Friction loss in the bearings varies with the $^3\!/_2$ power of the peripheral velocity of the shaft in the bearings, up to velocities of about 1800 ft. per min.; at higher velocities it varies directly

* ADAMS, LANIER, POPE, and SCHOOLEY, *Trans. A.I.E.E.*, **28**, 1133, 1909.
† A. GRAY, "Electrical Machine Design," p. 102.

with the velocity. The windage loss, as in the case of fans, varies as the third power of the speed. But in both cases these losses are independent of the load on the machine.

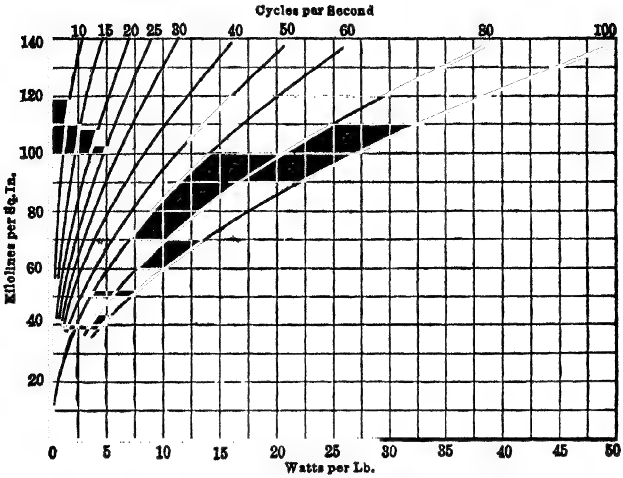


FIG. 15 —Total core loss.

2. *Commutator Friction Loss.*—Let

d_{com} = diameter of the commutator in inches.

A_b = total area of brush contact in square inches.

p_c = brush pressure in pounds per square inch.

f = coefficient of friction.

Then the brush friction loss (in watts) is

$$P_{bf} = \frac{\pi d_{com} n}{12} \cdot \frac{A_b p_c f}{33{,}000} \times 746 = 0.0059 d_{com} n A_b p_c f \qquad (30)$$

Ordinarily the value of p_c is 1.5 to 2 lb. per sq. in.; the coefficient of friction is about 0.3 for hard carbon brushes and about 0.2 for medium brushes; but exact figures cannot be given since the friction varies with the amount of oxide on the commutator and numerous other factors.*

12. Stray Load Losses. 1. *Eddy Currents and High-frequency Effects in the Armature Conductors.*—When large, solid armature conductors are used in open slots, different portions of the same section of the conductor may be simultaneously in fields of differ-

* J. NEUKIRCHEN, "Carbon Brushes," pp. 73–76, 81, 83.

ent strength. Under these conditions e.m.fs. of different magnitudes will be generated from point to point of the cross-section, and eddy currents will result. The loss due to these eddy currents may be 5 to 15 per cent of the loss due to the ohmic resistance. That is, so far as the armature copper loss is concerned, the effective armature resistance is 5 to 15 per cent greater than the true resistance. This loss may be minimized by stranding the conductors or by using smaller conductors in parallel.

Another source of loss in armature conductors is due to "skin effect," a phenomenon usually associated with alternating currents in conductors of large cross-section. Though it is customary to think of the current in the winding of a d-c machine as unidirectional, it is in reality alternating, as may be seen by

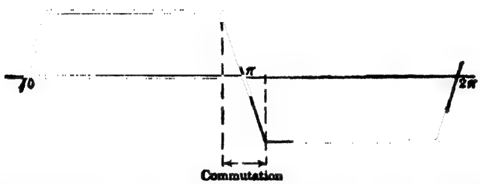


FIG. 16.—Alternating current in d-c armature winding.

referring to Fig. 16. This diagram represents in idealized manner the variation of current in the armature conductors of a d-c machine; it differs in form from the sinusoid usually associated with alternating current, but by Fourier's theorem the trapezoidal wave form of Fig. 16 can be resolved into a series of sinusoidal currents which have frequencies equal to 1, 3, 5, 7, 9, . . . times that of the trapezoidal wave. It follows that the actual current is equivalent to a series of superposed alternating currents of steadily increasing frequency. The higher the frequency, the greater is the tendency of the current to crowd to the outer layers of the conductor. Measurements* of the additional loss due to this effect indicate that in some cases it is equivalent to an increase of nearly 50 per cent in the effective resistance of a conductor of large section. The obvious remedy is to use stranded conductors where large cross-section is required.

2. *Miscellaneous Losses Due to Short-circuited Currents, etc.*—These are minor losses and cannot be computed. In testing,

* W. V. LYON, E. WAYNE, and M. L. HENDERSON, Heat Losses in Direct-current Armature Conductors, *Trans. A.I.E.E.*, **47** (No. 2), 589, 1928.

they are absorbed in the amount attributed to the stray load losses.

13. Service Conditions, Duty Classification, and Rating.*— The A.I.E.E. Standards* for d-c rotating machines (except generators of less than 560 watts output and fractional-horsepower motors) are based upon service conditions in which the temperature of the cooling medium does not exceed 40°C. and the altitude does not exceed 3300 ft. (1000 m.). In general, motors must operate successfully at rated load at any voltage not more than 10 per cent above or below the rated voltage, though not necessarily in accordance with the standards established for operation at normal rating. Generators having a 40°C. continuous rating (that is, designed for a temperature rise of 40°C.) and general-purpose motors must be able to carry continuously a load 1.15 times the rated load; the factor 1.15 is called the *service factor.*

The *rating* of a machine is the output, or designated operating limit based on definite conditions, assigned to it by the manufacturer, together with such other characteristics as speed, voltage, and current. The rating, which depends ultimately upon the temperatures reached by the materials used in the machine, varies for a given frame and type of construction with the type of *duty cycle* to which the machine is to be subjected. Thus, *continuous duty* demands operation at substantially constant load for an indefinitely long time; *short-time duty* calls for operation at substantially constant load for a short and definitely specified time; *intermittent duty* implies alternate periods of (1) load and no load, (2) load and rest, or (3) load, no load, and rest; *periodic duty* means intermittent duty in which the load conditions are regularly recurrent; *varying duty* implies operation at loads and for periods of time that may both be subject to wide variation.

The recognized kinds of rating are as follows:

Continuous rating defines the load that may be carried for an indefinite time without exceeding any of the established limitations.

Short-time rating defines the load that may be carried for the time specified in the rating without exceeding established limita-

* American Standards for Rotating Electrical Machinery, C-50, 1936.

tions; standard periods for short-time ratings are 5, 10, 15, 30, 60, and 120 min.

Nominal rating defines the constant load that, having been carried without causing further measurable increase in temperature rise, may be increased 50 per cent in amperes at a specified voltage, for a period of 2 hr., without causing any of the limitations for nominally rated machines to be exceeded.

Continuous with 2-hr. 25 per cent overload rating of a generator defines the load that can be carried continuously, immediately followed by a 25 per cent overload for 2 hr., without causing any of the established limitations to be exceeded.

The rating of a machine for intermittent, periodic, or varying duty is expressed in terms of the continuous, short-time, nominal, or continuous with 2-hr. 25 per cent overload rating that produces thermal effects as nearly as possible the same as those of the actual service.

In the absence of any specifications as to the kind of rating, the continuous rating is implied for all purposes except in the case of machines carrying traction loads, for which the nominal rating is implied. Generator ratings are expressed in watts or kilowatts available at the terminal at a specified speed and voltage, and motor ratings are expressed in horsepower available at the shaft at a specified speed and voltage.

The limitations referred to in the foregoing definitions of standard ratings relate to maximum temperatures and to commutation, overspeed, and insulation requirements. These are discussed in ensuing articles.

14. Temperature Limitations.—Theoretically, the output of a generator is limited only by the possibility of sufficiently reducing the resistance of the receiver circuit, at the same time maintaining the generated e.m.f. and supplying the driving power; practically, however, the capacity of the machine is limited by the ability of the insulation to withstand, without deterioration and for long periods, the maximum temperature caused by the heating due to I^2R and other losses, though in some cases the load limit may be determined by commutating conditions. For each kind of insulating material there is a limiting temperature above which deterioration is very rapid, but so far as the useful life of the insulation is concerned there seems to be no particular advantage in operating at temperatures below the safe

limits. In case the machine is designed to operate at temperatures well within the safe limits, there will be a margin between its rating and its capacity; hence, these terms are not synonymous. If the safe limits of temperature are exceeded, the deterioration of the insulation is rapid, the damage increasing with the duration and extent of the excess temperature. The life of insulating materials depends not only upon temperature and its duration, but upon electric stress and associated effects (such as corona formation), vibration and varying mechanical stress, repeated expansions and contractions, and exposure to moisture, air, and fumes.

In the original Standardization Rules of the A.I.E.E., it was specified that the allowable *rise* of temperature of the parts of a machine (excepting railway motors) should be as follows: armature and field windings, 50°C.; commutator, 55°C.; bearings, 40°C. These rises of temperature were based upon standard conditions of room temperature of 25°C., barometric pressure of 760 mm., and normal conditions of ventilation. It was further provided that, if the room temperature differed from 25°C., the observed rise of temperature should be corrected by ½ per cent for each degree difference between room temperature and 25°C., the correction to be added to the observed rise if the room temperature was below 25°C. and subtracted if it was higher.

In later rules, now in force, emphasis is placed upon the highest permissible temperature of the hottest spot as well as upon the maximum rise of temperature. The rise of temperature of air-cooled machines is in all cases based upon an ambient temperature not exceeding 40°C., and preferably not lower than 10°C.; but it is assumed that the rise of temperature is the same for all cooling air temperatures between these limits.

The standard temperature limitations differ with the different kinds of insulating materials, of which three* classes are recognized in the case of d-c machinery†:

* A fourth class, designated as Class C, includes inorganic materials such as pure mica, porcelain, and quartz; this type of insulation is used, for example, in synchronous machines, but the classification does not enter into specifications for d-c machines.

† A.I.E.E. Standards, C-50, 1936.

Class O consists of cotton, silk, paper, and similar organic materials when neither impregnated nor immersed in oil.

Class A consists of cotton, silk, paper, and similar organic materials when impregnated or immersed in oil; also enamel as applied to conductors.

Class B consists of inorganic materials such as mica and asbestos in built-up form combined with binding cement. If Class *A* material is used in small quantities for structural purposes only, in conjunction with Class *B* insulation, the combined material may be considered as Class *B*, provided the electrical and mechanical properties of the insulated winding are not impaired by the application of the temperature permitted for Class *B* material. The word "impair" is here used in the sense of causing any change which could disqualify the insulating material for continuous service.

An impregnated insulation is one in which

. . . a suitable substance replaces the air between its fibers, even if this substance does not completely fill the spaces between the insulated conductors. The impregnating substance, in order to be considered suitable, must have good insulating properties; must entirely cover the fibers and render them adherent to each other and to the conductor; must not produce interstices within itself as a consequence of evaporation of the solvent or through any other cause; must not flow during the operation of the machine at full working load or at the temperature limit specified; must not unduly deteriorate under prolonged action of heat.

In the case of d-c machines, the standard procedure for measuring temperature is the *thermometer method.* This consists in the determination of the temperature by mercury or alcohol thermometers, by resistance thermometers, or by thermocouples, any of these instruments being applied to the hottest part of the machine accessible to mercury or alcohol thermometers.

The *resistance method* consists in the determination of temperature by comparison of the resistance of a winding at the temperature to be determined with the resistance at a known temperature, as explained in Art. 20, Chap. I. This method is among those accepted as standard for synchronous machines (except synchronous converters).

The embedded-detector method consists in the determination of the temperature by thermocouples or resistance temperature

detectors, built into the machine as specified in the section of the A.I.E.E. Standards dealing with the specific kind of machine. This method is not used with d-c machines.

In the case of d-c generators and motors, temperatures are in all cases to be taken by the thermometer method, except that for railway motors, although the thermometer method is permissible, the resistance method is to be considered the rule. Heat runs, for the determination of thermal limitations, may be made at any temperature of the cooling air, preferably not below 10°C. (50°F.); but without regard to the actual ambient temperature, the temperature rise is to be considered the same between the limits of 10° and 40°C. If the test is made at altitudes greater than 1000 m. (3300 ft.), it has been provisionally agreed that the temperature rise at altitudes less than 1000 m. will be the temperature rise observed at the higher altitude reduced by 1 per cent for each 100 m. by which the altitude of the test site exceeds 1000 m.

The ambient temperature is to be measured by means of several thermometers placed at different points around and halfway up the machine at a distance of 1 to 2 m. and protected from drafts and abnormal heat radiation. To this end the thermometers are to be immersed in oil in a suitable heavy metal cup, for example, a massive metal cylinder with a hole drilled partly through it. This hole is filled with oil and must be sufficiently deep to ensure complete immersion of the bulb of the thermometer. The smallest size of oil cup permitted by the rules consists of a metal cylinder 25 mm. (1 in.) in diameter and 50 mm. (2 in.) high, but the size of the oil cup must be increased with that of the machine under test. The object of thus increasing the size of the oil cup is to avoid errors in the calculations of temperature rise due to the time lag between changes of temperature of the machine and the surrounding air, this time lag being greater, the greater the size of the machine.

Where machines are partly below the floor line in pits, the temperature of the armature is referred to a weighted mean of the pit and room temperatures, the weight assigned to each being based on the relative proportions of the machine in and above the pit. The temperature of the portion of the field structure constantly in the pit must be referred to the ambient temperature in the pit.

In taking the temperatures of parts of the machine, the thermometers are placed directly in contact with them, with the bulbs covered by felt pads cemented to the machine by oil putty, or by cotton waste; the felt pads are standardized as to size for large machines, the dimensions being 1½ by 2 by ⅛ in. thick. Temperature readings are to be taken, so far as practicable, during the heat run as well as immediately after shutdown, the

Table I

Item	Generators and general-purpose motors	Totally enclosed and totally enclosed fan-cooled motors		Nominally rated generators at end of 2-hr. overload		Generators having 2-hr. 25 per cent overload rating				All other generators and motors	
						At continuous load		At end of 2-hr. overload			
	Class of insulation										
	A	*A*	*B*	*A*	*B*	*A*	*B*	*A*	*B*	*A*	*B*
1. Armature windings, wire field windings, and all windings other than item 2	40	55	75	55	75	40	60	55	75	50	70
2. Single-layer field windings with exposed uninsulated surfaces and bare copper windings	50	65	85	65	85	50	70	65	85	60	80
3. Cores and mechanical parts in contact with or adjacent to insulation	40	55	75	55	75	40	60	55	75	50	70
4. Commutators and collector rings*	55†	65	85	65	85	55	70	65	85	65	85

5. Miscellaneous parts (such as brush holders, brushes, and pole tips) may attain such temperatures as will not injure the machine in any respect.

* The class of insulation refers to insulation affected by heat from the commutator, which insulation is employed in the construction of the commutator or is adjacent thereto.

† In generators for electrolytic service, change to 50°C.

highest reading being the accepted value. In case some time elapses between the instant of shutdown and the time of taking the first thermometer reading, a correction is to be applied by plotting a temperature-time curve and extrapolating back to the moment of shutdown (except that if successive readings show increasing temperature after shutdown, the highest value is taken).

Subject to these specifications defining the methods of measuring temperature, the permissible rise of temperature for various types of machine is shown in Table I. The reduced temperature rises in the first column are intended to provide a margin of safety where the load conditions are unknown.

The temperature rise in fractional-horsepower motors, in which insulation of Classes *O* and *A* is used, is defined in Table II.

TABLE II

Item	All motors not included in last two columns		General-purpose motors	Totally enclosed and totally enclosed fan-cooled motors
	Class *O*	Class *A*	Class *A*	Class *A*
1. Coil windings, cores, and mechanical parts in contact with or adjacent to insulation...	35	50	40	55
2. Commutators and collector rings..........	50	65	55	65
3. Squirrel-cage windings and miscellaneous parts (such as brush holders, brushes, and pole tips) may attain such temperatures as will not injure the machine in any respect.				

15. Temperature Limitations in Railway Motors.*—Operating conditions in railway motors are much more severe than in ordinary motors because of the restricted space and the variable (and sometimes unpredictable) nature of the duty. Wide variations in the temperature of the cooling air introduce another difficulty. Higher working temperatures are necessary because of the large amount of power that must be developed in relatively small space, and this requirement calls for the use of heat-resisting

* A.I.E.E. Standards, No. 11, 1937.

types of insulation and special provision for ventilation. Railway motors are either *self-ventilated*, in which case the cooling air is circulated by means integral within the machine itself, or *separately ventilated*, the cooling air being supplied by an independent fan or blower external to the machine.

The ratings of railway motors are established by *stand tests* made under prescribed conditions and are of three kinds:

Continuous rating (which in the absence of any special specification is to be understood) is the output at the motor shaft in horsepower (or kilowatts) that the motor can carry for an unlimited period on stand test at its rated voltage, and in the case of an a-c motor at its rated frequency, with the ventilation system as in service, without exceeding the permissible limits of temperature rise.

TABLE III

Rating	Part	Class of insulation	Temperature rise	
			By resistance	By thermometer
Continuous	Armature and field windings	*A*	85	65
1 hr. . . .	Armature and field windings	*A*	100	75
Continuous and 1 hr	Armature	*B*	120	90
	Field	*B*	130	95
All ratings. . . .	Commutator and collector	*A* and *B*	. .	90*

* Thermometer to be located at mid-length of commutator.

Direct-current, self-ventilated motors are also given a continuous rating in amperes, at one-half rated voltage, with the same temperature limits as when operated at full voltage. Non-ventilated, totally enclosed d-c motors (now rapidly becoming obsolete) are rated in amperes at one-half of rated voltage, with a permissible temperature rise on *stand test* 10°C. above the regularly specified limits, this leeway allowing for the cooling effect of the surrounding air when the car is in motion.

The 1-*hr. rating* of a railway motor is the shaft output in horsepower (or kilowatts) that the motor can carry for 1 hr. on stand test, starting with a motor winding temperature within 4°C. of the cooling air temperature, at its rated voltage (and rated frequency in a-c motors), with the ventilation system as in service, without exceeding the permissible limits of temperature rise.

The *thermal-capacity rating* is a measure of the ability of the motor to absorb heat. It is expressed in terms of the average time (in seconds) for the temperature of the machine to rise 1°C. during the transition from a normal operating temperature to the maximum permissible, when carrying a definite overload (1.6 times its continuous rated current at rated voltage). Further details are given in Art. 20.

The permissible observable temperature rise of the parts of a ventilated railway motor are shown in Table III.

16. Commutation Limitations.—Successful commutation, as defined by the A.I.E.E. Standards, is attained if neither the brushes nor the commutator are burned or injured in an acceptance test, or, in normal service, to such an extent that abnormal maintenance is required. The presence of some visible sparking is not necessarily evidence of unsuccessful commutation. This criterion of successful commutation applies to all loads not exceeding the 1-*min. load* and without adjustment of the brushes. The 1-min. load is a test load maintained for 1 min., during which commutation must be satisfactory and which is defined as follows:

1. In continuously rated machines, 150 per cent of the amperes corresponding to the continuous rating, the field rheostat being kept set for the rated load excitation.
2. In generators having a 2-hr., 25 per cent overload rating, 200 per cent of the amperes corresponding to the continuous rating, the field rheostat being kept set for the excitation at the continuous rating.
3. In nominally rated generators, 200 per cent of the amperes corresponding to the nominal rating.

The 1-min. load does not apply to machines for short-time, intermittent, periodic, or varying duty; in these ratings, commutation must be satisfactory under the specified operating conditions.

17. Overspeed Limitations.—Generators, except water-wheel-driven and steam-turbine-driven generators, equipped with emergency governors, must in general safely withstand an overspeed of 25 per cent. Water-wheel-driven generators must safely withstand the maximum runaway speed that can be attained by the combined unit. The overspeed limitation in the case of steam-turbine-driven generators, equipped with emergency governors, is 20 per cent.

Shunt- and compound-wound motors must have a safe overspeed limit of 25 per cent above the no-load speed. Series-wound motors, in the absence of special guarantees, having short-time ratings of 1 hr. or less, must safely withstand the speed corresponding to one-fourth rated load.

18. Dielectric Test.—To ensure safety against breakdown of insulation, all machines, before shipment from the factory, are subjected to an alternating voltage of which the effective value is in general 1000 volts plus twice the rated voltage of the circuit to which the machine is to be connected. The standard duration of this test is 1 min. An exception is made if machines are designed for use on circuits of 25 volts or lower, in which case the alternating test voltage is 500 volts. Machines produced in large quantities, and for which the standard test voltage is 2500 volts or less, may be tested for 1 sec. with a test voltage 20 per cent higher than for the 1-min. test. Dielectric tests made after the final installation of a machine that has previously passed the factory test, provided that the windings have not in the meantime been disturbed, are made with a test voltage of 75 per cent of the standard voltage.

The frequency of the alternating test voltage is 60 cycles per sec., and the crest value of the voltage must be $\sqrt{2}$ times the specified effective value (the effective value is the reading given by a voltmeter of the dynamometer type).

19. Output Equation.—A definite relation, originally derived by G. Kapp, exists between the rating, speed, and dimensions of the armature. This relation, when expressed in algebraic form is commonly referred to as the *output equation*. Thus, let

V = rated terminal voltage.
I_a = rated armature current.
ψ = ratio of pole arc to pole pitch.
q = ampere-conductors per unit length of armature periphery.

Since

$$V = \frac{p}{a}\frac{\Phi Z n}{60 \times 10^8} \quad \text{(nearly)}$$

and

$$\Phi = B_g b l = \frac{\pi d \psi}{p} B_g l \quad \text{(nearly)}$$

the power output of the machine (in kilowatts) is

$$Kw = \frac{VI_a}{1000} = \frac{\pi^2 \psi B_g q}{60 \times 10^{11}} d^2 l n = \xi d^2 l n \tag{31}$$

where

$$\xi = \frac{\pi^2 \psi B_g q}{60 \times 10^{11}} \tag{32}$$

is called the *output coefficient.* The numerical value of this coefficient depends upon the "design constants" of the machine,

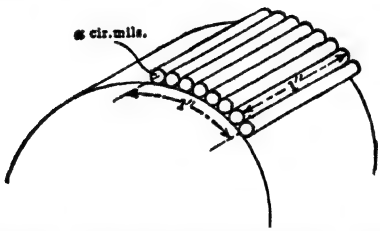


Fig. 17.—Calculation of copper loss per square inch of armature surface.

ψ, B_g, and q, but principally upon B_g and q since the range of values of ψ is limited. B_g is clearly a measure of the degree of utilization of the magnetic material of the machine; similarly, q is in part a measure of the specific utilization of armature copper, for it is closely related to the thermal characteristics, as has been shown by Adams.* Thus, let q be expressed in ampere-conductors per inch of armature periphery, and let h be the current density in the armature conductors expressed in circular mils per ampere. Let Fig. 17 represent a portion of the armature surface (shown as a smooth-core type for convenience) 1 in. square. Each conductor will carry a current of I_a/a amp., and

* *Trans. A.I.E.E.*, **24**, 653, 1905.

its cross-section will be $I_a h/a$ cir. mils; its resistance per inch of length is

$$r = \rho \frac{\text{length}}{\text{area}} = 1 \times \frac{1}{\frac{I_a}{a}h} = \frac{a}{I_a h} \quad \text{ohms}$$

since the specific resistance of copper at the working temperature of the armature is very nearly 1 ohm per cir.-mil-inch. The I^2R loss per conductor is then

$$\left(\frac{I_a}{a}\right)^2 \frac{a}{I_a h} = \frac{I_a}{ah} \quad \text{watts}$$

But the number of conductors per inch of armature periphery is $\frac{q}{I_a/a} = \frac{aq}{I_a}$; hence the I^2R loss per square inch of armature surface is

$$\frac{I_a}{ah} \times \frac{aq}{I_a} = \frac{q}{h} \quad \text{watts per sq. in.} \tag{33}$$

The value of q varies from about 400 in machines of 20 kw. or less, up to 850 or more in machines of 1000 kw. capacity. The ratio q/h (watts per sq. in. due to copper loss) is generally in the neighborhood of unity for ordinary peripheral velocities of 2500 ft. per min. but may be as high as 2.5 in large machines running at high peripheral speeds (6000 ft. per min.) where the ventilation is more effective. Values of B_g range from about 40,000 lines per sq. in. in small machines up to 60,000 lines per sq. in. in large machines. The value of ξ generally lies between 0.000015 (small machines) and 0.000056 (large machines).

20. Heating and Cooling Curves.—The energy losses in any machine are converted into heat and cause a rise of temperature of which the final value depends upon the heat capacity of the materials of the structure and upon the facility with which the heat may be radiated or otherwise dissipated. The temperature will become stationary when the rate of heat generation becomes equal to the rate of dissipation.

If is of interest to derive the law of heating and cooling of a homogeneous body for the reason that it throws light on the conditions obtaining in the more complex structure of a generator or motor.

Let

Q = heat generated per second, in kilogram-calories.
s = specific heat of the substance = amount of heat required to raise 1 kg. 1°C.
W = weight of the body in kilograms.
A = radiating surface in square centimeters.
α = coefficient of cooling = amount of heat in kilogram-calories dissipated per second per square centimeter of radiating surface per degree difference of temperature between body and surrounding medium.
θ = temperature of body in degrees C.
θ_1 = temperature of surrounding medium in degrees C.

1. *Heating of the Body.*—In a time dt the temperature will increase by $d\theta°$. During this interval the heat liberated amounts to $Q\,dt$ kg.-cal., and the body absorbs $sW\,d\theta$ kg.-cal. The remainder will be dissipated, to the amount $A\alpha(\theta - \theta_1)dt$ kg.-cal., so that

$$Q\,dt = sW\,d\theta + A\alpha(\theta - \theta_1)dt \tag{34}$$

Transposing,

$$dt = \frac{sW\,d\theta}{Q - A\alpha(\theta - \theta_1)}$$

Assuming that $\theta = \theta_1$ when $t = 0$,

$$\int_0^t dt = sW\int_{\theta_1}^{\theta}\frac{d\theta}{Q - A\alpha(\theta - \theta_1)}$$

which gives

$$\theta - \theta_1 = \frac{Q}{\alpha A}\left(1 - \epsilon^{-\frac{\alpha A}{sW}t}\right) \tag{35}$$

and when $t = \infty$,

$$(\theta - \theta_1)_{t=\infty} = \frac{Q}{\alpha A} \tag{36}$$

which is the limiting temperature rise of the body. Equation (36) may be written

$$Q = \alpha A(\theta - \theta_1)_{t=\infty} \tag{37}$$

which expresses the fact that when the temperature becomes stationary the rates of heat production and dissipation are equal.

2. *Cooling of the Body.*—In this case no heat is developed; consequently $Q = 0$, and the fundamental equation becomes

$$0 = sW\,d\theta + A\alpha(\theta - \theta_1)dt \tag{38}$$

If the temperature is $\Theta°$ when $t = 0$,

$$\int_0^t dt = -sW\int_\Theta^\theta \frac{d\theta}{A\alpha(\theta - \theta_1)}$$

or

$$\theta - \theta_1 = (\Theta - \theta_1)\epsilon^{-(\alpha A/sW)t} \tag{39}$$

is the equation of the cooling curve. If $\Theta - \theta_1 = (\theta - \theta_1)_{t=\infty} = Q/\alpha A$, that is, if the temperature at the beginning of cooling is

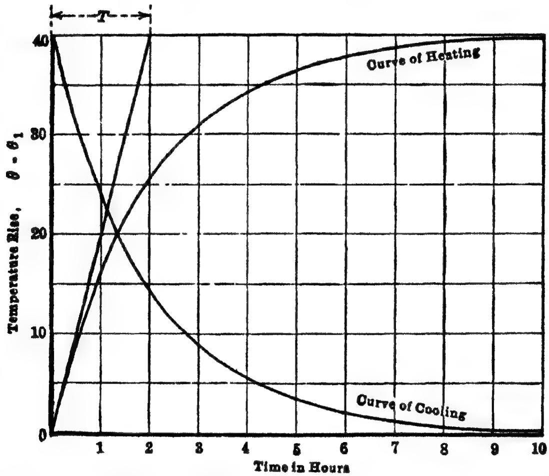


FIG. 18.—Heating and cooling curves.

equal to the limiting temperature at the end of the heating period, the equation of the cooling curve is the same as the variable part of the heating equation, but with a change of sign. In this case, the heating and cooling curves are of the same logarithmic shape, but one is turned upside down with respect to the other, as shown in Fig. 18.

Differentiating the heating equation

$$\theta - \theta_1 \quad \frac{Q}{\alpha A}\left(1 - \epsilon^{-\frac{\alpha A}{sW}t}\right)$$

and substituting $t = 0$ in the result, the slope of the curve at the origin is found to be

$$\left(\frac{d\theta}{dt}\right)_{t=0} = \frac{Q}{sW} \tag{40}$$

that is, dependent upon the mass and material of the body, but not upon its cooling area or the nature of the radiating surface. In fact, at the first instant, all the heat is absorbed and none is radiated; hence, the slope of the curve at the origin gives the rate at which the temperature would rise if all the heat were absorbed. If the temperature continued to rise at this rate, the limiting temperature rise $Q/\alpha A$ would be reached in a time $T = sW/\alpha A$ sec.; T is called the *time constant* of the body. The heating equation in terms of T can be written

$$\theta - \theta_1 = \frac{Q}{\alpha A}\left(1 - \epsilon^{-\frac{t}{T}}\right)$$

To substitute numerical values in the above equations, the following relations obtain:

$$\begin{aligned} Q &= 0.2386 \times (\text{loss in kw.}) \quad \text{kg.-cal. per sec.} \\ &= 0.527 \times (\text{loss in kw.}) \quad \text{lb.-cal. per sec.} \\ s &= 0.11 \text{ for iron.} \\ s &= 0.09 \text{ for copper.} \end{aligned}$$

The value of α may be found from the experimentally determined* fact that when air is blown across the bare (or thinly varnished) surface of an iron core its surface temperature will rise 1°C. when the radiation is $0.0038(1 + 0.25v)$ watts per sq. cm. of surface, where v is the velocity of the air in meters per second; this value is equivalent to $0.0245(1 + 0.00127v)$ watts per sq. in. if v is in feet per minute. Thus, it follows that

$$\begin{aligned} \alpha = 0.906(1 + 0.25v) \times 10^{-6} \text{ kg.-cal. per sec. per} \\ \text{sq. cm. per 1°C.} \end{aligned} \tag{41}$$

where v is given in meters per second; or

$$\begin{aligned} \alpha = 12.89(1 + 0.00127v) \times 10^{-6} \text{ lb.-cal. per sec. per} \\ \text{sq. in. per 1°C.} \end{aligned} \tag{42}$$

where v is expressed in feet per minute.

*LUDWIG OTT, *London Electrician*, p. 805, 1907.

The experiments of Ott also showed that if the surface of the core is coated with a thick double layer of varnish the radiation is $0.0030(1 + 0.107v)$ watt per sq. cm. per 1°C. rise of surface temperature, v being in meters per second; or $0.0194(1 + 0.00054v)$ watt per sq. in. per 1°C., v being in feet per minute.

Temperature rise computed from the above formula will not agree in general with the observed rise in actual machines because it neglects the transfer of heat from the winding to the core, or vice versa; it also neglects the irregular distribution of heat evolution and the thermal capacity of the insulation. But in general it will be true that the ultimate rise of temperature can be expressed by the equation

$$\theta - \theta_1 = \text{constant} \times \frac{\text{watts dissipated}}{\text{radiating surface}} \tag{43}$$

where the constant is in each case to be determined by experiment.

The principles outlined in this article underlie the procedure specified in Rule 11-500 of the A.I.E.E. Standards* for the determination of the thermal-capacity rating of railway motors. The procedure involves the following steps:

1. Following the continuous rating test, measurements of temperature by the resistance method shall be made and continued for the establishment of a cooling curve.

2. As the motor cools down and the cooling curve for the armature approaches 75°C. for Class *A* insulated motors or 100°C. for Class *B*, the curve shall be extrapolated to that

TABLE IV.—LIMITS OF OBSERVABLE TEMPERATURES

Part	Insulation class	Method of temperature measurement	Peak value	Normal value
Armature and field........	*A*	Resistance	125	110
		Thermometer	100	90
Armature................	*B*	Resistance	160	145
		Thermometer	125	115
Field....................	*B*	Resistance	170	155
		Thermometer	130	120

* A.I.E.E. Standards, No. 11, 1937.

temperature and the motor started *as late as possible* but soon enough so that, at the instant the temperature as shown by the curve reaches this value, the motor shall be loaded* to 1.6 times its continuous rated current at its rated voltage and run until its limiting temperature reaches the peak value in Table IV, ±10°C.

3. The motor shall be shut down, the temperatures again measured by resistance, and the cooling curve extrapolated back to the instant of shutdown.

The duration of the overload, in seconds, divided by the rise in temperature in degrees centigrade during the overload gives the thermal-capacity rating in seconds per degree centigrade.

21. Heating of the Armature.—The experimental results of Ott, referred to in Art. 20, may be changed to a form applicable to the rotating part of the machine. If we take the value of the radiation for bare or thinly varnished surfaces, the temperature rise is

$$\theta - \theta_1 = \frac{w}{a} \frac{460}{1 + 0.25v} \tag{44}$$

where w = total watts dissipated,
a = total radiating surface in square centimeters,
v = peripheral velocity of armature in meters per second,
and, for a heavily varnished surface,

$$\theta - \theta_1 = \frac{w}{a} \frac{333}{1 + 0.107v} \tag{45}$$

Accordingly, the rise of temperature for a radiation of 1 watt per sq. cm. is found by putting $w/a = 1$. The temperature rise (in degrees C.) for a radiation of 1 watt per sq. in., expressing v in feet per minute, is

$$\frac{71.3}{1 + 0.00127v} \quad \text{for a bare surface}$$

$$\frac{52}{1 + 0.00054v} \quad \text{for a heavily varnished surface}$$

* When motors are force-ventilated, the blowers shall be started as soon as the overload is applied to the motor.

Other writers give the value of this constant as follows:

	Metric units	English units
Kapp.....	$\dfrac{550}{1 + 0.1v}$	$\dfrac{85}{1 + 0.00051v}$
Arnold....	$\dfrac{300}{1 + 0.1v}$	$\dfrac{46.5}{1 + 0.00051v}$
Esson.....	$\dfrac{354}{1 + 0.0006v}$	
Wilson....	$\dfrac{640}{1 + 0.18v}$	$\dfrac{99}{1 + 0.00091v}$
Thompson.	$\dfrac{645}{1 + 0.3\sqrt{v}}$	$\dfrac{100}{1 + 0.0213\sqrt{v}}$

These tabulated expressions are embodied in Fig. 19, which shows the rise of temperature per watt per square inch as a function of the peripheral velocity in feet per second. It will be observed that

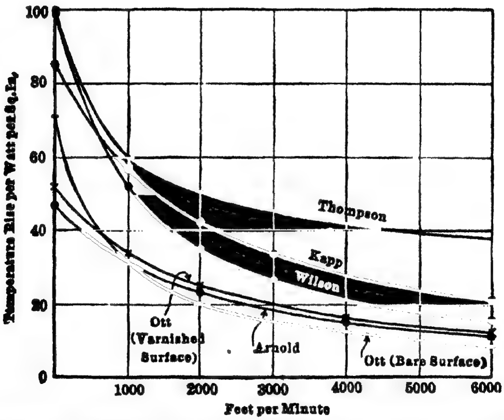


FIG. 19.—Relation between temperature rise and peripheral velocity of armature.

the curve represented by Arnold's formula lies nearly midway between the two corresponding to Ott's researches, at least for values of v within the usual limits of practice.

It should be noted that all the preceding formulas for computing rise of temperature are more or less uncertain, unless applied to a machine of the same type as that from which the constants were experimentally determined. A large part of the differences between the curves of Fig. 19 is due to the fact that there is no

absolute agreement as to what constitutes the radiating surface. Some writers specify the outer cylindrical surface only, but including the surface of the end connections as well as of the core; others include the exposed sides of the core in addition to the outer cylindrical surface. Evidently all exposed surfaces are useful in radiating heat, but not to the same extent. The magnitude and direction of the flow of heat from the interior to the exterior of a mass will depend upon the heat conductivity in different directions; and since the conductivity along the laminations is much greater than across them (Ott found it to be from fifty to one hundred times greater) it follows that unless the core is very deep the greater part of the heat will be dissipated from the cylindrical surface. It has been pointed out* that a rational equation for the rise of temperature of an armature should be of the form

$$\theta - \theta_1 = \frac{w}{\Sigma a_1 + c\Sigma a_2} \cdot \frac{C}{1 + bv} \tag{46}$$

where Σa_1 = sum of cylindrical cooling surfaces.
Σa_2 = sum of end surfaces.
c = a variable coefficient less than unity.

The value of c will be smaller, the greater the ratio of heat conductivity along the laminations to that across them.

The Arnold formula for rise of armature temperature is

$$\theta - \theta_1 = \frac{w}{a} \cdot \frac{(40 \text{ to } 70)}{1 + 0.00051v} \tag{47}$$

where a and v are expressed in square inches and feet per minute, respectively. The numerical coefficient in the numerator is to be taken near the lower limit of its range when the ventilation is good. In using this formula, however, it should be noted that w does not include the watts dissipated in the end connections, nor does a include their surface; in other words, the rise of temperature of the armature core is to be distinguished from that of the end connections. Consequently, to estimate the rise of temperature of the core, the value of w to be inserted in the formula is

$$w = \text{total core loss} + \frac{\text{embedded length of winding}}{\text{total length of winding}} \times I_a^2 R_a \tag{48}$$

* LUDWIG OTT, *London Electrician*, p. 805, 1907.

The value of a recommended by Arnold is the cylindrical surface of the core, plus the two end surfaces, plus half the lateral area of the walls of the ventilating ducts; or (Fig. 20)

$$a = \pi dl + \pi\, d_{aver.} h(2 + n_v) \tag{49}$$

In the case of the end connections,

$$\left.\begin{aligned} w &= \frac{\text{free length of winding}}{\text{total length of winding}} \times I_a^2 R_a \\ a &= 2\pi d l_e \end{aligned}\right\} \tag{50}$$

In semienclosed machines the temperature rise is about 50 per cent greater than that given by the above formula, and in enclosed machines it is about twice as great as that given by the formula.

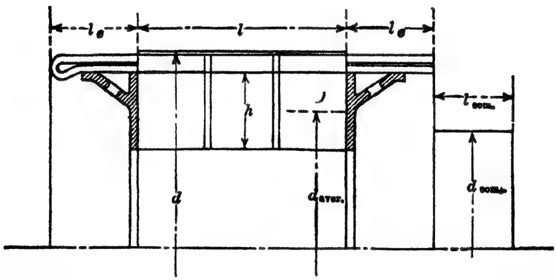


FIG. 20.—Dimensions of radiating surfaces.

22. Heating of Field Coils.*—The field coils are heated not only by the I^2R losses that occur in them but also by the losses in the pole faces, caused by eddy currents, and by radiation from the armature. The heat is dissipated in three ways: (1) by convection in the surrounding air; (2) by conduction through the pole cores and yoke; and (3) by direct radiation. The temperature inside the coil varies from point to point in a manner depending upon the depth of the winding and upon the nature of the insulation, being highest near the middle of the cross-section of the coil and lowest on the exposed surface. Impregnated coils run cooler than ordinary coils, for the insulating compound is a better heat conductor than the air it replaces. Measurement of the rise of temperature by the increase of resistance gives the

* For the results of elaborate studies of field-coil heating, see articles by Neu, Levine, and Havill, *Elec. World*, **38**, 56, 1901, and by Ludwig Ott, *London Electrician*, p. 805, 1907.

average rise of temperature of the winding as a whole, the *maximum* rise at the hottest point being 12 to 20 per cent greater than the average rise. The average rise of temperature of the entire winding is 40 to 60 per cent greater than the average rise of temperature of the exposed surface; the latter is determined by taking the mean of thermometer readings at the middle and ends of the exposed cylindrical surface.

Formulas for computing the rise of temperature of field coils are of the form

$$\theta - \theta_1 = C\frac{\text{watts lost in coil}}{\text{radiating surface of coil}} \tag{51}$$

Different writers assign various values to the constant C, depending upon the selection of what constitutes the radiating surface. Obviously, C means the rise of temperature due to a radiation of 1 watt per unit area. If the radiating surface is expressed in square inches and is taken to mean the outer cylindrical surface exclusive of the exposed end, the value of C for open-type machines, and under standstill conditions, is 70 to 80, with an average of 75. The value of C decreases with increasing peripheral velocity of the armature, due to fanning action, by approximately 5 per cent per 1000 ft. per min., or

$$C = 75(1 - 0.00005v) \tag{52}$$

where v is peripheral velocity in feet per minute. This is an average value, the decrease of C being somewhat greater if the coils are short, because of the cooling effect of the yoke, and somewhat less if the coils are long. In machines of the protected type, C is approximately 50 per cent greater than the foregoing value and in enclosed machines from two to three times greater than that given by Eq. (52).

Field coils of the ventilated type of construction are made of concentric parts with an open space of about ½ in. between them. The greater surface presented to the air by reason of this construction permits of increased radiation; however, the internal surfaces of the ducts are not so effective as an equal area on the outside. For a given temperature rise the ventilated coil will radiate about 50 per cent more watts per square inch than an ordinary coil; or, what amounts to the same thing, C may be taken as equal to 50.

23. Heating of Commutator.—The commutator is heated by the losses due to brush friction P_{bf} and by the flow of the current across the contact resistance between commutator and brushes. The rise of temperature can be computed from the formula

$$\theta - \theta_1 = 20\frac{W}{A}\,\frac{1}{1 + 0.00051v} \tag{53}$$

where W = total loss at the commutator.

$A = \pi\, d_{com.} l_{com.}$

v = peripheral velocity of commutator in feet per minute.

24. Rating of Enclosed Motors.—If a motor of the open type is converted into one of the enclosed type, its rating must be reduced to avoid excessive temperature rise. Experience shows that a reduction in horsepower rating of about 30 per cent, accompanied by an increase of speed of 20 per cent, will give a temperature rise within standard limits. The reduction in horsepower rating decreases the current and consequently the I^2R losses, and the increase of speed permits a reduction of the flux per pole, the excitation loss and the core loss being thereby lowered. The core loss decreases notwithstanding the increase of speed, for the effect of reduced flux density more than outweighs the effect of increased frequency of the magnetic reversals (see Fig. 15); the core loss varies nearly as the square of the flux density, and approximately as the first power of the speed, since the hysteresis loss is always greater than the eddy-current loss.

CHAPTER XIV

SPECIAL MACHINES AND APPLICATIONS

1. Boosters.—A *booster* is a dynamoelectric machine the armature of which is connected in series with a circuit, its generated e.m.f. being added to or subtracted from that of the circuit, depending upon the polarity of its excitation. Boosters may be driven by any form of prime mover but are generally direct-connected to a motor taking current from constant-potential mains.

2. Series Booster.—An obvious use for a booster is to raise the voltage of a generator or of a section of the bus bars of a central station by an amount sufficient to compensate the ohmic drop in a feeder supplying a distant load, in case the load is of such character as to require the same voltage as receiving devices at or near the source of supply. Since the line drop is directly proportional to the current, the voltage of the booster should also be proportional to the current; or the booster should have an external characteristic consisting of a straight line through the origin. It is impossible exactly to realise this form of characteristic without auxiliary devices, but it may be approximated sufficiently closely for practical purposes by designing the booster as a series-wound generator with flux densities well within the point of magnetic saturation. The hysteresis effect illustrated in Fig. 7, Chap. VIII, is especially objectionable in boosters, and should be reduced to a minimum. Further, if the excitation is of such character that the main flux is subject to wide variations, the magnetic circuit must be laminated throughout, in order that eddy currents set up by a change in the flux may not be of sufficient magnitude to retard the change of flux and so make the machine sluggish in its action.

The compensation, by means of a series booster, of the drop of potential in a circuit due to its ohmic resistance is equivalent to a complete cancellation of the resistance of the circuit. Ordi-

narily, if the resistance of a circuit is to be reduced, the reduction would be made by an increase of the cross-section, and therefore of the weight and cost of the line. Up to a certain point, which may readily be computed for a given set of conditions, it will be cheaper to save energy by adding copper than to install a booster equipment; beyond that point the booster will be more economical.

The apparent cancellation of the resistance of a circuit by means of a series booster is sometimes utilized in electric railways employing a ground return to mitigate the *electrolysis* of underground structures such as water and gas mains and telephone cables. The return circuit of the ordinary street-railway system consists of the track and the surrounding earth, the current dividing between these paths in the inverse ratio of their resistances. Even with well-bonded tracks a considerable flow of current may take place through the earth along paths of low resistance afforded by underground metallic structures, damage resulting wherever stray currents leave these paths to return through moist earth to the track or to the grounded bus at the powerhouse. The danger of electrolysis in such systems can be reduced by installing insulated *negative feeders* or cables that connect points along the track directly to the negative bus of the generating station, thereby draining the track current away from the stray paths. These negative feeders are clearly the more effective, the lower their resistance. If a series booster is connected in such a feeder so that its e.m.f. acts in the direction from the track to the negative bus, the equivalent resistance of the feeder may be reduced nearly to zero, and most of the current will return to the station by way of the feeder. Boosters used in this way are called *negative* or *track-return* boosters.

3. Shunt Booster.—In constant-potential systems of relatively small capacity in which the load changes gradually but covers a considerable percentage range, it is common practice to use a storage battery in parallel with the bus bars. The battery may then be used to carry the entire load at times of light load and in parallel with the generator at the time of peak load. At other times the battery takes charging current from the generator, a fairly uniform load on the generator during its working period being thus ensured, with consequent economy in cost of fuel.

In a system of this kind a *shunt booster* is used to force charging current into the battery, the connections being shown in Fig. 1. The field winding of the booster is connected across the main bus bars, never across its own armature; hence, the machine is really separately excited. The booster armature is in series with the battery during the charging period and is called upon to supply a relatively small e.m.f., hence the connection of the field winding to the bus bars. The booster voltage is manually controlled by means of the field rheostat, adjustment being made when the readings of the ammeters indicate that it is necessary. By inserting a reversing switch in the field circuit or by using a reversing rheostat, the booster e.m.f. may be added to that of the battery, the battery being thus assisted

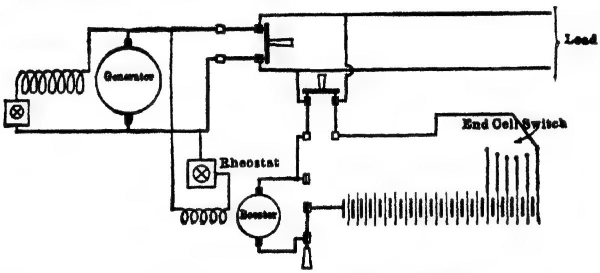


Fig. 1.—Connections of shunt booster.

to discharge if its voltage is low or if the demand for current is unusually heavy.

A lead storage battery when discharged to the permissible limit gives 1.8 volts per cell, and when fully charged requires an impressed voltage of 2.65 volts per cell to give it the "over-charge" that is periodically required to keep it in good condition. If, therefore, the voltage of the system is V, the total number of cells required is $V/1.8$ to provide for the contingency that the battery alone, when nearly exhausted, may be used to carry the load. At the end of a prolonged charge the battery voltage will have risen to $2.65 \times (V/1.8)$ and hence the booster must be capable of generating $V\left(\frac{2.65}{1.8} - 1\right) = 0.47V$ volts; thus, in a 110-volt system, the maximum booster voltage will be 52 volts. The design of the booster is completely determined when the maximum discharge rate of the battery is known.

The capacity of the motor that drives the booster need be only from two-thirds to three-fourths of the volt-ampere capacity of the booster for the reason that when the latter delivers its maximum current the voltage is low and when the voltage is highest, during the periods of overcharge, the current must be considerably reduced. The normal (8-hr.) discharge rate of a lead battery is defined as that current which, flowing uniformly for 8 hr., will reduce the battery voltage to the minimum value of 1.8 volts per cell; the current during overcharge should be not greater than one-half of the 8-hr. rate.

In Fig. 1 the cells shown at the right-hand end of the battery are the *end cells* which are cut in and out of circuit by means of an end-cell switch. Their purpose is to adjust the battery voltage

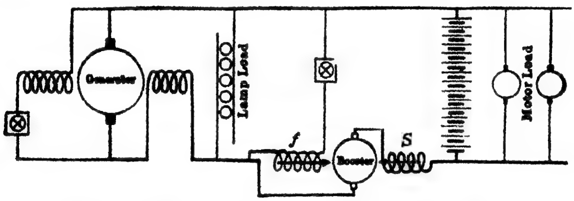


FIG. 2.—Constant-current or nonreversible booster.

to the line requirements to compensate for the changes in voltage due to varying conditions of charge and discharge. Thus, in a 110-volt system, the number of cells required is 110/1.8 = 61, when fully discharged; but, when a fully charged battery begins to discharge, its terminal voltage is 2.15 volts per cell, 110/2.15 = 51 cells being thus required. Consequently, in such a system, 61 cells would be installed, 10 of them as end cells. The number of end cells may be reduced if the booster field is provided with a reversing switch, for in that case the booster e.m.f. can be made to oppose that of the battery to a sufficient extent to bring the terminal voltage to the proper value.

4. Constant-current, or Nonreversible, Booster.—In isolated plants supplying a lamp load and a fluctuating motor load, as in hotels and office buildings, it is necessary to maintain a constant lamp voltage, and it is permissible or even desirable to allow the voltage of the power circuit to fall when there is a heavy rush of current, as on starting an elevator. Figure 2 represents a type of installation occasionally used in such a case. The shunt

field winding of the booster f is connected across the constant-potential lighting bus, and its magnetizing effect is opposed by that of the series winding S, as indicated by the arrows. The excitation due to f is normally the greater of the two, and the resultant differential excitation produces a booster voltage that acts in the same direction as the generator and that is normally 10 to 15 volts. At normal load the adjustments are such that the battery neither charges nor discharges, or the sum of the voltages of generator and booster equals the open-circuit voltage of the battery. The entire lighting and power load is then carried by the main generator. If the motor load is suddenly increased, there is an initial tendency to draw the increased current from the generator, but this results in an increased excitation of the series winding of the booster and a reduction of its generated e.m.f.; the original condition of balanced voltage at the battery terminals is therefore disturbed, and the battery discharges and relieves the generator of the current in excess of the normal amount. Conversely, a decrease of the motor load results in a momentary weakening of the series excitation of the booster, and a charging current flows into the battery. The current through the armature and series field of the booster is therefore not constant, as the term constant-current booster might imply, but it is substantially so, the total variation of a few per cent being no more than is sufficient to cause the battery to take up the fluctuations of current above and below the average value. When a storage battery charges or discharges, its terminal voltage rises or falls by an amount nearly proportional to the current. Thus, if the current is equal to the 8-hr. rate, the change of voltage is 0.05 volt per cell; and at the 1-hr. rate* (equivalent to four times the current at the 8-hr. rate), the variation is 0.2 to 0.21 volt per cell, provided that the battery is initially fully charged. The function of the booster is to produce a change of voltage at the battery terminals corresponding to the charge or

* If the capacity of a storage battery is C_8 amp.-hr. when discharged at the 8-hr. rate, its capacity is greatly decreased if it is discharged at greater rates. Thus, if the current is such that the voltage per cell falls to 1.8 volts in 1 hr., the current is said to be the 1-hr. rate, and the capacity falls to $\frac{1}{2}C_8$ amp.-hr. The reduced capacity is due to the fact that the high rates of discharge produce chemical changes of great velocity in a thin surface film of the active material, the electrolyte being thus prevented from pene-

discharge rate demanded by the load. For example, assume that the voltage of the motor circuit is 230, requiring, say 115 cells when charged to a normal voltage of 2 volts per cell. If the load calls for a supply of current equivalent to the 8-hr. discharge rate of the battery, over and above the normal supply of $I_{aver.}$ amp., the booster voltage must be lowered by $115 \times 0.05 = 5.75$ volts. This can be accomplished by so proportioning the series winding that an increase of the current through it from $I_{aver.}$ to $I_{aver.}\,(1 + p)$ will produce the necessary change in field excita-

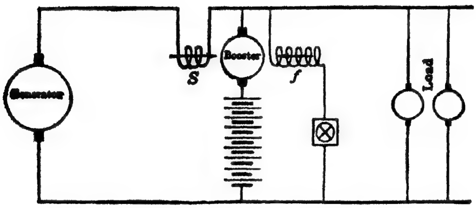


FIG. 3.—Differential or reversible booster.

tion, where $p \times 100$ is the prescribed percentage variation of booster current. The linear variation of booster voltage requires that the magnetic circuit be worked on the straight part of the magnetization curve. Change of battery voltage with varying conditions of charge can be compensated by manual regulation of a rheostat in the shunt field of the booster; but in the type of service to which the nonreversible (and other automatic) boosters are adapted the fluctuations of load causing alternate charge and discharge are so rapid that the general condition of the battery changes very little. The nonreversible booster is suited to systems in which the average motor load is small and the fluctuations are considerable.

trating to fresh material. The relation between discharge rate (n) and the corresponding capacity in ampere-hours (C_n) is given approximately by the formula

$$C_n = \frac{C_8}{2}\sqrt[3]{n}$$

If i_8 is the current corresponding to the 8-hr. rate and i_n the current corresponding to the n-hr. rate, it is clear that $C_8 = 8i_8$ and $C_n = ni_n$; whence, substituting the above approximate relation between C_n and C_8,

$$i_n = \frac{4i_8}{\sqrt[3]{n^2}} \text{ (nearly)}$$

5. Reversible Booster.—The reversible booster (Fig. 3) was formerly used in street-railway systems having a large average load, its purpose having been to permit the battery to carry the fluctuations above and below the normal load. Accordingly, at normal load, the battery current was adjusted to zero value by regulation of the shunt field f, which opposed the series winding S. Under these conditions an increase in load caused S to overpower f, and the resultant booster e.m.f., acting in the same direction as the battery, caused discharge to take place. Similarly, a decrease of load caused f to overpower S, thus reversing the polarity of the booster and causing a charging current to flow into the battery.

6. Auxiliary Control of Boosters.—The boosters illustrated in Figs. 2 and 3 required the use of series windings of large cross-

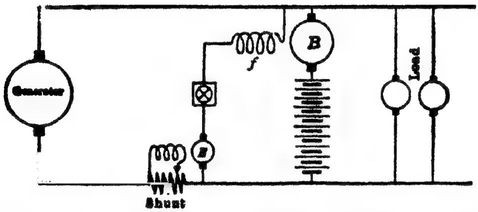


FIG. 4.—Hubbard counter e.m.f. system of booster regulation.

section, which added materially to the size, weight, and cost of the machine. Moreover, a given change of current in the series winding always resulted in the same change in the booster voltage, without regard to the condition of the battery, which might be either close to exhaustion or fully charged. Several automatic systems, briefly described in the following paragraphs, were developed to overcome these difficulties; at the present time, however, large storage batteries are seldom used in street-railway systems, and the need for elaborate regulating systems is not so great as it once was.

The Hubbard counter e.m.f. system (Fig. 4) employs a separately excited booster B, of which the field winding f is in series with the armature of a small motor-driven exciter E, the field of E being in turn excited by a fractional part of the main generator current. At normal load the e.m.f. of E is equal and opposite to that of the line, so that the booster field is unexcited, and the battery neither charges nor discharges provided that the battery

voltage is equal to the line voltage. An increase of load above the average value results in an increase of the current through the series coil of the exciter, the generated e.m.f. of the latter then exceeds the line voltage, and a flow of current is established through the booster field winding in the proper direction to generate in the booster armature an e.m.f. that assists the battery to discharge. Conversely, a decrease of load weakens the field of the exciter, the polarity of the booster reverses, and the battery then takes a charging current.

The Entz system (Fig. 5) employs a booster B of which the field winding f is separately excited by a small motor-driven exciter E. The field winding f' of the exciter is connected into the Wheatstone-bridge circuit formed by the carbon piles R_1

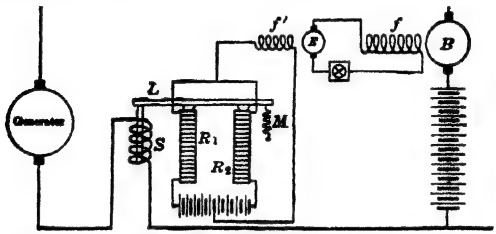


FIG. 5.—Automatic booster regulation, Entz system.

and R_2 and a small auxiliary battery. The pressure on the carbon piles is controlled by a lever L, actuated by the core of an electromagnet S excited by the main generator current. Under normal load conditions the resistances R_1 and R_2 are adjusted to equality, so that the winding f' is unexcited, booster B develops no e.m.f., and the main battery floats on the line. Any change in the main generator current disturbs the balance of the bridge circuit, causing current to flow one way or the other in f' and thus producing a booster e.m.f. of proper polarity to produce a charge or discharge of the main battery as demanded by the initial change in the generator current.

7. Balancers.—Figures 6*a*, *b*, and *c* represent three possible methods of connecting a balancer set for the purpose of maintaining equality, or approximate equality, between the voltages on the two sides of a three-wire system (see Art. 21, Chap. X). If, with the connections shown in Fig. 6*a*, the load becomes unbalanced, the voltage on the more heavily loaded side falls,

whereas that on the more lightly loaded side rises. Under these conditions the unit on the heavily loaded side acts as a generator, thus checking the extent of the voltage drop, and the other unit acts as a motor and limits the rise of voltage on that side; but the drop in speed of the balancer, due to the load on the motor element, prevents the generator element from assuming a sufficient part of the unbalanced load to maintain the potential of the neutral as nearly constant as would be the case were the speed to remain constant. A partial compensation of this shift of the neutral may be effected by the system of field connections shown in Fig. 6*b*; in this case the drop in voltage on the heavily loaded side weakens the field of the motor, thus tending to

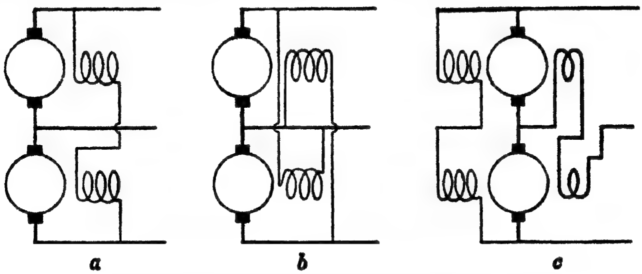


FIG. 6.—Connections of balancer set, three-wire system.

increase its speed, and at the same time the rise in voltage on the lightly loaded side strengthens the field of the generator element, thus tending still further to balance the voltage on the two sides of the system; but the balance cannot be perfect for the reason that the automatic response of the balancer depends for its inception upon an actual unbalancing of the voltage. Perfect regulation is possible, however, if the units comprising the balancer are compound-wound as in Fig. 6*c*, where the series windings are connected in such a manner that the current in the neutral excites the generator cumulatively, while in the motor it acts differentially. The voltage on the heavily loaded side is therefore kept up by the combined effect of increased excitation and increased speed; but, whereas in the system of Fig. 6*b* this automatic action was dependent upon an unbalanced *voltage*, in the system of Fig. 6*c* it depends upon the unbalanced *current*, and it is therefore possible to adjust the compounding

to maintain perfect equality of voltage on both sides of the neutral.

8. Train Lighting.*—The condition to be satisfied by any system of train lighting is that the lamp voltage shall be maintained at a constant value independently of the number of lamps in use and of the speed and direction of motion of the train. In the case of steam railroads three methods of electric lighting are in use:

1. The *straight storage* system, in which each car is equipped with its own storage battery.

2. The *head-end* system, in which a single constant-voltage generator placed in the baggage car or on the locomotive supplies current to the entire train.

3. The *axle-lighting* system, in which a small generator mounted under each car is driven directly from the axle.

The straight storage system was used in the earliest installations of electric lighting on steam railroads. It has the disadvantage that the gradual exhaustion of the battery results in inferior illumination toward the end of long runs. The batteries must be charged at terminal or division points or else be replaced by fully charged batteries.

In the head-end system a single compound-wound generator is driven by a turbine taking steam from the locomotive; in some cases the turbo-generator unit is installed in the baggage car, and in others it is mounted on the locomotive. The complete equipment must include storage batteries, generally one for each car, in order that the lights may be operated when the train is parted, as during switching, and at low train speed. The standard (lead) battery equipment for Pullman cars consists of 16 cells, corresponding to a nominal lamp voltage of 30 volts. The variation of battery voltage between the extremes of full discharge and full charge requires the use of an automatic regulator in order to maintain constant voltage at the lamps.

In the axle-lighting system the maintenance of constant voltage is complicated by the fact that not only will the speed of a generator positively driven from the car axle vary through wide limits, but also the machine must be capable of operating in either direction. Generators of the ordinary types do not possess inherent operating characteristics suitable for such service; and

* See *Trans. A.I.E.E.*, **21**, 129–227, 1903.

to make a machine of ordinary type conform to the requirements, more or less elaborate regulating devices must be used. Naturally, axle-driven generators must be used in connection with storage batteries in order that the lights may not go out when the train is stationary or when the speed is so low that the generator voltage is less than the normal lamp voltage.

The design of generators for automobile lighting is similar to that of axle-driven machines for train lighting except that there is no need to provide for reversal of the direction of rotation. This follows from the fact that in automobiles the generator is driven from the engine, which always runs in the same direction.

9. Voltage Regulation in Train-lighting Systems.—To prevent objectionable variation of the candle power of the lamps, automatic regulation must be provided to compensate for the variation of battery voltage between the extremes of full charge and full discharge and, in the case of axle-driven generators, to overcome voltage variations due to change of speed. The various methods of regulation may be classified as either *mechanical* or *electrical* (or electromagnetic). In some systems the maintenance of constant voltage also involves regulation for constant-current output from the generator, and hence the latter, when in use, delivers constant power; such regulation is not entirely satisfactory, for it takes no account of the fact that the charging current of a lead battery should "taper," that is, become gradually less, as the battery approaches the fully charged condition. It is possible to arrange the regulating devices in such a manner that the voltage and current output of the generator are controlled by the battery voltage or else to make the generator control the line voltage and therefore also that of the battery.

Under the heading of *mechanical* methods of regulation may be included those axle-lighting systems in which the generator voltage is controlled by the slipping of the driving belt, as in the Stone generator, or by a slipping clutch. In these systems the speed of the generator is maintained constant when the load increases above a definite predetermined load that causes slipping to occur. In the Stone system the generator is provided with an automatic device, consisting of a rocker arm on the shaft, for reversing the polarity of the generator terminals when the

direction of rotation is reversed; there is also an automatic, centrifugally operated switch arranged to establish the connection between the generator and the battery when the speed and generator voltage have reached predetermined pickup values and to break the connection when the speed is below the assigned limit.

Under *electrical*, or *electromagnetic*, methods of regulation may be grouped all systems in which voltage control is obtained (*a*) by the automatic variation of resistance in the lamp circuit or in the exciting circuit of the generator; or (*b*) by the utilization of the armature reaction of the generator to secure the desired characteristics. Examples of these methods are given in the following articles.

10. Resistance Regulation.—Figure 7 illustrates diagrammatically a type of automatic regulator that operates by varying

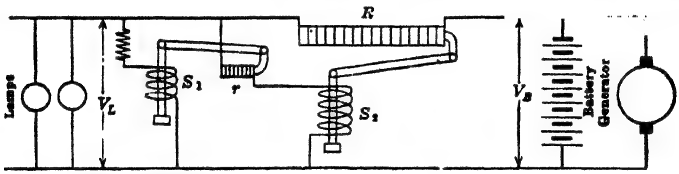


FIG. 7.—Voltage regulation by means of resistance in main line (Gould system).

the resistance of a pile of carbon disks connected in the main lamp circuit. If the battery voltage V_B rises above normal, as during charging, the lamp voltage V_L tends to increase also. This increase causes an increased flow of current through the solenoid S_1, and the movement of its plunger increases the pressure on the carbon pile r, thus reducing its resistance and permitting an increased flow of current through solenoid S_2. The motion of the plunger of S_2 releases the pressure on the carbon pile R, and its resistance is increased to a sufficient extent to absorb the greater part of the increase of V_B as an ohmic drop in R. Since the response of the solenoids S_1 and S_2 is dependent upon a variation of V_L, the lamp voltage cannot be held absolutely constant, but the variation is small; the use of the solenoid S_1 and pile r increases the sensitiveness of the response of S_2 to a change in V_L. The lamp regulator is used in conjunction with a generator regulator described in Art. 11.

11. Generator Field Regulation.—Figure 8 illustrates a method of regulating the generator voltage by the variation of a resistance

R in its field circuit. The carbon pile R is acted upon by the two solenoids V and B, the former responding to changes of the generator voltage due to change of speed, and the latter to variations of the battery current. For example, if we assume that the contact K is closed and that the generator is charging the battery, any increase of speed will tend to increase both the generator voltage and the charging current. As the charging current increases, the upward pull of solenoid B relieves the pressure normally exerted upon R by the weight of the plunger of B, thus increasing the field resistance of the generator and lower-

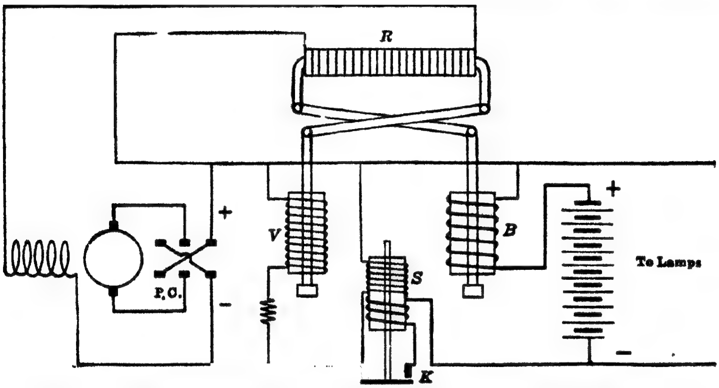


FIG. 8.—Voltage regulation by means of field rheostat (Gould system).

ing its voltage. To prevent excessive overcharge of the battery due to high generator speed, the plunger of solenoid V is arranged so that the increased line voltage causes it to relieve the pressure on the right-hand side of R, thus increasing the field resistance and lowering the generator voltage.

The automatic switch K is closed, and the connection between generator and battery is established, when the speed of the generator is sufficiently high to generate a voltage capable of actuating the solenoid S; the generator current, flowing through the series winding of switch K, reinforces the pull of the shunt winding S. When the speed falls below this pickup speed, the battery voltage overpowers that of the generator and a reverse current flows through the series winding of K, with the result that the net force acting upon the plunger is not sufficient to hold the contacts closed against the gravitational pull. The entire load is then

carried by the battery alone. The pole changer is represented diagrammatically at *PC*.

Another method, somewhat similar to that of Fig. 8, but involving a vibrating contact analogous to that of the Tirrill regulator, is illustrated in Fig. 9. The automatic switch *K* operates in the same manner as in Fig. 8, but the solenoids *B* and *V* act

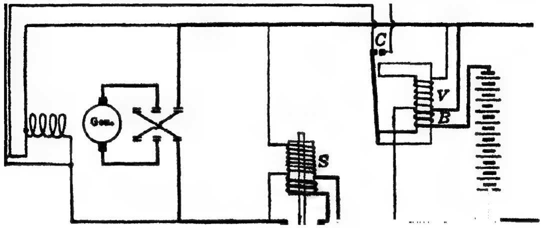


FIG. 9.—Voltage regulation using vibrating contact.

on the same magnetic circuit and open and close the contact *C*. Thus, if the battery-charging current exceeds the safe limit, coil *B* closes contact *C* and momentarily short-circuits the field winding of the generator, thus reducing the generator voltage. Coil *V* operates similarly if the generator voltage rises too high because of high rotative speed.

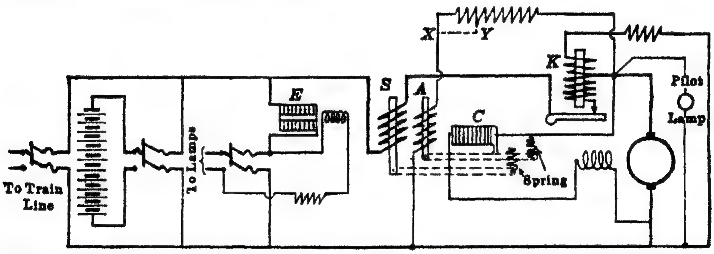


FIG. 10.—Combined field and line regulation. (*Safety Car Heating and Lighting Co.*)

12. Field and Line Regulation.—Figure 10 is a diagram of connections of a system of train lighting which, like the system described separately in Arts. 10 and 11, includes independent regulation of generator voltage and of lamp voltage.

The resistance of the carbon pile *C*, in series with the shunt field winding of the generator, is controlled by the pressure exerted upon it by levers operated by the plungers of coils *S* and

A. Coil S carries the entire generator current and is adjusted to hold the current at its full rated value. Coil A, shunted across the line, is set to hold the generator voltage at 39 volts on equipments having 16-cell lead batteries (2.45 volts per cell) and at 78 volts on nominal 60-volt equipments having 32 lead cells. If Edison batteries are used, these voltages are set at 43 volts and 86 volts, respectively, by opening the short circuit XY on part of the resistance in series with the voltage coil A. The object of limiting the generator voltage to 2.45 volts per cell of lead battery is to prevent excessive overcharging of the battery; when the battery is fully charged, the charging current will automatically taper down to a safe value. The limitation of generator current imposed by coil S prevents overloading of the generator due to lamp load or to charging an exhausted battery.

The lamp voltage is controlled by the pair of carbon piles E, in series with the lamps, the two piles being connected in parallel with each other. The pressure upon these piles is due to a system of levers and a toggle joint actuated by a coil connected across the lamp mains. The pull of this electromagnet is opposed by a spring, the design being such that the armature of the electromagnet will remain in any position within the limits of its travel when the lamp voltage is normal.

In this system the armatures of the magnets controlling the generator and the lamp circuit are provided with air dashpots having graphite plungers. The effect of variation of temperature upon the voltage coils of the generator and lamp regulators is compensated by means of resistors, having zero temperature coefficients, placed in series with these coils.

The automatic switch K for establishing the connection between generator and battery at train speeds above the pickup speed is similar to others already described. The shunt coil lifts the pivoted armature when the generator voltage equals the battery voltage, thus bringing into action the series coil, which assists the shunt coil in holding the switch tightly closed and which accelerates the opening of the switch when the generator voltage falls below battery voltage.

The four brush arms of the generator are mounted on a rocker ring carried on ball bearings, the ring being free to rotate through 90 deg. between a pair of stops. When the machine is running in one direction, the friction of the brushes against the commuta-

tor holds the rocker ring against one of the stops and the brushes are then in the proper position for sparkless commutation. Reversal of the direction of rotation causes the rocker ring to be turned through 90 deg. against the other stop, the original polarity of the generator being thus preserved.

Figure 11 shows in diagrammatic form another system, which includes a lamp regulator and a generator field regulator F. The automatic switch K is closed in response to the pull of the voltage coil when the generator speed and voltage have attained their proper values, the battery being thus connected to the generator. The charging current, flowing through the series coil of the regulator F, tends to be maintained at constant value by the

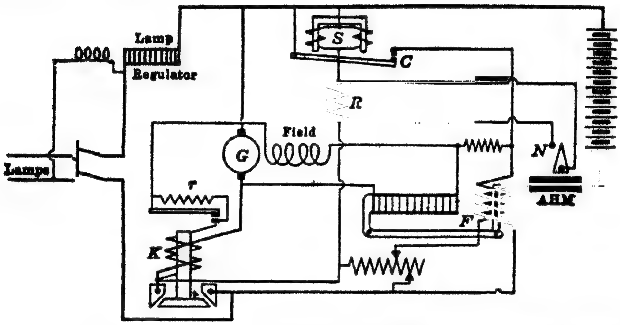


FIG. 11.—U. S. Lighting and Heating Company system of train lighting.

action of the carbon-pile rheostat in circuit with the generator field. At the same time the ampere-hour meter AHM is running in the direction of charge; and when the battery is charged and the contact needle N has reached its point of contact, the resistance R is short-circuited. Thereupon the switch S is energized, contact C is closed, and current flows through the shunt coil of the regulator F. The pull of the shunt coil adds to that of the series coil, so that there results a sudden reduction of generator voltage and the battery is thereby "floated" on the line; that is, the generator supplies current directly to the load, and the battery neither charges nor discharges.

13. Regulation by Means of Armature Reaction.—Regulation of generator voltage by making use of armature reaction under load conditions is exemplified in the Rosenberg train-lighting generator (Art. 14) and in the third-brush type of generator

used for automobile lighting (Art. 17). This type of regulation, since it is dependent upon the inherent characteristics of the generator, may be classed as electromagnetic.

14. Rosenberg Train-lighting Generator.—The Rosenberg generator, first described* in 1905, embodies a number of interesting structural features and has operating characteristics that make it suitable for train-lighting service. Its distinctive properties are: (1) that it develops an e.m.f. the direction of which is independent of the direction of rotation; and (2) that it produces a current which, beyond a certain speed, remains

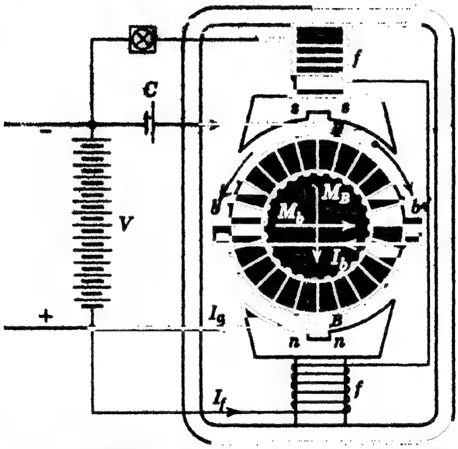


FIG. 12.—Diagram of Rosenberg train-lighting generator.

practically constant no matter how much the speed is increased. The diagram of connections of a bipolar machine is shown in Fig. 12, but it will be understood that with suitable modifications the principle is applicable to multipolar machines. The battery V, which must be used in connection with the generator if the latter is to function properly, supplies current to the lamps when the train is at rest and also to the shunt field winding ff, producing the polarity indicated by nn, ss. The axis of commutation of the brushes BB is in line with the axis of the poles, instead of being at right angles thereto as in the ordinary type of machine.

* *Elektrotech. Z.*, **26**, 393, 1905.

The brushes BB are connected to the battery terminals through an aluminum cell C which offers a very high resistance to the flow of current from the battery to the armature and only a very small resistance in the direction from the armature to the battery; this property of the aluminum cell prevents the discharge of the battery through the armature when the train is at rest or running at a speed below that at which the generator picks up its load. In addition to the main brushes there is a pair of short-circuited auxiliary brushes bb, placed at right angles to the polar axis, that is, in the same position as the main brushes of an ordinary generator.

Rotation of the armature through the magnetic field set up by ff produces a flow of current through the short-circuited armature along axis bb, thereby creating a powerful cross field M_b, the lines of force of this field finding a path of low reluctance through the pole shoes. As is clear from the figure, clockwise rotation results in a cross field directed from left to right; the motion of the armature conductors through this cross field then generates an e.m.f. and current along the BB axis in such a direction that the armature m.m.f., represented by the arrow M_B, opposes the excitation due to the field winding ff. In case the direction of rotation is reversed (that is, becomes counterclockwise) the direction of the cross field M_b also reverses, and the effect of this double reversal is to preserve the original polarity of the brushes BB. The fact that the armature m.m.f. M_B opposes the excitation due to the field winding means that the field flux parallel to the BB axis is small, and this fact in turn prevents excessive current through the short circuit bb. The machine differs widely from the ordinary generator in that what is usually the main field is of secondary importance relative to the cross field. The weak field in the BB axis obviates commutation difficulties that would otherwise arise due to the short-circuiting of winding elements under the middle of a pole face; such difficulty as might still exist is further overcome by notching the pole faces opposite the main brushes. It is clear that there is a definite limit beyond which the main current delivered by the brushes BB cannot increase, this limit being reached when the armature m.m.f. M_B neutralizes the field excitation due to ff; for in that case there would be no e.m.f. and current in the bb axis, and hence no e.m.f. in the main brush axis. It follows,

therefore, that beyond a certain speed the machine will deliver a practically constant current. Any desired limit to the current may be set by adjusting the rheostat in the field circuit *ff*. The generator may be driven either by a belt from the car axle or by mounting the armature directly on the axle itself.

On the basis of the foregoing qualitative study of the physical phenomena occurring in the machine, Kuhlman and Hahnemann* have developed the quantitative relations between the speed and current output of the machine operating as a generator. Thus, let

n = speed of the armature in revolutions per minute.
I_f = constant exciting current in field winding *ff*.
I_a = main current output.
I_b = short-circuit current in axis *bb*.
N_fI_f = ampere-turns due to *ff*.
N_aI_a = effective ampere-turns of armature in axis *BB*.
N_bI_b = effective ampere-turns of armature in axis *bb*.
V = terminal voltage of line, assumed constant.
E_b = e.m.f. generated in short circuit *bb*.
Φ_B = field flux in axis *BB*.
Φ_b = field flux in axis *bb*.
R_a = armature resistance (including brushes).

If saturation of the magnetic circuit is neglected, so that the flux may be considered to be proportional to the m.m.f. that produces it, the following relations will hold:

$$E_b = c_1\Phi_B n \tag{1}$$

$$I_b = \frac{E_b}{R_a} = \frac{c_1}{R_a}\Phi_B n \tag{2}$$

$$\Phi_B = c_2(N_fI_f - N_aI_a) \tag{3}$$

$$\Phi_b = c_3N_bI_b \tag{4}$$

$$V = c_1\Phi_b n - I_aR_a \tag{5}$$

Upon substituting Eqs. (2) and (3) in (4), there results

$$\Phi_b = c_1c_2c_3\frac{N_b}{R_a}(N_fI_f - N_aI_a)n \tag{6}$$

and substituting this value of Φ_b in Eq. (5)

$$V = c_1^2c_2c_3\frac{N_b}{R_a}(N_fI_f - N_aI_a)n^2 - I_aR_a \tag{7}$$

* *Elektrotech. Z.*, **26**, 525, 1905.

and

$$I_a = \frac{c_4 N_f I_f n^2 - V}{R_a + c_4 N_a n^2} = \frac{\dfrac{N_f I_f}{N_a} - \dfrac{V}{c_4 N_a} \dfrac{1}{n^2}}{1 + \dfrac{R_a}{c_4 N_a} \dfrac{1}{n^2}} \tag{8}$$

where

$$c_4 = c_1^2 c_2 c_3 \frac{N_b}{R_a}$$

From Eq. (8) the following conclusions may be drawn:

1. If $n = 0$,

$$I_a = -\frac{V}{R_a}$$

which means that were it not for the aluminum cell C the armature, at standstill, would be a dead short circuit on the line (or battery), the negative sign of I_a indicating a flow of current into the armature from the line.

2. If $n = \infty$,

$$I_a = \frac{N_f I_f}{N_a} \qquad \text{or} \qquad N_a I_a = N_f I_f$$

which means that at infinite speed the armature m.m.f. M_B would exactly neutralize the field excitation. This condition therefore determines the limiting current output of the machine running as a generator, or

$$(I_a)_{max.} = \frac{N_f I_f}{N_a}$$

This result also shows how the rheostat in the field circuit controls $(I_a)_{max.}$ by fixing the value of I_f.

3. If the machine is to act as a generator, I_a must be positive; hence,

$$\frac{N_f I_f}{N_a} = (I_a)_{max.} \geq \frac{V}{c_4 N_a} \frac{1}{n^2}$$

or

$$\frac{(I_a)_{max.} R_a}{V} \geq \frac{R_a}{c_4 N_a} \frac{1}{n^2}$$

But the term $(I_a)_{max.} R_a / V$ is the ratio of the maximum ohmic drop in the armature to the line voltage; and since this ratio must be

small from considerations of efficiency, it follows that $(R_a/c_4N_a)\cdot(1/n^2)$ is still smaller in comparison with unity and that with increasing speed it rapidly approaches zero. Therefore the denominator of Eq. (8) may be considered equal to unity and the expression for I_a becomes, with only slight error,

$$I_a = (I_a)_{max.} - \frac{V}{c_4N_a}\cdot\frac{1}{n^2} \tag{9}$$

Eqs. (8) and (9) show that the current is zero when

$$n = n_0 = \sqrt{\frac{V}{c_4N_a(I_a)_{max.}}} = \sqrt{\frac{V}{c_4N_fI_f}} \tag{10}$$

and that it rapidly approaches $(I_a)_{max.}$ as a limit as the speed increases.

For example, suppose that the generator is to supply a maximum current of 50 amp. at a terminal voltage of 50 volts and that it is to pick up its load at a speed of 300 r.p.m. From Eq. (10)

$$300 = \sqrt{\frac{50}{c_4N_a \times 50}}$$

$$c_4N_a = \frac{1}{90{,}000}$$

and, from Eq. (9),

$$I_a = 50 - \frac{4.5 \times 10^6}{n^2}$$

This is the equation of the curve shown in Fig. 13. The manner of variation of I_b (the current in the short-circuited path *bb*) is determined by combining Eqs. (2), (3), (9), the result being

$$I_b = \frac{V}{c_1c_3N_b}\cdot\frac{1}{n} \tag{11}$$

which represents an equilateral hyperbola. It is seen that I_b is a function of c_3 and this is dependent upon the reluctance in the path of the cross field Φ_b. The curve showing I_b in Fig. 13 is based on the assumption that $I_b = 60$ amp. when $n = 300$, or $I_b = 18{,}000/n$. The dashed portions of the curves of Fig. 13, computed from Eqs. (9) and (11), correspond to negative values of I_a (indicating motor action); and though not entirely accurate

because of the neglect of terms involving R_a in the presence of small values of n, they depart only slightly from the correct curves within the range shown in the diagram.

An examination of Fig. 12 shows that in two of the quadrants of the armature winding the currents I_a and I_b flow in the same direction in the conductors, and in the other two quadrants

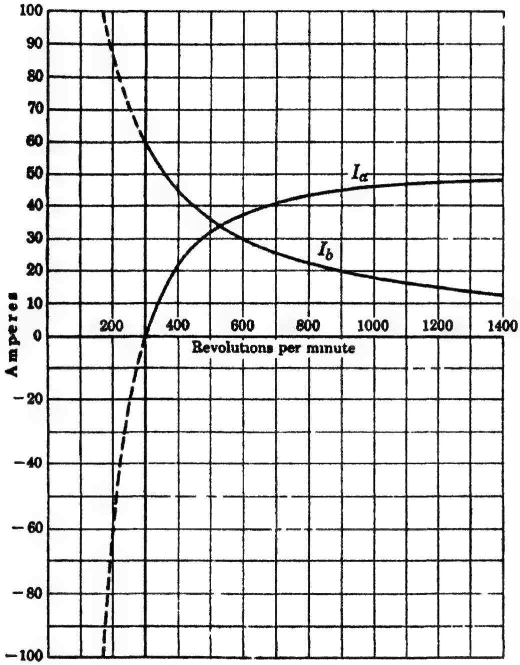


FIG. 13.—Relation between current and speed, Rosenberg generator.

they flow in opposite directions. The total current in the former case is

$$I_a + I_b = (I_a)_{max} - \frac{V}{c_4 N_a} \cdot \frac{1}{n^2} + \frac{V}{c_1 c_3 N_b} \cdot \frac{1}{n}$$

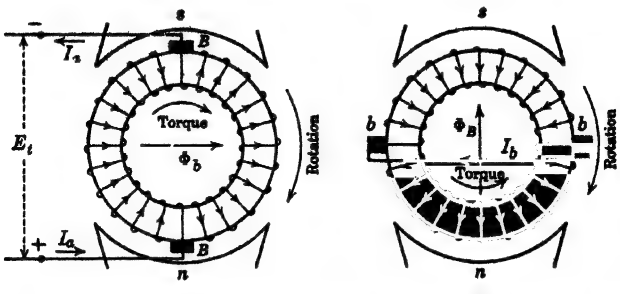
and this current reaches a maximum value when

$$\frac{d(I_a + I_b)}{dn} = \frac{2V}{c_4 N_a} \cdot \frac{1}{n^3} - \frac{V}{c_1 c_3 N_b} \cdot \frac{1}{n^2} = 0$$

or when $n = 2c_1c_3N_b/c_4N_a$. Upon substituting the values used above, the speed corresponding to this maximum current in

the conductors is 500 r.p.m., and the currents themselves are $I_a = 32$ and $I_b = 36$.

15. Operation of Rosenberg Machine as Motor.—The Rosenberg machine when supplied with current from an external source will operate as a motor, but it has no torque at standstill. The reason for this absence of starting torque is clear from Fig. 12; for the armature current and the field flux due to *ff* have their axes in the same direction and cannot react upon each other, and there can be no cross field to react upon the armature current until rotation through Φ_B produces current and flux in the *bb* axis. But if the armature is given a start in either direction it will continue to run in that direction. Thus, let



(a) (b)

FIG. 14.—Relation between currents and fluxes in Rosenberg machine operating as motor.

Fig. 14 represent the same machine shown in Fig. 12 but taking current from, instead of supplying it to, the line, and let the starting impulse be in the clockwise direction. The figure is drawn in two parts in order to show with greater clearness the effect of the two pairs of brushes; the arrows on the armature conductors of part (*a*) show the direction of flow of I_a and those in (*b*) serve similarly for I_b. The direction of I_b is determined by applying Fleming's right-hand rule for generator action; had the initial rotation been counterclockwise, the direction of I_b, and therefore also of Φ_b, would have been opposite to that shown. In either case the reaction between Φ_b and I_a produces a torque in the same direction as the initial rotation and, therefore, serves to accelerate the armature; and the torque due to the reaction between Φ_B and I_b always opposes the rotation. The resultant torque is the difference between these two opposing torques.

Analytically, the characteristics of this motor when supplied from constant-potential mains are involved in the equations derived in the preceding article for the case of the generator. It is necessary only to interpret negative values of I_a in those equations as current input to the motor, but some care should be used in applying Eqs. (9) and (11), especially at low speeds, because the term involving R_a in Eq. (8) is then not negligible as has been assumed. Thus, if $R_a = 0.10$ ohm, corresponding to a maximum armature drop of 10 per cent when the constants are those used in the foregoing discussion, the standstill current computed from Eq. (9) is $I_a = -\infty$, whereas the true value from Eq. (8) is $I_a = -500$. The range of speed through which motor action occurs is from $n = 0$ to $n = \sqrt{V/c_4N_fI_f}$ (0 to 300 r.p.m., Fig. 13). Without going into further particulars it will be seen that the speed characteristic is similar to that of a cumulative-compound motor.

16. A modification of the Rosenberg type of generator, together with a special method of voltage control, developed by the Electric Storage Battery Company, is illustrated in Fig. 15. Instead of connecting the shunt field winding across the machine terminals, as in Fig. 12, it is connected between opposite points of a Wheatstone-bridge circuit (marked "control bridge" in Fig. 15). There is also a compensating winding, marked "series field," for the purpose of neutralizing the armature reaction due to the main generator current. Two of the bridge arms, marked R, consist of ordinary resistors, and the other two, marked IR, have negative temperature coefficients. The junction points of the bridge not connected to the control field are connected directly across the line. A variation of generator voltage will then alter the difference of potential between the control-field terminals, and the field current will change to a sufficient extent to readjust the generator voltage. If it is desired to give the battery an overcharge, the overcharge switch which normally short-circuits the resistance R' is opened, the lamp circuit having been previously disconnected. These connections have the effect of reducing the voltage impressed on the bridge, in the same manner as though the generator voltage had itself decreased; hence, the readjustment of bridge currents increases the main field excitation and raises the terminal voltage. The extent of the rise of voltage and, therefore, the magnitude of

the charging current are determined by the value of the resistance R'. It will be observed that the closing of the lamp circuit through the triple-pole switch short-circuits R', thus reducing the generator voltage to the normal lamp voltage and preventing damage to the lamps because of high voltage during charging. It follows, therefore, that overcharging of the batteries must be accomplished during daylight runs.

The automatic switch, in addition to the usual shunt and series coils, has a third coil connected between the generator and the battery. The pull due to the main shunt coil is insufficient to close the switch, or to keep it closed without the pull due either

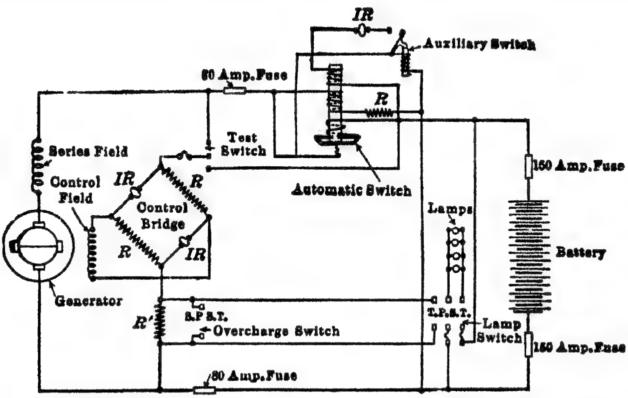


FIG. 15.—Rosenberg generator and control circuits. (*Electric Storage Battery Co.*)

to the auxiliary coil or to the series coil. The auxiliary coil, therefore, determines the closing of the switch by the difference between the voltages of generator and battery, and the switch will open when the current in the series coil drops to zero.

17. Third-brush Generator.—In an English patent (No. 9364) issued to W. B. Sayers in 1896, there is disclosed a system of connections, illustrated in Fig. 16, designed to produce automatic compounding action in a constant-speed generator of the shunt type by utilizing the reaction of the armature current upon the main magnetic field. The shunt winding, instead of being connected to the main brushes in the usual manner, is connected between an auxiliary brush B_3, located about halfway between the main brushes B_1 and B_2, and that one of the main brushes which will cause the shunt winding to subtend the trailing half

of the pole faces. Under load conditions the armature reaction increases the flux under the trailing half of the pole faces, thus increasing the e.m.f. generated in that part of the armature winding included between the terminals of the shunt-field winding and hence giving rise to increased excitation. The principles inherent in the Sayers' generator are embodied in the third-brush type of generator now widely used as part of the electrical equipment of automobiles, but with the difference (1) that the terminals of the shunt winding subtend the *leading* half of the pole faces, and (2) that the machine must be adapted to variable-speed operation since it is driven through suitable gearing directly from the engine of the automobile.

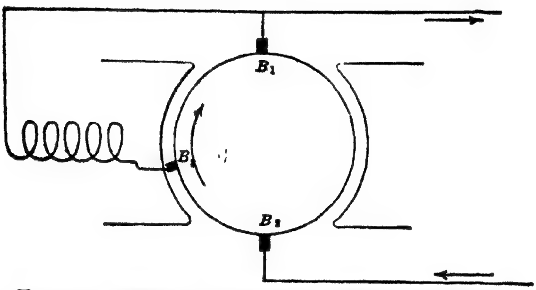


FIG. 16.—Diagram of connections of Sayers' generator.

With reference to Fig. 16, it will be seen that the connections automatically set a limit to the magnitude of the current the machine will deliver even though the speed be increased indefinitely, provided that the battery voltage remains substantially constant. For if the machine is delivering current at some particular speed, an increase of speed will tend to increase both the generated e.m.f. and the current; but the increased armature current shifts the flux away from the leading pole tip because of the greater cross-magnetizing action, the e.m.f. generated in the armature between brushes B_1 and B_3 being thus reduced and the shunt excitation weakened, or at any rate an increase being prevented to the extent that would otherwise result from the increase of speed. This action continues with each increment of speed until the main field has become so weakened that a further increase of current causes a disproportionately large reduction of excitation; thus not only is any further increase of current prevented, but actually the current falls off to smaller

values as the speed is still further raised. Accordingly, it is to be expected that the machine will develop a current that at first increases with rising speed and, after reaching a maximum value, falls off again.

The demagnetizing action due to the reaction of the main armature current is not the only effect of the kind that must be taken into account; it must also be considered that the auxiliary brush short-circuits an element of the armature winding which lies well under the pole face and in which there is generated an appreciable e.m.f. (see Fig. 17); this e.m.f. produces a con-

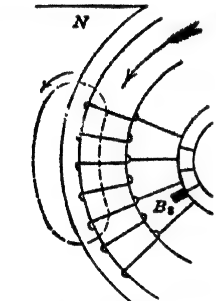


FIG. 17.—Demagnetizing effect due to short-circuited coil.

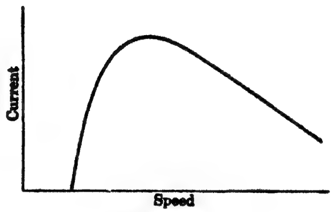


FIG. 18.—Characteristic of third-brush generator.

siderable current in the short-circuited element, and the direction of this current is such that it sets up an additional demagnetizing action in the airgap under the leading half of the pole faces, reducing the voltage in that part of the armature winding which supplies the excitation of the shunt winding.

Some type of automatic switch or cutout must be connected between the generator and the battery to prevent the battery from discharging through the generator under standstill or low speed conditions. The cutout usually employed is similar to those already described in connection with train-lighting systems, and it does not operate to close the circuit until the speed has reached a value sufficient to develop in the generator an e.m.f. equal to that of the battery. The speed-current characteristic may therefore be expected to have the general form of Fig. 18, and the effect of shifting the auxiliary brush will be to alter the points of pickup and maximum current, as well as the magnitude of the maximum current.

18. Arc-welding Generators.—Figure 19 is a diagram of a special generator* of the third-brush type which has a voltage-current characteristic suitable for single-operator, arc-welding outfits. Outwardly, the machine has the appearance of an ordinary four-pole generator with two commutating poles. The diagram shows, however, that the polarity of the poles has the order N, N, S, S, so that in reality the machine has two split poles, and the armature is accordingly provided with a two-pole winding. The axis of the main brushes coincides with the neutral axis between adjacent north and south poles and a third brush

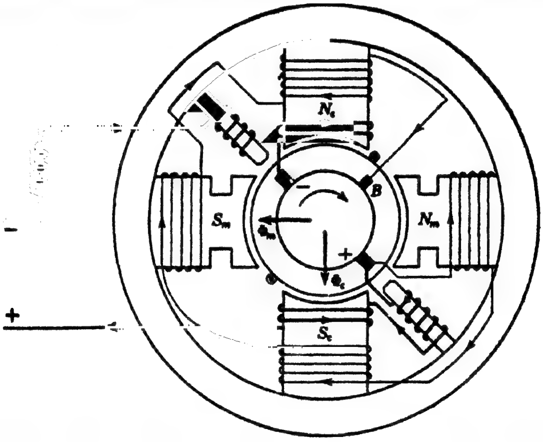


FIG. 19.—Special arc-welding generator.

B is located halfway between them. The shunt winding is connected between the main positive brush and B, as may be seen somewhat more clearly in the schematic diagram, Fig. 20. The series winding, which is placed on only two of the polar projections, opposes the magnetizing action of the shunt winding on the same poles.

If we assume clockwise rotation of the armature, referring to Fig. 19, the armature m.m.f. is directed along the 45-deg. line joining the main brush and in the direction + to −. It may be resolved into two components, one directed horizontally from right to left, coinciding with the m.m.f. due to the "main" poles N_m, S_m; the other directed vertically, in opposition to the

* S. R. BERGMAN, A New Type of Arc-welding Generator, *Gen. Elec. Rev.* **23**, No. 5, 442. 1920.

m.m.fs. due to the "cross" poles N_c, S_c. The path of the main flux Φ_m is constricted so that magnetic saturation exists at all loads; thus, the magnitude of Φ_m is not materially affected by the armature reaction, and Φ_m therefore remains substantially constant at all loads. The cross flux Φ_c, the magnetic circuit of which is not saturated, decreases with increasing current; for it is due to the steady m.m.f. of the shunt winding on N_c, S_c, less the sum of the m.m.fs. of the series winding and the cross-component of armature reaction. If the series winding is made sufficiently powerful, the cross flux Φ_c may even reverse its direction when the current is large.

It will be noted from the diagrams that the shunt field winding taps that portion of the armature winding which cuts the main flux Φ_m; since the latter remains practically constant, the e.m.f. generated in the part of the armature winding between the positive brush and B will also remain constant, constant shunt excitation being thus ensured. But the e.m.f. generated in the part of the armature winding between B and the negative brush decreases as Φ_c decreases and will even reverse its sign if Φ_c reverses. The total e.m.f. between the two main brushes is therefore made up of two parts, one of which is constant, the other of which decreases with increasing current. In the machine, as actually built, the no-load e.m.f. in each of the two halves of the armature winding is 30 volts, corresponding to a no-load terminal voltage of 60 volts. With a load current of 200 amp., the e.m.f. between B and the negative brush reverses and becomes -10 volts, the terminal voltage then being $30 - 10 = 20$ volts, which is what is needed to maintain the arc at the welding terminals.

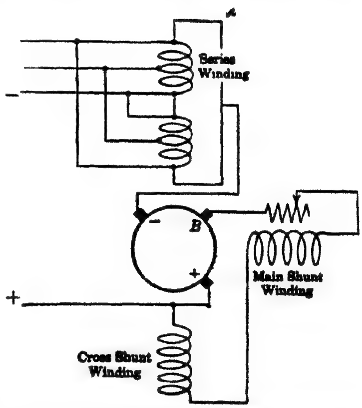


FIG. 20.—Schematic diagram of special arc-welding generator.

In Fig. 20 the series field winding is shown with a number of taps. In this way the welding current can be adjusted to any one of a series of values.

The volt-ampere characteristic of a d-c welding generator should have the drooping form illustrated in Fig. 21. The upper part of the characteristic is similar to that of a shunt generator

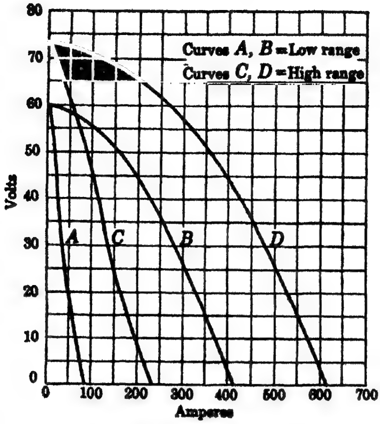


FIG. 21.—Characteristics of Wilson welder, 40 volts, 300 amp.

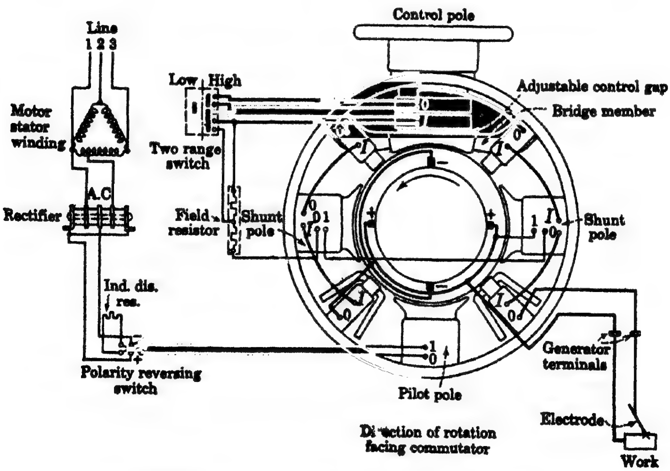


FIG. 22.—Connection diagram, Wilson welder.

with exaggerated armature reaction, and the lower part is similar to that of a series generator with excessive armature reaction. A generator of simple shunt type will not function satisfactorily because under short-circuit conditions the magnetic field col-

lapses, thus preventing recovery of excitation when the short circuit is removed; and a series-type generator is inappropriate because the open-circuit voltage is either zero or too small to strike an arc.

To meet these conditions the Wilson Welder and Metals Company has developed a special d-c generator of which the connection diagram is shown in Fig. 22. The generator is driven by a motor which may be either an a-c or a d-c machine, depending upon the available supply circuit. The generator frame

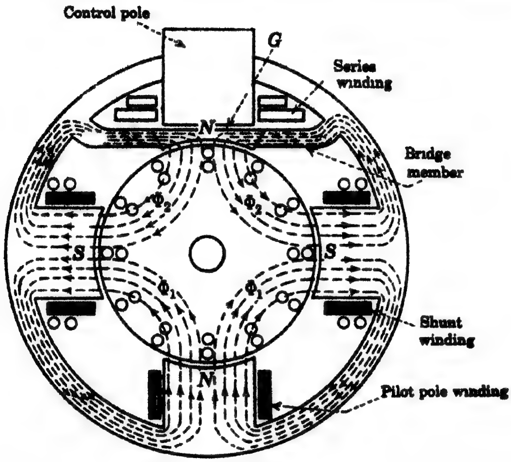


FIG 23a —Flux distribution, open circuit

has the general appearance of a four-pole, interpole-type machine, and the armature has a two-circuit (wave) winding wound for four poles The bottom (pilot) pole is separately excited from a constant-potential source—a rectifier in case of a-c supply, or directly from the line in case of d-c supply. The top (control) pole, excited by a winding in series with the armature, has a movable core by means of which the adjustable control gap can be varied at will; the pole face of the control pole is fixed in position, being supported by the bridge member shown in Figs. 22 and 23. The two horizontal poles are shunt-wound.

Figure 23a, which represents the flux distribution under open-circuited conditions, shows that the pilot-pole excitation sets up the flux Φ_1, which takes the shortest available path through the two adjacent horizontal poles. If this flux were the

only one present, it would still suffice to generate e.m.f. in the two-circuit armature winding; this e.m.f. produces an exciting current in the shunt field winding, thereby setting up flux Φ_2 and at the same time supplementing the excitation of the pilot pole in maintaining the flux Φ_1. By suitably proportioning the windings and the flux paths, the resultant flux distribution is the same as in any four-pole machine, and the voltage builds up immediately, and with certainty concerning its polarity, because of the constant separate excitation of the pilot pole.

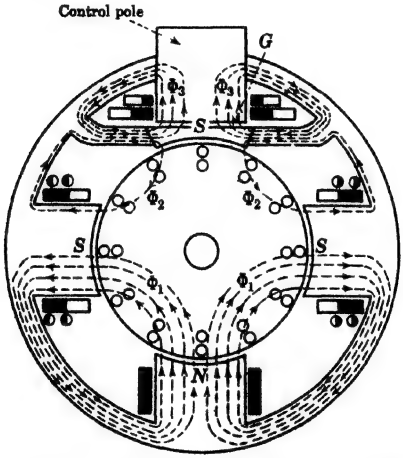


FIG. 23b.—Flux distribution, load conditions.

Under load conditions (Fig. 23b), the polarity of the control pole reverses because of the magnetizing action of the series winding, with the result that flux Φ_2 is reduced. This reduction of flux decreases the e.m.f. developed in the armature, thus still further weakening the shunt excitation, and so lowering the terminal voltage until a condition of equilibrium is reached.

When the current delivered to the load reaches a value such that flux Φ_2 practically disappears, as in Fig. 23c, the flux Φ_1 produced by the pilot pole is mostly diverted away from the shunt poles and completes its path by way of the control pole, the machine being thus changed from a four-pole to a predominantly two-pole type. Inasmuch as the armature is wound for four poles, there can be no e.m.f. developed in it under these

circumstances, and the terminal voltage becomes zero, corresponding to a condition of complete short circuit (as when the welding rod is in direct contact with the work).

Because of the fact that the pilot pole is always energized, partial or complete opening of the short circuit causes the flux Φ_1 to resume its path through the shunt poles, prompt recovery of the terminal voltage being thus ensured.

Since the series winding of the control pole is practically surrounded by iron, its inductance is high; this feature is desirable

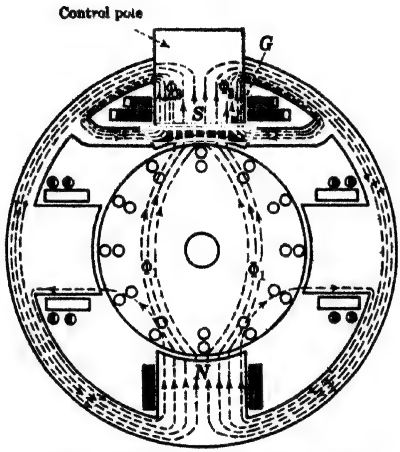


FIG. 23c.—Flux distribution, short circuit.

because inductance in the main circuit tends to suppress fluctuations in the magnitude of the welding current.

The high and low ranges indicated in Fig. 21 are obtained by means of the two-range switch shown in Fig. 22, which serves to connect the two halves of the control winding in series or in parallel. With a given setting of this switch, the form of the external characteristic can be adjusted anywhere between the limits indicated by curves *A* and *B*, or by *C* and *D*, Fig. 21, by changing the length of the gap *G* in the magnetic circuit of the control pole; for it will be seen from Fig. 23*b* that a given voltage calls for a definite distribution of the magnetic fluxes, and if the gap *G* is large, more current must flow in the series winding, and therefore to the weld, than when the gap is small.

19. Diverter-pole Generator.*—This machine, built by The Electric Products Company, is designed for battery-charging service in telephone exchanges, and in power stations where batteries are used to operate relays and circuit breakers. The construction is shown in Figs. 24 and 25. The pole punchings are so formed as to comprise in one piece a main pole and a secondary, or diverter, pole, the latter serving the purpose of a commutating pole. The main poles are excited by a shunt winding, and the commutating section by a series winding.

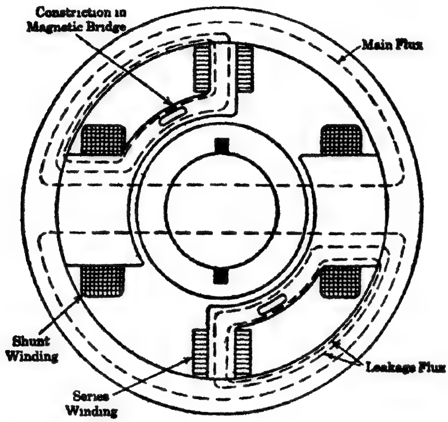


FIG. 24.—Diverter-pole generator. Flux distribution at no load.

In Fig. 24, it will be seen that at no load a part of the magnetic flux produced by the shunt excitation is diverted away from the armature, completing its path through the diverter pole. With increasing load the series excitation of the diverter pole rediverts more and more of this flux back to the armature, thus increasing the generated e.m.f. so as to compensate for the ohmic drop in the generator and for speed changes in the driving motor, and at the same time providing a commutating field.

The magnetic bridge between the main pole and the diverter has a constricted section which serves the double purpose of (1) limiting the magnetic leakage from the main pole, and (2) acting

* E. D. SMITH, The Diverter-pole Generator, *Jour. A.I.E.E.*, January, 1929, p. 11.

as a choke to control the flux entering the armature from the inner surface of the diverter pole.

The main pole and the yoke are designed for moderate flux densities, well below saturation; consequently the flux variations in the diverter pole, so far as these affect the main magnetic circuit, have little influence upon the excitation required to maintain the main flux. The flux densities in the diverter pole are also kept at low values, except in the constricted section; hence, the machine as a whole operates on the nearly straight

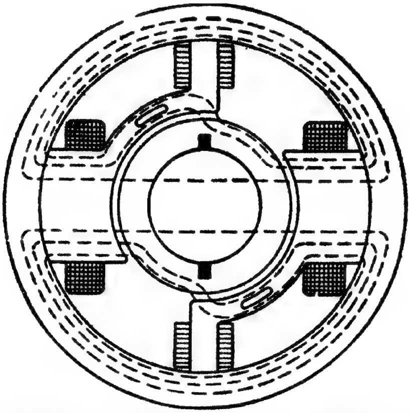


FIG. 25.—Diverter-pole generator. Flux distribution at full load.

portion of the magnetization curve, and this eliminates most of the curvature of the generator characteristic generally found in machines of the usual types.

In ordinary generators, the effect of the series excitation is to increase the polar flux as the load current becomes greater, whereas in this particular design the reverse is true. The effect is to invert the curvature of the magnetization curve as it is reflected in the generator characteristic, the volt-ampere curve being concave instead of convex. If, as may happen in battery charging, the battery feeds back to the generator, causing it to run as a motor, there is here no tendency to speed up or to reverse the polarity. This fact is due to the main flux being maintained substantially constant, since the constricted section of the magnetic bridge prevents the reversed series excitation from diverting additional leakage flux. Any tendency of the

diverter pole to assume a polarity opposite to that of the main pole is checked by the saturation of the diverter pole that precedes such reversal.

As the current output of the generator increases, a point will be reached at which the series ampere-turns on the diverter pole become equal to the shunt ampere-turns on the main pole.

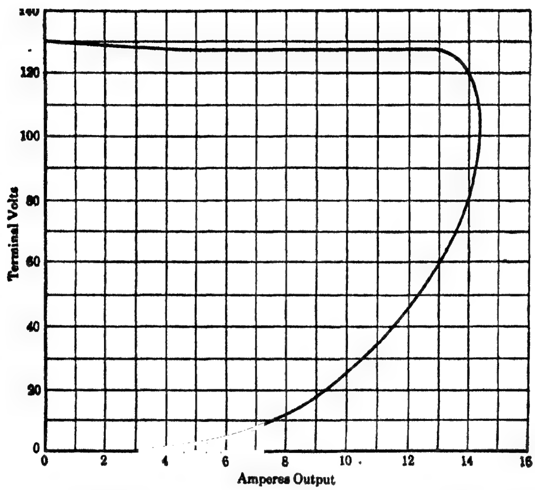


FIG. 26.—Characteristic of 1½-kw., 130-volt diverter-pole generator.

At this point all the leakage flux from the main pole will have been diverted back to the armature, so that there is no further reserve of flux to draw upon to maintain the terminal voltage. Even a slight increase of current beyond this point will cause the diverter-pole flux to collapse, and this, in turn, reduces the terminal voltage, and therefore also the main flux. The combined effect of these several reactions is to produce a sharp cutoff in the volt-ampere characteristic, as indicated in Fig. 26.

APPENDIX A

STANDARD ANNEALED COPPER WIRE, SOLID AMERICAN WIRE GAGE (B. & S.)*

Gage number	Diameter, mils	Cross-section: Circular mils	Cross-section: Square inches	Ohms per 1000 ft.: 25°C.	Ohms per 1000 ft.: 65°C.	Ohms per mile, 25°C.	Pounds per 1000 ft.
0000	460	212,000	0.166	0.0500	0.0577	0.264	641.0
000	410	168,000	0.132	0.0630	0.0727	0.333	508.0
00	365	133,000	0.105	0.0795	0.0917	0.420	403.0
0	325	106,000	0.0829	0.100	0.116	0.528	319.0
1	289	83,700	0.0657	0.126	0.146	0.665	253.0
2	258	66,400	0.0521	0.159	0.184	0.839	201.0
3	229	52,600	0.0413	0.201	0.232	1.061	159.0
4	204	41,700	0.0328	0.253	0.292	1.335	126.0
5	182	33,100	0.0260	0.319	0.369	1.685	100.0
6	162	26,300	0.0206	0.403	0.465	2.13	79.5
7	144	20,800	0.0164	0.508	0.586	2.68	63.0
8	128	16,500	0.0130	0.641	0.739	3.38	50.0
9	114	13,100	0.0103	0.808	0.932	4.27	39.6
10	102	10,400	0.00815	1.02	1.18	5.38	31.4
11	91	8230	0.00647	1.28	1.48	6.75	24.9
12	81	6530	0.00513	1.62	1.87	8.55	19.8
13	72	5180	0.00407	2.04	2.36	10.77	15.7
14	64	4110	0.00323	2.58	2.97	13.62	12.4
15	57	3260	0.00256	3.25	3.75	17.16	9.86
16	51	2580	0.00203	4.09	4.73	21.6	7.82
17	45	2050	0.00161	5.16	5.96	27.2	6.20
18	40	1620	0.00128	6.51	7.51	34.4	4.92
19	36	1290	0.00101	8.21	9.48	43.3	3.90
20	32	1020	0.000802	10.4	11.9	54.9	3.09
21	28.5	810	0.000636	13.1	15.1	69.1	2.45
22	25.3	642	0.000505	16.5	19.0	87.1	1.94
23	22.6	509	0.000400	20.8	24.0	109.8	1.54
24	20.1	404	0.000317	26.2	30.2	138.3	1.22
25	17.9	320	0.000252	33.0	38.1	174.1	0.970
26	15.9	254	0.000200	41.6	48.0	220.0	0.769
27	14.2	202	0.000158	52.5	60.6	277.0	0.610
28	12.6	160	0.000126	66.2	76.4	350.0	0.484
29	11.3	127	0.0000995	83.4	96.3	440.0	0.384
30	10.0	101	0.0000789	105.0	121.0	554.0	0.304
31	8.9	79.7	0.0000626	133.0	153.0	702.0	0.241
32	8.0	63.2	0.0000496	167.0	193.0	882.0	0.191
33	7.1	50.1	0.0000394	211.0	243.0	1114.0	0.152
34	6.3	39.8	0.0000312	266.0	307.0	1404.0	0.120
35	5.6	31.5	0.0000248	335.0	387.0	1769.0	0.0954
36	5.0	25.0	0.0000196	423.0	488.0	2230.0	0.0757
37	4.5	19.8	0.0000156	533.0	616.0	2810.0	0.0600
38	4.0	15.7	0.0000123	673.0	776.0	3550.0	0.0476
39	3.5	12.5	0.0000098	848.0	979.0	4480.0	0.0377
40	3.1	9.9	0.0000078	1070.0	1230.0	5650.0	0.0299

* From *Circ.* 31, U.S. Bureau of Standards.

Table above is based upon a resistivity of 0.15328 ohm (meter-gram) at 20°C. The temperature coefficient is $\alpha_{20} = 0.00393$, or $\alpha_0 = 0.00427$. The density is 8.89 g. per cc.

The resistivity of hard-drawn copper may be taken as about 2.7 per cent higher than that of annealed copper.

APPENDIX B

BARE CONCENTRIC LAY CABLES OF STANDARD ANNEALED COPPER

A.W.G.	Circular mils	Ohms per 1000 ft.		Pounds per 1000 ft.	Standard concentric stranding		
		25°C.	65°C.		Number of wires	Diameter of wires, mils	Outside diameter, mils
	2,000,000	0.00539	0.00622	6180	127	125.5	1631
	1,700,000	0.00634	0.00732	5250	127	115.7	1504
	1,500,000	0.00719	0.00830	4630	91	128.4	1412
	1,200,000	0.00899	0.0104	3710	91	114.8	1263
	1,000,000	0.0108	0.0124	3090	61	128.0	1152
	900,000	0.0120	0.0138	2780	61	121.5	1093
	850,000	0.0127	0.0146	2620	61	118.0	1062
	750,000	0.0144	0.0166	2320	61	110.9	998
	650,000	0.0166	0.0192	2010	61	103.2	929
	600,000	0.0180	0.0207	1850	61	99.2	893
	550,000	0.0196	0.0226	1700	61	95.0	855
	500,000	0.0216	0.0249	1540	37	116.2	814
	450,000	0.0240	0.0277	1390	37	110.3	772
	400,000	0.0270	0.0311	1240	37	104.0	728
	350,000	0.0308	0.0356	1080	37	97.3	681
	300,000	0.0360	0.0415	926	37	90.0	630
	250,000	0.0431	0.0498	772	37	82.2	575
0000	212,000	0.0509	0.0587	653	19	105.5	528
000	168,000	0.0642	0.0741	518	19	94.0	470
00	133,000	0.0811	0.0936	411	19	83.7	418
0	106,000	0.102	0.117	326	19	74.5	373
1	83,700	0.129	0.149	258	19	66.4	332
2	66,400	0.162	0.187	205	7	97.4	292
3	52,600	0.205	0.237	163	7	86.7	260
4	41,700	0.259	0.299	129	7	77.2	232

APPENDIX C

RESISTIVITY AND ELECTRICAL PROPERTIES OF MATERIALS

Material	Resistivity at 20°C. Microhms per centimeter-cube	Resistivity at 20°C. Ohms per cir. mil-ft.	Temperature coefficient of resistance, 20°C.
Aluminum	2.828	17.02	0.0039
Antimony	41.7	251.0	0.0036
Bismuth	110.0	663.0	0.004
Carbon:			
Amorphous	3800 to 4100	....	(—)
Graphite	720 to 812	....	(-)
Copper (drawn)	1.724	10.37	0.00393
Gold	2.44	14.7	0.0034
Iron:			
Electrolytic	9.96	59.9	
Cast	74.4 to 97.8	448–588	
Lead	20.4(0°)	123(0°)	0.00387
Mercury	94.07(0°)	566(0°)	0.00072
Nickel	6.93(0°)	41.7(0°)	0.0062
Platinum	10.96(0°)	66.0(0°)	0.003
Silver	1.629	9.8	0.0038
Steel:			
Soft	15.9	95.8	0.0016
Glass-hard	45.7	275	
Silicon 4 per cent	51.15	308	
Transformer	11.09	66.8	
Tungsten	5.51	33.2	0.005
Zinc	5.75(0°)	34.6(0°)	0.0037
Brass	7.0	42.1	0.002
Constantan (Cu 65; Ni 40)	49.0	295.0	0.000005
German silver (Ni 18)	33.0	198.5	0.0004
Iala	49.0	283.0	0.000005
Ideal	50.0	296.0	0.000005
Manganin (Cu 84, Mn 12, Ni 4)	44.0	265.0	0.000006
Monel metal	42.0	256.0	0.0019
Nichrome (Fe Ni Cr)	109.0	660.0	0.00019
Phosphor bronze (Cu, Sn 2 to 6; Ph 0.005 to 0.13)	3.95	23.7	

APPENDIX D

RELATIVE PERMITTIVITY OF DIELECTRICS

Material	Dielectric Constant K
Air	1.0006
Asphalt	2.7
Cloth, varnished	3.5 to 5.5
Ebonite	1.9 to 3.5
Fiber	2.0
Fuller board	7.5 (at 100°C.)
Fuller board, varnished	2.9 (at 25°C.)
Glass	5.5 to 10.0
Gutta-percha	3.0 to 5.0
Mica	2.5 to 6.0
Micanite	2.5 to 6.0
Paraffin wax	1.9 to 2.3
Porcelain	4.0 to 6.0
Rubber	2.0 to 3.0
Shellac	2.75
Slate	6.0 to 7.0
Transformer oil	2.5

PROBLEMS

Chapter I

1. Draw a horizontal line to represent a time axis, to the scale of 250 years per in., beginning with 600 B.C. and extending to A.D. 2000. Indicate on this line the most important events in electrical history.

2. Repeat this construction beginning with the year A.D. 1600 using a scale of 50 years per inch.

3. Three point charges Q_a, Q_b, and Q_c, each equal to $+20$ statcoulombs, are placed in air at the vertices a, b, c of an equilateral triangle whose sides are 10 cm. long. What is the force F on each charge, and what is its direction?

4. If, in Prob. 3, $Q_a = Q_b = +20$ and $Q_c = -20$, other conditions remaining the same, what are the forces F_a, F_b, F_c? Indicate their directions.

5. In Probs. 3 and 4, what is the field intensity at the point of intersection of the perpendiculars dropped from the vertices to the opposite sides?

6. Five point charges, in air, are distributed in a plane in accordance with the following table:

Point	Charge	Coordinates
A	-15	$(6, 2)$
B	$+20$	$(3, 4)$
C	$+10$	$(-3, 3)$
D	$+50$	$(-2, -4)$
E	-20	$(-1, -1)$

Find the resultant field intensity, and its direction, at a point which has the coordinates $(+5, -4)$.

7. What is the absolute potential, in statvolts, at the center of the circumscribing circle of the triangle of (*a*) Prob. 3, (*b*) Prob. 4?

8. The triangle of Prob. 3 is inscribed in a sphere of radius 50 cm. (*a*) What is the potential at the center of the sphere? (*b*) How much work is required to carry a statcoulomb from the center of the sphere to the center of the triangle?

9. From the result of part (*b*) of Prob. 8, compute the average force between the terminal points of the travel. How does this average compare with the forces at the two ends of the travel? How do you account for the result?

10. A charge of 50 statcoulombs is distributed uniformly around the circumference of a circle of radius 10 cm. What is the potential at a point 40 cm. from the plane of the circle and directly opposite the center of the circle?

11. *It is estimated that there are approximately* 5×10^{22} *free electrons* per cu. cm. of copper. Taking the charge of an electron as 4.803×10^{-10} statcoulomb, what is the average speed with which the electrons drift along a conductor having a cross-section of 1 sq. cm. when the current is 150 amp.? How can the result be reconciled with the fact that an electrical impulse travels along a wire with a speed that approaches that of light?

12. The atomic weight of cupric (bivalent) copper is 63.1. How many grams of copper should be deposited from copper sulphate solution per hour when a current of 10 amp. flows through the solution between copper electrodes?

13. A dry cell, when new, has an e.m.f. of 1.4 volts and an internal resistance of 0.08 ohm. (*a*) What is the initial difference of potential between the terminals of the cell? (*b*) If the terminals of the cell are connected by a short copper strap of negligible resistance, how much current will flow momentarily through the strap, and what is then the potential difference between the terminals of the cell? (*c*) If the potential difference in (*b*) differs from that in (*a*), what accounts for the difference? (*d*) If the cell remains short-circuited as in (*b*) for an appreciable time before being restored to open-circuit conditions, what will be the effect upon the e.m.f. and internal resistance of the cell?

14. The ends of a strip of German silver, 2 in. wide and 100 ft. long, having a resistance of 0.8 ohm, are connected to the terminals of a lead storage cell which has an e.m.f. of 2 volts and an internal resistance of 0.005 ohm. The connections between the battery terminals and the strip are made by means of heavy clamps which grip the German silver entirely across its 2-in. width. What potential difference will be found (*a*) between points 10 ft. apart, measured along the center line of the 2-in. strip? (*b*) Between points 1 in. apart along the center line? (*c*) Between points 1 in. apart measured at right angles to the center line of the strip?

15. Five resistors, whose resistances are 1, 2, 3, 4, and 5 ohms, are connected in series, and the two end terminals are connected to a 6-volt storage battery which has an internal resistance of 0.03 ohm. What is the difference of potential across each resistor and across the battery terminals?

16. Make a sketch of the five resistors of Prob. 15, assuming that they are strung out along a horizontal line whose ends A and B are connected to the battery terminals. A is the positive terminal and B the negative terminal; if terminal B is grounded, what are the potentials of the successive junction points between resistors, assuming that their order, in the direction from A to B, is as stated in Prob. 15. Erect ordinates at each junction point so that the ordinate represents the potential to a scale of 2 volts per in.

17. Repeat the construction of Prob. 16 if the order of the resistors from A to B is 2, 5, 4, 1, 3.

18. The pivoted coil of a millivoltmeter is designed to give full scale deflection of 50 divisions when it carries a current of 0.02 amp. The resistance of the coil, measured between the binding posts on the base of the instrument, is 2.5 ohms. Two identical instruments of this kind are connected in series with each other and with an external resistor of 4995

ohms, and the entire combination is connected across a 75-volt supply line. What is the deflection of each instrument?

19. A millivoltmeter having a resistance of 2.5 ohms and which gives full scale deflection (50 divisions) with a coil current of 0.02 amp. is in series with another millivoltmeter which has a resistance of 1 ohm and which requires a coil current of 0.05 amp. to give full scale deflection of 50 divisions. What resistance must be connected in series with the two instruments so that if the combination is connected across 100 volts the first instrument will give a deflection of 40 divisions? What will be the corresponding deflection of the second instrument? What is the drop of potential across each instrument and across the external resistor?

20. Twenty-five dry cells, each having an e.m.f. of 1.4 volts and an internal resistance of 0.08 ohm, are connected in series to a 10-ohm resistor. The current is found to be 2.45 amp. What conclusion may be drawn that will account for these data? All connections are perfectly solid.

21. A 3-cell storage battery has an e.m.f., when discharged, of 1.78 volts per cell; the internal resistance is 0.005 ohm per cell, and the normal charging current is 15 amp. What resistance must be connected in series with the battery to make it charge at normal rate from a 115-volt circuit? Toward the end of this charging period the e.m.f. per cell rises to 2.5 volts; what is the final charging rate?

22. What resistance in series with the battery of Prob. 21 will draw a current of 100 amp from the battery when its e.m f. is 2.0 volts per cell?

23. Three resistors of 10, 15, and 20 ohms, having maximum current-carrying capacities of 5, 3, and 2 amp., respectively, are in parallel and are to be supplied with current from a 220-volt circuit What is the minimum resistance which must be connected in series with the parallel combination so that the current-carrying capacity may not be exceeded in any one of the three resistors? What current will then be supplied from the supply circuit, and what is the drop of potential across the three parallel resistors?

24. *A* and *B* are two points between which resistors of 100 ohms and 200 ohms are connected in parallel. Between point *B* and a third point *C* three resistors of 25, 50, and 100 ohms are in parallel. Between points *A* and *C* is a branch circuit consisting of a 15-ohm resistor in series with two parallel-connected resistors of 10 and 20 ohms, respectively. Compute the equivalent resistance between points *A* and *C*. If the current in the 15-ohm resistor is 2 amp , what is the drop of potential in the 25-ohm resistor?

25. In Fig. 15, Chap. I, let the resistances of wires *a* and *b* be equal to 0.4 ohm, that of wire *c* 0.8 ohm, and that of load *B* 4 ohms. Load *A* is a storage battery having an e.m.f. of 110 volts and an internal resistance of 0.40 ohm, the polarity being such that the e.m f. of battery *A* opposes that of the supply line. Find the current in each part of the circuit and the potential differences at *A* and *B*. Construct the potential diagram of the circuit.

26. Assume that in the circuit defined in Prob. 25 the 110-volt battery is removed but that its place is taken by a 0.4-ohm resistor. Compute the current in branch *A*. Next assume that the 110-volt battery is restored to branch *A* but that the two 115-volt batteries are replaced by passive resistors

each of resistance 0.5 ohm, and compute I_A. Show that the algebraic sum of these two values of I_A is the same as the result of Prob. 25.*

27. The results of Probs. 25 and 26 may be generalized as the *principle of superposition,* namely: In a circuit in which the resistances are constant, each electromotive force produces an effect which is entirely independent of other electromotive forces distributed in any manner in the network. Deduce a proof by means of Kirchhoff's laws.

28. Resistors R_1, R_2, R_3 . . . R_n are connected in parallel between two points A and B. The circuit of resistor R_j is opened, and a potential difference of V volts is introduced into the opening, thereby setting up current I_k in R_k. Next, remove V from R_j and introduce it in the circuit of R_k, thus setting up current I_j in R_j. Prove that $I_j = I_k$.

29. Prove by means of Kirchhoff's laws that Prob. 28 is a particular case of the reciprocity theorem: If a current is produced at one point of a network by a given e.m.f. at another point, the same current will be produced at the second point by the given e.m.f. at the first point. Is there any change in the relative directions of the e.m.f. and current?

30. Four points A, B, C, D are arranged to form a square. Resistors are connected between these points as follows: A to B, 2 ohms; B to C, 3 ohms, C to D, 2 ohms; D to A, 3 ohms; B to D, 5 ohms. A variable resistor R is shunted across points A and C, and a 6-volt battery having an internal resistance of 0.2 ohm is also connected to points A and C. By means of Kirchhoff's laws find the value of R which will cause the current in R to be half the battery current. What is the corresponding current in the 5-ohm resistor between B and D?

31. Solve Prob. 30 by replacing the network on the right of the diagram by a single equivalent resistance.

32. Six equal resistors, each having a resistance of 10 ohms, are connected to form the sides of a tetrahedron. What is the equivalent resistance measured between any pair of apices?

33. Twelve equal resistors, each having a resistance of 10 ohms, are connected to form the sides of a cubical figure. Compute the equivalent resistance between diagonally opposite corners of the cube.

34. What is the equivalent resistance of the cubical arrangement of Prob. 33 when the measurement is made between two adjacent corners?

35. Three batteries of which the e.m.fs. are E_1, E_2, E_3 and the internal resistances are R_1, R_2, R_3, respectively, are connected in parallel with their positive terminals connected to point A and their negative terminals to point B. What is the current through each battery and what is the potential difference between A and B? Solve for the case where E_1, E_2, E_3 are 2, 4, 6 volts and R_1, R_2, R_3 are 0.05, 0.1, and 0.15 ohm, respectively.

36. If the polarity of the 6-volt battery in Prob. 35 is reversed, what are the individual battery currents and what is the potential difference between A and B?

37. Points A, B, C, D are the four corners of a square, and O is its middle point. Resistors are connected between these points in the following

* Principle of superposition.

manner: A to B, 5 ohms; B to C, 10 ohms, C to D, 25 ohms; D to A, 15 ohms; O to B, 2 ohms, O to C, 4 ohms; O to D, 3 ohms. If a difference of potential of 100 volts is impressed between points A and C, what is the current in the resistor OC? Use the star-delta method.

38. How much power is consumed in the circuit of Prob. 37 between terminals A and C?

39. The potential difference across the terminals of a circuit varies in accordance with the relation $v = 110\sqrt{2} \sin \omega t$ and the current through the circuit follows the law $i = 10\sqrt{2} \sin (\omega t - \pi/6)$. What is the formula for the instantaneous power consumed in the circuit? Plot the curves showing v, i, and power from $\omega t = 0$ to $\omega t = 2\pi$. What is the average power consumed in the circuit?

40. A water-rheostat containing 40 gal. of water is connected to a 220-volt circuit and takes a current of 50 amp. Neglecting radiation losses and the heat capacity of the rheostat, how long will it take to raise the temperature of the water from 20° to 90°C.? What is the cost if electrical energy is worth 4 cents per kw-hr.?

41. A water proof coil of 250 ohms, carrying a current of 1.5 amp., is placed in a vertical glass tube through which water flows from bottom to top at a controlled rate. The temperature of the water at the bottom of the tube is 58°F. and the temperature of the room is 70°F. By suitably adjusting the rate of flow of the water the outlet temperature of the water is held constant at 82°F. At what rate does the water flow? How can this experiment be used to measure the mechanical equivalent of heat? What is the purpose of adjusting the rate of flow until the initial and final temperatures of the water are equally below and above the room temperature?

42. Assume that gas burners and electric heating elements may be installed in such a way that all the heat developed is effectively utilized and that possible differences in first cost of equipment may be neglected. Which is the cheaper, gas costing 85 cents per 1000 cu. ft. and containing 850 B.t.u. per cu. ft., or electrical energy at 1.5 cents per kw.-hr.? At what rate must electrical energy be purchased to bring about equality under the conditions stated?

43. The pivoted coil of a millivoltmeter is designed to give full deflection of 150 scale divisions when it carries a current of 0.02 amp. The coil resistance is 2.5 ohms. (*a*) How should the instrument be connected to convert it into a direct reading voltmeter (*i.e.*, giving full scale deflection when 150 volts are impressed on the terminals)? (*b*) What must be the resistance of an external shunt so that the instrument will give full scale deflection when inserted in a circuit carrying 50 amp., the two leads between the shunt and the coil having a combined resistance of 0.01 ohm?

44. (*a*) A voltmeter constructed as in part (*a*), Prob. 43 and an ammeter constructed as in part (*b*) are used to measure the power consumed in a resistor R. The voltmeter is connected across the terminals of R and the ammeter is in series with the line leading to this parallel combination. If the instruments read 110 volts and 45 amp., how much power is consumed in R? (*b*) The connections are changed so that the ammeter is directly in series with R and the voltmeter is connected across the outside terminals of the

resultant series combination. If the instruments give the same readings as before, how much power is consumed in R?

45. A semicircular ring of annealed copper has an internal diameter of 5 cm. and an external diameter of 10 cm. and is 2 cm. thick. What is the resistance of the semicircular ring at 20°C. measured from one end face to the other, upon the assumption that the stream lines of the current are concentric semicircles?

Chapter II

1. Two slender bar magnets A and B, respectively 30 cm. and 10 cm. long, are held parallel to each other and to the magnetic meridian and are 3 cm. apart in the same horizontal plane, in air. Both magnets are placed so that their north poles face toward the magnetic north, and the two north poles are 5 cm. apart, so that the projection of B on A lies wholly within the length of A. The pole strengths of A and B are 250 and 100 c.g.s. units, respectively. What is the moment of the couple acting on B due to A? If magnet B is pivoted at its middle point, what is the magnitude and direction of the reaction between B and its pivot?

2. Magnet B of Prob. 1 is turned end for end, its position being otherwise unchanged. What is the moment of the couple acting upon it, and the magnitude and direction of the pivot reaction?

3. In Prob. 1 let the north pole of magnet A be the origin of a pair of coordinate axes x and y, the former perpendicular to A, the latter aligned with A. Compute the magnitude and direction of the field intensity at the points of which the coordinates (measured in centimeters) are $(-5, 0)$, $(-5, -15)$, $(-5, -30)$, $(0, 5)$, $(8, 0)$, $(8, -14)$, $(8, -30)$, $(0, -35)$, and from the results make a sketch of the lines of force surrounding magnets A and B.

4. Repeat Prob. 3, but with magnets A and B occupying the positions specified in Prob. 2.

5. A spherical glass flask of radius 5 cm. has a narrow neck through which may be inserted a slender magnetized bar. The spherical part of the flask is filled with a solution of chloride of iron which has an absolute permeability of 1.00064. The magnetized bar may be considered to have concentrated poles of 250 c.g.s. units at its ends, and its length is sufficiently great so that when the positive pole is at the center of the sphere the negative pole is too far away to have any effect. What is the strength of the distributed pole at the boundary of the liquid?

6. Construct an accurate drawing of the lines of induction surrounding a bar magnet 10 cm. long which may be assumed to have concentrated point poles of 100 c.g.s. units at its ends.

7. How many (conventionalized) lines of force per square centimeter will there be at points 5 cm. from the north pole, and 8 cm. from the south pole, of Prob. 6? Specify the direction of the field at these points relative to the axis of the magnet.

8. Two identical bar magnets, each having concentrated point poles of 100 c.g.s. units at its ends, are placed in line with each other, the north poles being 10 cm. apart and the two south poles being at the remote ends

of the system. Make an accurate drawing of the lines of induction produced by the adjacent north poles, the effect of the two south poles being ignored.

9. In Prob. 1, magnet B is turned through 90 deg. in a plane perpendicular to the original plane through the two magnets A and B. The axis of rotation passes through the middle point of magnet B. How much work is required thus to turn magnet B, and what is the difference in the amount of work required to turn it first through 90 deg. in one direction, then 90 deg. in the other? What agency does the work?

10. How much work is required to turn magnet B of Prob. 1 through 180 deg., that is, end for end?

11. Through a point O are drawn three mutually perpendicular axes of reference, OX directed to the north, OY toward the west, and OZ vertically upward. A right cone (made of a piece of wood), which has a circular base of 30 cm. diameter and a slant height of 30 cm., is placed with its vertex at O and its axis along OZ. Three slender bar magnets A, B, and C are attached to the surface of the cone so that each magnet constitutes an element of the cone. Magnet A, 30 cm. long, having poles of 100 c.g.s. units, is placed with its north pole at O so that it lies in the east-west plane through O and on the east side of the cone. Magnet B, 20 cm. long and having poles of 150 c.g.s. units, is placed on the north side of the cone in a north-south plane, its south pole being 5 cm. from O. Magnet C, 25 cm. long and having poles of 200 c.g.s. units, is attached to the south-west side of the cone in a northeast-southwest plane, so that its north pole is 5 cm. from O. How much work is required to place a magnet D, 20 cm. long, having poles of 200 c.g.s. units, on the circular section of the cone, so that the center of D shall be on axis OZ and its north pole pointing directly north?

12. An iron bar which has a cross-section 2 by 2 in. is bent to form a U-shaped magnet with the two end faces lying in the same plane. It is magnetized by a coil of wire through which a current flows. A soft-iron armature is placed across the end faces and it is found that a central pull of 295 lb. is required to overcome the force of attraction. What is the pole strength of the horseshoe magnet per square centimeter of end-face area?

13. A bar magnet of rectangular section ⅜ by 1 in. and 15 in. long has poles of 2000 c.g.s. units that may be assumed to be uniformly distributed over its end surfaces. What is the approximate flux density in gausses, at the middle section of the bar? What is the intensity of magnetization?

14. A point P, r cm. from a bar magnet of length l cm., lies on a line perpendicular to l that is midway between the poles $+m$ and $-m$. Prove that if r is large in comparison with l the field strength at P varies inversely as the cube of r. For given values of r and l, what is the percentage error in the inverse cube law?

15. A short bar magnet NS (Fig. 27) deflects needle ns through 30 deg. when the distance between centers of NS and ns is 20 cm. Magnet NS is 10 cm. long, and its circular cross-section has a diameter of 2 cm.; it is made of steel of density 7.6. When magnet NS is suspended on a fine fiber and is caused to oscillate as a torsional pendulum, it is found that the time of a complete swing is 29.1 sec. What is the horizontal component of the earth's field?

Chapter III

1. A circular loop of wire has a radius of 20 cm. and is placed with its plane in the magnetic meridian. At the center of the coil there is placed a small compass needle. On passing a current through the coil, the compass is deflected through an angle of 30 deg., the north pole of the compass being deflected toward the west. If the horizontal component of the earth's magnetic field is 0.2 gauss, what is the strength of the current and what is its direction around the coil? What is the effect of reversing the current in the coil?

2. A straight wire AB, 20 cm. long, carries a current of 15 amp. Find the intensity of the magnetic field at a point P which is 10 cm. from the wire and directly opposite end A of the wire. It may be assumed that the leads to which wire AB is connected have the directions PA and PB.

3. A wire is bent to form a circular arc of radius 10 cm. The center of curvature is placed at point P, Prob. 2, and one end of the curved wire is placed at point A, Prob. 2. The other end of the curved wire lies on line PB, as defined in Prob. 2. How much current must flow in the curved wire, and in what direction, to annul the magnetic field at P produced by the current of 15 amp. in the straight wire AB?

4. A coil of fine wire having 20 turns is wound on a rectangular frame 1 cm. wide and 3 cm. long and is mounted on pivoted bearings between the pole faces of a magnet that produces a uniform magnetic field of 500 gausses. The pivots are at the middle points of the 1-cm. sides, and the 3-cm. sides of the coil are perpendicular to the lines of induction. The plane of the coil is initially at right angles to the pole faces and is held in that position by springs of such strength that they will permit an angular twist of 10 deg. on applying a torque of 0.04 g.-cm. How much current must flow through the coil to produce a rotation of 30 deg.?

5. Other conditions remaining the same as in Prob. 4, what will be the effect of replacing the rectangular coil by a circular coil of 20 turns which has a diameter of 3 cm.?

6. The sides of a rectangular coil of 1 turn measure 20 cm. and 40 cm. The plane of the coil is placed parallel to the magnetic meridian so that the 40-cm. sides are horizontal. A small compass needle is placed 10 cm. from the plane of the coil at a point opposite the center of the rectangle. If the horizontal component of the earth's magnetic field is 0.2 gauss, what current in the coil will deflect the compass through 30 deg.?

7. A circular coil of 2 turns and radius 15 cm. is placed parallel to the rectangular coil of Prob. 6 so that the centers of the two coils are on a straight line through the pivot of the compass, which lies between the two coils. If the circular coil carries a current of 10 amp., flowing in a direction opposite to the current in the rectangular coil, how far must it be placed from the compass so that the needle will not be deflected?

8. Solve the bus bar problem of page 111 on the assumption (*a*) that the current in each bus bar is uniformly distributed in a vertical sheet through the central axis; (*b*) that the current in each bar is uniformly distributed over the entire cross-section.

9. A trolley wire running in an east-west direction carries a current of 300 amp. flowing toward the west. The earth's magnetic field has a flux density of 0.42 gauss, the angle of dip being 60 deg. and the declination being 5 deg. east of north. What is the total force on a length of 50 ft. of trolley wire, and what is its direction?

10. A steady current of 750 amp. is uniformly distributed over the circular cross-section of a copper bar which has a diameter of 1 in. What is the field intensity at the boundary of the wire? At the center of the section? At a point halfway between center and boundary?

11. Compute the total magnetic flux within the copper bar of Prob. 10, the length of the bar being 10 in.

12. What is the field intensity in the plane of a single-turn circular coil of 15 cm. radius at a point 3.75 cm. from the center, if the current through the coil is 25 amp.?

13. Make an estimate of the total flux that passes inside the circular coil of Prob. 12, if the diameter of the section of the wire is 0.15 cm.

14. A solenoid of 250 turns is wound uniformly on a cylindrical core 25 in. long and which has a diameter of 2 in. If the current in the coil is 10 amp. what are the values of the field intensity at the center, at the ends, and at points midway between the center and the ends?

15. At what points along the axis of the solenoid of Prob. 13 is the field intensity less than that at the center (*a*) by 10 per cent, (*b*) by 20 per cent?

16. At a certain point the flux density due to the earth's magnetic field is 0.42 gauss. The angle of dip of the field is 60 deg. below the horizontal, and the declination of the compass is 5 deg. east of north. An all-metal airplane having a wing spread of 75 ft. is flying toward the geographical northwest at a speed of 250 miles per hr. and is descending on a course that dips 30 deg. below the horizontal. What difference of potential is developed between the tips of the wings?

17. The e.m.f. between the wing tips of the airplane of Prob. 16, if applied to a millivoltmeter, would give a reading proportional to the speed of the plane. Why is this idea impracticable?

18. A rectangular coil having a single turn of No. 14 wire, measuring 20 by 40 cm., is dropped vertically between frictionless guides so that its plane remains horizontal, the 40-cm. sides being aligned in a geographical east-west direction. If the earth's magnetic field is defined by the data in Prob. 16, what current will flow in the coil 3 sec. after it is released?

19. A concentrated circular coil of 10 turns and radius 15 cm. is revolved around a horizontal axis at the rate of 5 r.p.s. If the earth's magnetic field is defined as in Prob. 16, is there a particular orientation of the coil axis that will yield an average e.m.f. of maximum value? If so, what is this orientation, and what is the maximum instantaneous e.m.f. developed in the coil?

20. A slender bar magnet 20 cm. long having concentrated poles of 200 c.g.s. units at its ends is placed on the axis of a 1-turn circular coil of radius 25 cm. The north pole of the magnet lies nearest to the plane of the coil and is originally 10 cm. away from it. The coil carries a current of 20 amp. flowing in a counterclockwise direction when viewed from the magnet.

How much work must be done to move the magnet along the axis of the coil until it occupies a symmetrical position on the other side of the coil?

21. Given the same coil and bar magnet as in Prob. 20, except that the south pole is nearest the plane of the coil in the initial position. How much work is required to turn the bar magnet end for end?

22. What is the average e.m.f. developed in the coil of Prob. 21 when the magnet is turned end for end, and what is its direction relative to the current in the coil? If the current is to be held constant at all times by means of a balancing e.m.f. from an external source, how much energy must be supplied from this source?

23. Derive a formula for the magnetic potential at a point P on the axis of a 1-turn square coil which has sides a cm long, the point P being p cm. from the plane of the coil. By differentiating the resultant expression for the potential with respect to p as variable, find the intensity of the magnetic field at point P in the axial direction, and check the result by finding the field intensity at P by means of the Biot-Savart law.

24. A toroidal ring of circular cross-section has an inside diameter of 6 in. and an outside diameter of 8 in. Half of the ring is made of magnetic material that has a relative permeability of 100, and the other half has a relative permeability of 400. The ring is wound with a uniformly distributed winding of 250 turns in which a current of 10 amp. is caused to flow. Compute the total flux in the ring core and the magnitude of the field intensity in each half. Of the total excitation applied to the ring, how many ampere-turns are consumed in each half? What is the value of H in each of the two parts of the ring?

25. If the ring core of Prob. 24 is replaced by a wooden core having the same dimensions, what is the value of H, and how much flux passes through the ring?

26. With reference to Fig 39, page 146, the following data characterize the test ring: Inside diameter, 6 in, outside diameter, 8 in., cross-section circular; $N_1 = 200$, $n_1 = 4$. In the calibrating solenoid, $l_2 = 30$ in., diameter of core 2 in., $N_2 = 720$, $n_2 = 20$. On reversing a current of 10 amp. in the calibrating coil the galvanometer deflects through 11 divisions, and on reversing a current of 2 amp. in the ring winding the galvanometer swings through 19.5 divisions. Compute the relative permeability of the ring core on the assumption that the deflection of the galvanometer obeys a linear law.

27. A cast-iron ring has inside and outside diameters of 6 in. and 8 in., and the cross-section is square. The ring is made of two semicircular halves which are ground and polished at the contact surfaces. An exciting winding of 150 turns wound uniformly around the ring carries a current of 10 amp. From the curves of Fig 37, compute the flux produced in the ring and the relative permeability of the core.

28. The two halves of the ring of Prob. 27 are clamped together with nonmagnetic spacing blocks ⅛ in. thick between their end faces. What is the flux in the ring and the relative permeability of the core?

29. The core shown in Fig. 43, page 151, is made of sheet-steel punchings which have a stacking factor of 0.9; that is, the net thickness of core is 90 per cent of the gross thickness. If the exciting winding on the middle leg

of the core produces a flux of 360,000 maxwells (0.0036 weber) across the $\frac{1}{8}$-in. airgap, what will be the flux across the $\frac{1}{4}$-in. gap? How many ampere-turns of excitation must the coil supply?

30. How many ampere-turns must be applied to the core specified in Prob. 29 so that the total flux through the middle leg of the core shall be 730,000 maxwells? What is then the flux through each airgap?

31. A cast-iron ring of circular cross-section has a mean diameter of 10 in., and the diameter of the cross-section is 1 in. If the ring is wound uniformly with a coil of 500 turns, how much current must flow to develop a relative permeability of 250 in the iron core? What is the self-inductance of the ring when this current is flowing? If the current is reduced to zero at a uniform rate in 0.001 sec., what e.m.f. will be developed in the winding?

32. Two No. 0000 wires are spaced 6 in. apart with their axes parallel. What is the inductance of the line per mile of length?

33. The cusps of the hysteresis loop of Fig. 50, page 158, lie on a curve which can be represented by the equation

$$\Phi = \frac{400{,}000i}{(2.5 + i)}$$

If the exciting winding of the magnetic core has 200 turns, what is the average inductance of the coil when the current varies between -5 and $+5$ amp.?

34. A coil which has a resistance of 5 ohms and an inductance of 10 henrys is connected to a 50-volt battery of negligible resistance. How much current will flow in the coil (*a*) 0.1 sec. after closing the circuit? (*b*) 1 sec. after closing the circuit?

35. If in an inductive circuit of R ohms and L henrys the current increases at a uniform rate for a time equal to the time constant of the circuit and the current is then found to have reached a value E/R amp., at what rate does the current increase?

36. The current in the coil of Prob. 34 has the steady value of 10 amp. An additional e.m.f. of 75 volts is then suddenly introduced by connecting in series with the first battery another battery of negligible resistance. How much current will flow in the circuit 0.1 sec. after the second battery is connected?

37. The coil of Prob. 34 is connected to a 50-volt and a 75-volt battery (both of negligible resistance) in series. When the current is steady, the 50-volt battery is suddenly cut out of the circuit, and 0.2 sec. later the 75-volt battery is cut out of the circuit. What current will flow in the coil 0.1 sec. after the second battery is removed?

38. The field winding of a shunt motor has a resistance of 25 ohms and an inductance of 50 henrys. It is supplied with current from a 125-volt circuit, and a voltmeter which has a resistance of 10,000 ohms is connected across its terminals. If the supply circuit is disconnected while the voltmeter is still connected to the winding terminals, what difference of potential will be developed across the voltmeter terminals? How much energy, in foot-pounds, is stored in the magnetic field?

39. Two circular coils A and B are mounted concentrically, one inside the other. A current in A produces a flux 75 per cent of which links with B, and a current in B produces a flux 80 per cent of which links with A. When the two coils are in series so that they magnetize in the same direction, the total self-inductance is 0.5 henry, and when they magnetize in opposite directions the total self-inductance is 0.04 henry. Find (a) the self-inductance of each coil; (b) the coefficient of mutual inductance; (c) the amount of work required to turn coil B through 180 deg., starting from the position in which they magnetize in the same direction, assuming that the coils are in series and that the current through them is 50 amp.

40. Coils A and B of Prob. 39 are arranged to magnetize in the same direction. The resistance of A is 25 per cent greater than that of B. If they are connected in parallel, what is their joint self-inductance? If B is turned through 90 deg., what is their joint self-inductance?

41. Coil A of Prob. 39 is made of two halves A_1 and A_2, mounted side by side so that 90 per cent of the flux due to A_1 links with A_2, and vice versa. Coil B is likewise made of two halves B_1 and B_2, so that 92 per cent of the flux due to B_1 links with B_2, and vice versa. Find the values of the inductance that can be obtained (a) when A_1 and A_2, in series, are connected in series with B_1 and B_2, in parallel; (b) A_1 and A_2, in parallel, are in series with B_1 and B_2, in series; (c) A_1 and A_2, in parallel, are in series with B_1 and B_2, in parallel; (d) A_1, A_2, B_1, B_2 are all in parallel.

42. In Prob. 27, what is the force of attraction between the two halves of the cast-iron ring, expressed in pounds?

43. What is the force of attraction, in pounds, between the two halves of the ring of Prob. 28?

44. The cast-iron core of Fig. 41, page 148, is magnetized until the flux is 160,000 maxwells. Compute the pull in pounds, between the pole faces at the airgap on the assumption that all of the flux passes around the circuit without fringing (bulging) at the airgap and without leakage through the air between the legs of the core. The pull produces a deflection of the 4-in. legs which tends to reduce the airgap and increase both the flux and the deflection. What condition determines the ultimate size of the gap and the magnitude of the magnetic pull?

45. Two cast-steel rods, each 1 in. in diameter, are placed parallel to each other, and 6 in. apart, center to center. If the relative permeability of the steel may be assumed to have the constant value of 4000, what is the inductance of the two rods per centimeter of length?

46. The magnetization curve of the iron composing the rods of of Prob. 45 may be represented by the equation $B = 17{,}200H/(5.75 + H)$. Make an estimate of the inductance of the rods per foot of length if the current in the rods is 10 amp., uniformly distributed over the cross-section.

Chapter IV

1. Brass tubing having rectangular cross-section is used in a pneumatic system for transporting paper tickets. If the paper becomes electrified by friction against the inner walls of the tube, can the electrification of the paper be removed by grounding the tubing?

2. A positive point charge of 1000 e.s.u. is placed at the center of a hollow spherical shell made of paraffin of relative permittivity 2.1. The inner and outer radii of the shell are 5 cm. and 8 cm. What is the magnitude of the charge that will be induced on the surface of the shell? What is the intensity of the electric field midway between the faces of the shell? How does this magnitude compare with the intensity at the same point if the shell were removed?

3. What is the capacitance of a parallel-plate condenser made of sheets of tinfoil 12 by 12 in. attached to the opposite faces of a glass plate $\frac{1}{16}$ in. thick. Assume that the relative permittivity of the glass is 8.0. Express the result in microfarads.

4. Concentric cylinders of sheet brass, having diameters of 6 in. and 8 in., are immersed in transformer oil having a dielectric constant of 2.5. What is the capacitance in microfarads per foot of length? What must be the areas of two parallel plates separated by the same distance as the two cylinders, to give this same capacitance when the plates are immersed in oil? How does this area compare with the corresponding amount of metal in the cylindrical capacitor?

5. Two concentric spheres made of sheet metal have diameters of 6 in. and 8 in. What is their capacitance when the space between them is filled with transformer oil? How does the capacitance per square foot of metal compare with that of cylindrical and parallel-plate condensers having the same total area of metal and the same separation, transformer oil being used as dielectric in all cases?

6. How much energy, in joules, is stored in the condenser of Probs. 4 and 5 when the potential difference in all cases is 1000 volts?

7. Three capacitors, of 5, 10, and 15 microfarads, are connected in Y between a point O and points A, B, and C, respectively. What is the capacitance measured between points A and B, B and C, and C and A? Derive expressions for the capacitances of a delta-connected set of capacitors equivalent to a Y-connected set which has capacitances C_A, C_B, C_C.

8. Derive expressions for the capacitances of a Y-connected set of condensers equivalent to a delta-connected set C_{AB}, C_{BC}, C_{CA}.

9. Points A, B, C, D are at the corners of a square, taken in sequence. Point O is the middle point of the square. Condensers are connected between these points as follows: A to B, 1 microfarad; B to C, 2 microfarads; C to D, 3 microfarads; D to A, 4 microfarads; O to A, 2 microfarads; O to B, 2 microfarads; O to C, 3 microfarads; O to D, 4 microfarads. What is the equivalent capacitance measured between points A and B; between points A and C?

10. Two point charges, $+Q_1 = 25$ e.s.u. and $-Q_2 = 5$ e.s.u., are placed 10 cm. apart. Specify the radius of the circle of zero potential and the position of its center.

11. Find the expression for the density of charge on the surface of the sphere of zero potential in Prob. 10 on the assumption that it is metallic and grounded. Construct a diagram of the circle of zero potential and show by an enveloping curve the variation of surface density, in such a way that the radial distance between the circle and the surrounding curve is everywhere proportional to the density.

12. A straight wire AB, 20 cm. long, has a charge of 5 e.s.u. per cm. of its length. Compute the magnitude and direction of the electric field 5 cm. from one end of the wire at a point P which lies on the line PB perpendicular to AB, the length PB being 5 cm.

13. The straight wire AB of Prob. 12 is replaced by a wire bent to form a circular arc of radius $PB = 5$ cm., the length of the arc being such that it subtends at P an angle equal to BPA in Prob. 12. If the charge on the curved wire is 5 e.s.u. per cm., prove that the magnitude and direction of the electric field are the same as in the case of the straight wire AB.

14. Two No. 10 wires are parallel and 0.125 in. apart, center to center. What is their capacitance in microfarads per foot of length. If their capacitance were computed by means of Eq. (50), what would be the percentage error in the result?

15. How far apart must be the wires of Prob. 14 in order that the error arising from the use of Eq. (50) may not be more than 2 per cent?

16. What is the capacitance to ground, in microfarads per mile, of a No. 00 wire mounted 20 ft. above the earth's surface?

17. A potential difference of 12,000 volts is established between the circular plates of an absolute electrometer which have a diameter of 5 cm. and which are 3 cm. apart. What is the force of attraction between the plates, expressed in milligrams?

18. Two flat, square metal plates, A and B, of negligible thickness, each measuring 12 by 12 in., are placed parallel to each other and 3 in. apart and are connected, respectively, to the positive and negative terminals of a 10,000-volt battery. A third plate C, of the same size as A and B, is placed midway between A and B and is connected to the midpoint of the battery. How much work is required to move plate C parallel to itself until it is $\frac{1}{2}$ in. from A and $2\frac{1}{2}$ in. from B?

19. Two parallel plates, placed 1 in. apart, are immersed in oil which has a dielectric constant of 2.5. A sheet of glass $\frac{1}{4}$ in. thick and having a dielectric constant 8 is placed midway between the plates. The breakdown voltage of the oil is 220 volts per mil, and that of glass is 300 volts per mil. Which material will fail first when the voltage between the plates is gradually increased?

20. The 10 successive layers of insulation of a condenser type bushing (Fig. 28, page 214) are all of equal thickness t. If the length of the outermost (shortest) layer is l, what are the lengths of the others if all are to be equally stressed?

21. How much time is required to charge a 10-microfarad condenser to three-fourths of full charge through a resistance of 2 megohms?

22. A 20-microfarad condenser, initially charged to a potential difference of 500 volts, is discharged through an unknown resistance. After 1 min. the potential difference at the terminals of the condenser is found to be 200 volts. What is the magnitude of the resistance? What type of voltmeter must be used in making these observations?

23. In Prob. 22, what is the initial current when the circuit is established?

24. What resistance must be put in series with an inductance of 50 millihenrys and a capacitance of 25 microfarads to give critical charge and

discharge conditions? If 500 volts is impressed on this circuit, what is the maximum current that will flow, and at what time after closing the circuit does the maximum current occur?

25. If the resistance in Prob. 24 is reduced to one-tenth of the critical value, what is the maximum charge in the condenser and what is the corresponding potential difference at its terminals? What is the natural frequency of oscillation of the circuit, expressed in cycles per second?

Chapter V

1. The kilogram-calorie is the amount of heat required to raise the temperature of 1 kg. of water by 1°C., and the B.t.u. is the amount of heat required to raise 1 lb. of water by 1°F. If experiment shows that 1 kg.-cal. is equivalent to 4.19 joules, show that 1 B.t.u. is equivalent to 778 ft-lb.

2. In Eq. (28), page 40, what is the reason for expressing the numerator as R_p^2?

3. By means of the procedure described in Art. 5, page 231, determine the ratio between the kilowatt-hour and the foot-pound. How many foot-pounds are equivalent to 50 kw.-hr.?

4. If in Eq. (18), page 234, the exponents a, b, f are eliminated, what grouping of the variables will result?

5. The resistivity of copper is 10.37 ohms per cir. mil-ft. What is the resistivity in abohms per meter-cube?

6. A wire 15 in. long, carrying a current of 100 amp. is at right angles to a magnetic field which has a flux density of 60,000 lines per sq. in. What is the force on the wire expressed in newtons? Express the flux density and the field intensity in m.k.s. units.

7. The absolute permeability of free space is unity in the c.g.s. electromagnetic system, and the absolute permittivity is unity in the c.g.s. electrostatic system. What is the absolute permittivity in the c.g s. electro-magnetic system, and what is the absolute permeability in the c.g.s. electro-static system?

8. An inductive winding of 500 turns is wound on a core which has a mean length of 30 in., a cross-section of 4 sq. in., and a relative permeability of 2000. Compute its inductance in henrys, using (*a*) c.g.s. units, (*b*) m.k.s. units. In each case write all terms in detail and state what difference, if any, exists in making the final reductions.

9. In Fig. 37, page 145, the scales of flux density and excitation are to be changed to m.k.s. units—webers per square meter, and ampere-turns per meter, respectively. Indicate these scales on the diagram.

10. What velocity of propagation of an electromagnetic wave is to be expected in glass which has a dielectric constant of 5.5?

11. Show by means of Tables III, IV, V that terms like VI and I^2R always have the dimensions of power; and that I^2Rt, $\frac{1}{2}LI^2$, $\frac{1}{2}CV^2$ always have the dimensions of energy.

Chapter VI

1. A concentrated coil of 50 turns is wound on a square frame which has sides 0.3 m. long. The coil is rotated at a speed of 10 r.p.s. about an axis

passing through the diagonal of the square frame. The coil is in a uniform magnetic field of flux density 0.31 weber per sq. m. directed at an angle of 30 deg. with the center line of the shaft. What are the average and maximum values of the e.m.f. developed in the coil, and what are the angular positions of the plane of the coil corresponding to each of these values of e.m.f ?

2. A second coil, identical with that of Prob. 1, is mounted in the same way on the same shaft, except that the plane of the second coil is 90 deg. from that of the first coil? If the two coils are connected in series, what is the maximum e.m.f. developed in the circuit, and at what angular position of the first coil does this maximum e.m.f. occur?

3. The eight-pole alternator of Fig. 2, page 268, has a field flux of 0.025 webers per pole, distributed sinusoidally around the armature. Each slot contains 16 conductors, all conductors being in series. The armature rotates at a speed of 900 r.p.m. What are the average and maximum values of the e.m.f. per alternation?

4. Given the same data as in Prob. 3, except that the field flux crosses the airgap on radially distributed lines. The poles cover 60 per cent of the armature surface. What are the average and maximum values of the e.m.f. per alternation?

5. A ring-wound armature has 600 uniformly distributed turns and rotates in a six-pole field having 0.0183 weber per pole. At what speed must it rotate to develop an e.m.f. of 220 volts?

6. The armature of Prob. 5 is provided with 300 commutator segments. What is the magnitude of the pulsation of voltage if the flux is sinusoidally distributed?

7. The winding of Prob. 5 is made of No. 8 A.w.g. wire and the length per turn is 2 ft. What is the resistance of the armature, measured between brushes, at a temperature of 75°C.?

8. The commutator of a machine which runs at 1150 r.p.m. has a diameter of 12 in. There are four sets of brushes, each set consisting of three carbon brushes each measuring 1.0 by 0.375 in. The contact pressure is 2.0 lb. per sq. in., and the coefficient of friction is 0.3. What is the brush friction loss, in watts?

9. Construct drawings, similar to Figs. 31, 32, 33, Chap. VI, showing the direction of current flow in all windings and the polarity of the poles, for (*a*) a series motor, (*b*) a shunt motor, (*c*) a cumulative-compound motor, the direction of rotation being counterclockwise.

10. A 115-volt shunt motor running without load takes a field current of 5 amp. and an armature current of 6 amp. When the armature is blocked, full-load current of 220 amp. is produced through it by impressing 4.85 volts upon the brushes. What are the resistances of the field and armature windings, and what is the counter e.m.f. at full-load armature current? What are the ohmic losses in the field and armature at no load and at full load? How do the ohmic losses at no load compare with the no-load power input, and how is the difference, if any, accounted for?

11. The machine of Prob. 10 is provided with a series field winding having a resistance of 0.008 ohm, and a field rheostat is connected in series with the

shunt field winding. The machine is then operated as a generator with a load current of 150 amp. and with a shunt excitation of 5.5 amp. which at the running speed develops an e.m.f. of 250 volts in the armature winding. Find the ohmic losses in the armature, in the shunt and series field windings, and in the shunt field rheostat when the generator is connected (a) long shunt, (b) short shunt.

12. The shunt field winding of a four-pole 115-volt motor must be designed to take an exciting current of 2.8 amp. when the working temperature is 75°C. Each field coil is wound on a cylindrical form and has a mean diameter of 7 in. If the wire is No. 15 A.w.g., how many turns must be wound on each coil?

13. A 125-volt compound generator like Fig. 33, page 306, is to be driven in the counterclockwise direction, and it is desired that the polarities of the brushes shall be the reverse of those shown. If there is no residual magnetism in the magnetic cricuit and the only source of current available is a half-dozen dry cells, what must be done to make the generator build up to full voltage?

14. A series-wound generator has a normal rating of 115 volts and 10 amp., and its field winding has a resistance of 1.3 ohms. If it is to be operated as a separately excited generator, how must the connections be made if there is available a 220-volt supply circuit?

Chapter VII

1. An armature has 96 coil sides which are to be arranged as a simplex lap winding for a six-pole field. Make a list of the theoretically possible combinations of front, back, and commutator pitches for both right-hand and left-hand windings, ranging from coils of full pitch down to two-thirds of full pitch. Prepare a complete winding table for one pair of front and back pitches, and make a complete drawing of the developed winding, showing the direction of current in each coil side.

2. An armature for a four-pole machine has 98 coil sides which are to be arranged as a simplex wave winding. State the possible combinations of front, back, and commutator pitches. Prepare a complete winding table for one set of pitches, also a drawing showing the direction of current in each coil side.

3. A drum armature has 450 conductors. What are the theoretically possible lap and wave windings, including simplex, duplex, and triplex windings, for a six-pole arrangement, provided that each element has one turn. State the front, back, and commutator pitches and the degree of reentrancy in each of these windings.

4. The armature core of a four-pole 550-volt railway motor has 57 slots and is to be provided with a two-circuit winding. If the flux per pole is approximately 0.03 weber when the motor speed is 1200 r.p.m., how many conductors must the winding have so that all the slots will be equally filled?

If the average voltage from segment to segment of the commutator is not to exceed 25 volts, how many turns must there be per element, and how many commutator segments must be provided? Indicate the front, back,

and commutator pitches that may be used. Is the use of a dummy coil necessary?

5. The name plate of a generator includes the following data: number of poles, 6; rating, 125 kw.; speed, 1175 r.p.m.; voltage, 230 volts. The armature winding has been stripped, and there are no data concerning its original arrangement except that there are 71 slots and 142 commutator segments. The area of each pole face is 26 sq. in., and it may be assumed that the flux density in the airgap is to be 50 kilolines per sq. in., plus or minus 10 per cent. Specify a winding that will meet these requirements, stating the type and front, back, and commutator pitches.

6. A 200-kw. 250-volt, eight-pole generator has a simplex lap-wound armature. There are 14 slots per pole and 224 commutator segments. The winding elements, each of 1 turn, are made of strip copper 0.1 by 0.4 in. in cross-section, the total length per element being 86 in. What is the resistance of the armature winding, measured between brushes, at the working temperature of 75°C., neglecting the contact drop at the brushes? What is the ohmic loss in the armature winding at full-load current (neglecting the shunt field current)?

7. A four-pole generator is rated as 15 kw. at 125 volts and 1170 r.p.m. The diameter of the armature is 12 in., the length of armature core is 5 in., and the pole arc is 70 per cent of the pole pitch. The armature has 47 slots, each containing four conductors, and there are 47 commutator segments. If it is assumed that the flux density in the airgap is about 50,000 lines per sq. in., how must the winding be arranged?

8. The average length of an end connection of the winding of Prob. 7 may be taken to be 1.5 times the pole pitch. If the winding is made of No. 4 A.w.g. wire, what is the armature resistance at 75°C.?

9. The armature core of a six-pole machine has 84 slots. The commutator has 252 segments. The winding is to have six coil sides per slot, arranged as a two-layer winding. What must be the front and back pitches so that the elements may be insulated in groups of three (*a*) if the winding is to be a simplex lap, (*b*) if it is to be a simplex wave?

10. Prepare a table showing the conditions that must be satisfied if equalizer connections are to be used on (*a*) lap windings, (*b*) wave windings, including simplex, duplex, and triplex windings, in machines having 2, 4, 6, . . . 24 poles; include the requirements as to number of conductors, number of conductors per slot, number of slots, degree of reentrancy.

11. Recheck the answers to Prob. 3 in the light of the considerations explained in Art. 21, Chap. VII.

12. A four-pole generator has a two-circuit (wave) armature winding. If the two south poles are deenergized, while normal current flows in the coils of the two north poles, what will be the terminal voltage at the brushes?

13. A six-pole armature has a simplex lap winding without equalizing connections. The resistance of the armature, measured between brushes, but not including the brush contact resistance, is 0.008 ohm. The connections between the brush sets are removed, the field is excited, and voltmeter readings are taken between adjacent brushes when the armature is rotated at normal speed. The readings are 219.6, 219.8, 220.0, 220.3, 220.7, 221.1

volts. The brush connections are reestablished, and the armature is connected to an external circuit that draws 400 amp. What is the terminal voltage of the generator, and how much current flows through each set of brushes, neglecting the contact drop at the brushes?

14. Solve Prob. 13 by taking into account that there is a drop of potential of 1 volt at each brush set.

Chapter VIII

1. The magnetization curve of a machine is determined by the following data:

Flux per pole, webers,	Total ampere-turns per pole	Ampere-turns per airgap	Ampere-turns per set of teeth
0 0482	1,955	1785	6 5
0 0730	3,025	2675	25
0 0970	4,268	3565	113
0 1220	6,335	4500	645
0 1330	8,160	4925	1440
0 1460	10,930	5400	2770
0 1580	14,235	5850	3850

Upon the assumption that the magnetization curve may be represented by Froelich's equation [Eq. (4), page 371] determine the constants a and b.

2. The 500-kw. machine which has the magnetization curve defined in Prob 1 has 12 poles and is wound to develop 550 volts at 100 r p m. The simplex lap-wound armature has 2484 conductors in 207 slots, and there are 621 commutator segments. Construct the saturation curve corresponding to rated speed.

3. If the brushes of the generator of Prob. 2 have a forward lead of two commutator segments, what is the number of demagnetizing ampere-turns per pole when the armature current is equal to rated current?

4. The generator of Probs 1 and 2 has an armature diameter of 94.5 in. and the percentage of polar embrace is 70. The airgap is 0 21 in. What is the flux density at the pole tips produced by the armature cross field when the armature current is 125 per cent of the rated full-load current of the machine, and the flux per pole, due to the field excitation, is 0 1330 weber?

5. Compute the demagnetizing effect caused by the cross-magnetization under the conditions specified in Prob. 4. Express the result in ampere-turns per pole.

6. When the machine specified in the preceding problems is delivering the armature current specified in Prob. 4, with a field excitation of 8160 ampere-turns per pole, what difference of potential may be expected between adjacent commutator segments at a point midway between adjacent brush sets? The resistance of the armature, measured between brushes, is 0.012 ohm.

7. The shunt field winding of the machine specified in the preceding problems has 1050 turns per pole. At no load the shunt field current is adjusted to develop an armature e.m.f. of 550 volts. What is the inductance of the entire shunt field winding, and how much energy, in joules, is stored in it under the stated conditions (magnetic leakage being ignored)?

8. What is the time constant of the shunt field winding under the conditions of Prob. 7? How much time is required for the shunt field current to fall to half value when the field winding is discharged through a noninductive resistance of 10 ohms?

Chapter IX

1. Estimate the leakage flux linked with the exciting winding of the cast-iron core of Fig. 41, page 148, under the conditions described in Art. 24, page 148.

2. A four-pole 120-volt shunt generator is rated at 25 kw. at 900 r.p.m. The armature has a simplex wave winding of 194 conductors, 1 turn per element. The shunt field winding has 800 turns per pole, and the no-load exciting current is 5.5 amp. Compute the saturation curve corresponding to rated speed, a tentative value of 1.15 being assumed for the leakage coefficient, given the following data:

External diameter of armature core	13.5 in.
Gross length of armature core	7.0 in.
Number of ventilating ducts	2
Width of each duct	$\frac{1}{4}$ in.
Radial depth of core below teeth	$2\frac{3}{4}$ in.
Number of slots	49
Slot dimensions	0.4 by 1.25 in.
Conductors per slot	4
Airgap (clearance)	$\frac{3}{16}$ in.
Ratio of pole arc to pole pitch	0.7
Diameter of cast-steel core	5.5 in.
Radial length of core (pole shoe to yoke)	6.0 in.
Thickness of pole shoe at center of pole face.	1.0 in.
Yoke, cast steel	1.5 by 10 in.
Diameter of commutator	9.0 in.

3. Compute the coefficient of dispersion of the machine specified in Prob. 2.

Chapter X

1. The saturation curve of a 10-pole 220-volt generator that has a rated output of 400 kw. at 200 r.p.m. is determined by the following data:

Field current I_s	0	2	4	6	8	10	12	14	16	18	20	22
Open-circuit e.m.f. E	5	41	82.5	117.5	145	165	182	195	207.5	219	230	240

Determine the constants a and b in Froelich's equation, $E = aI_s/(b + I_s)$, which best fit the given data in the working range of the flux.

2. The armature of the machine of Prob. 1 has a simplex lap winding of 810 conductors, 1 turn per element, and the brushes are advanced one segment beyond the neutral axis. The armature resistance, not including the brush contact resistance, is 0.003 ohm. The shunt field winding has 560 turns per pole, and its total resistance, exclusive of the regulating rheostat, is 10.4 ohms. If the shunt winding is separately excited from a 250-volt circuit, how much resistance must be inserted in series with the shunt winding to develop an open-circuit e.m.f. of 250 volts at a speed of 210 r.p.m.?

3. The machine of Probs. 1 and 2 is operated as a separately excited generator with a field excitation sufficient to develop an open-circuit e.m.f. of 230 volts at 205 r.p.m. What will be the terminal voltage when it is delivering an armature current of 1500 amp.? Assume that there is a drop of 1 volt at each brush set.

4. Plot the armature characteristic of the generator of Probs. 1 and 2 for a speed of 200 r.p.m. and a constant terminal voltage of 220 volts. Allow for a total brush contact drop of 2 volts except at zero current. Compute points on the curve for armature currents of 0, 500, 1000, 1500, 1850 amp.

5. A four-pole series generator, rated at 15 kw. at 125 volts and 1150 r.p.m., has a saturation curve which, within the working range and at rated speed, is represented by $E = 200I/(65 + I)$. The resistance of the field winding is 0.015 ohm, and that of the armature, exclusive of brush contacts, is 0.024 ohm. The brushes are set so that the armature demagnetizing ampere-turns per pole amount to 8 per cent of the field ampere-turns per pole. What is the terminal voltage when the generator is driven at 1200 r.p.m. and delivers 125 amp., the drop of potential at the brushes being 2 volts?

6. Solve Prob. 5 subject to the additional condition that the series field winding is shunted by a resistance of 0.05 ohm.

7. The generator specified in Probs. 1 and 2 is operated as a shunt generator at a speed of 200 r.p.m., with the regulating rheostat adjusted so that the terminal voltage at no load is 225 volts. What is the terminal voltage when the armature current is 1000 amp.?

8. The resistance of the shunt field rheostat of the generator of Prob. 7 is fixed at 5 ohms. Construct the theoretical external characteristic.

9. A 1200-r.p.m. shunt generator develops an open-circuit e.m.f. of 240 volts with a field current of 4 amp. If the speed is reduced to 1100 r.p.m., the open-circuit e.m.f. drops to 210 volts. What speed is required to develop 250 volts on open circuit? Neglect armature reaction and ohmic drop in the armature under no-load conditions.

10. The shunt generator of Prob. 7 is provided with a series field winding of 4½ turns per pole, the resistance of the entire series winding being 0.002 ohm. The windings are connected long-shunt. What must be the resistance of a shunt around the series field winding in order that the terminal voltage of the machine may be 230 volts when the armature current is 1850 amp.? Allow 2 volts brush drop.

11. Two shunt generators, rated at 150 kw. and 250 kw., are adjusted until each develops its normal open-circuit e.m.f. of 230 volts. The external characteristics may be assumed to be linear, the voltage regulation of the

150-kw. machine being 6 per cent and that of the 250-kw. machine 4 per cent, both in terms of their respective full-load terminal voltages. The generators are connected in parallel and supply a total load of 350 kw. What is the load on each machine, and what is their common terminal voltage?

12. It is desired to alter the degree of compounding of a 250-kw. 250-volt compound generator so that the voltage increases from 220 volts at no load to 250 volts at rated load. With the series field coils out of circuit and the shunt winding excited from an external source, a load test shows that the increase of voltage can be obtained by increasing the shunt field current from 18 amp. at no load to 32 amp. at full-load current. There are 400 shunt turns per pole and 5½ series turns per pole. The total series field resistance is 0.00238 ohm. What must be the resistance of the series shunt? Long-shunt connections are used, and it is permissible to neglect the series ampere-turns at no load and the drop in the series field winding at full load.

13. The speed of an under-compounded generator is increased 10 per cent and the terminal voltage at no load is adjusted to the value it had at normal rated speed. Will the degree of under-compounding be greater or less than that at normal speed?

14. Two identical over-compounded generators are connected in parallel to a common set of bus bars. Their saturation curves may be represented by the same equation (Froelich's). The resistances of armature windings and of shunt and series field windings are known, also the numbers of turns per pole in the shunt and series field windings. The connections to the switchboard are made by cables of sufficient size to provide 1 sq. in. of cross-section per 1000 amp. (adjusted so that the cables are of standard size), and the equalizer cables have the same cross-section as the main leads. One machine, A, is 40 ft. from the switchboard, and the other, B, is 65 ft. from the switchboard. If the total current supplied to the load is double the rated current of either machine alone, how much load does each machine carry? First outline the procedure in analytical form, then insert data representing the machine described in Probs. 1, 2, 7, 10.

15. An arc-welding generator develops an open-circuit e.m.f. of 100 volts and a short-circuit current of 200 amp. A second generator of similar type yields corresponding values of 90 volts and 220 amp. In both cases the load characteristics may be represented by equations of the type $V = E_0 - cI^2$. What is the maximum power output of each machine?

Chapter XI

1. A 230-volt 25-hp. shunt motor has an armature resistance of 0.136 ohm and a field resistance of 76.6 ohms. At full load the current input to the motor is 94.3 amp. What are the losses in the armature and field windings, the counter e.m.f. and the total mechanical power developed at full-load output? Make an allowance of 2 volts for the drop at the brushes.

2. The motor of Prob. 1 is provided with a series field winding which has a resistance of 0.025 ohm. If the load is adjusted so that the current input is 94.3 amp., solve for the same quantities as in Prob. 1 (a) for long-shunt connection, (b) for short-shunt connection.

3. The shunt motor of Prob. 1 takes a line current of 7.4 amp. at no load and runs at a speed of 1175 r.p.m. What is the (ideal) no-load speed?

4. The shunt motor of Prob. 1 has a magnetization curve that may be represented by Froelich's equation such that an exciting current of 1.5 amp. produces two-thirds as much flux as that produced by a field current of 3.0 amp. What is the ideal no-load speed if the shunt field resistance is increased to 85 ohms by inserting an auxiliary field rheostat?

5. The shunt motor of Probs. 1 and 3, when carrying its full-load current, develops a demagnetizing effect due to armature reaction equivalent to 8 per cent of the field ampere-turns. Compute the speed (*a*) at full load, (*b*) at a load that requires 150 per cent of full-load current.

6. What resistance must be inserted in series with the armature of the shunt motor specified in Probs. 1, 3, and 5 to make it develop (*a*) full-load torque at the moment of starting, (*b*) a starting torque 50 per cent greater than full-load torque?

7. A line tangent to the magnetization curve at the origin of coordinates represents the relation between flux and the excitation required for the airgap (explain why this is true). From this fact and from the data in Prob. 4, determine the ideal zero-load speed of the shunt motor of the preceding problems when the airgap is increased 10 per cent by boring out the pole faces.

8. What will be the speed of the motor specified in the preceding problems, after the airgap has been lengthened as in Prob. 7, when the total current input to the motor is 94.3 amp.?

9. The series field defined in Prob. 2, when connected long-shunt, produces 20 per cent as much excitation as the shunt field winding when the armature current has the full-load value specified in Prob. 1. Construct the speed and torque characteristics (*a*) when the compounding is cumulative, (*b*) when it is differential, a brush drop of 2 volts being allowed for.

10. The shunt motor of the preceding problems is operated successively with impressed voltages of 200, 220, 240, 260, 280 volts. Construct the speed characteristics by determining values of speed corresponding to zero armature current and to armature currents of 50 per cent and 100 per cent of full-load value.

11. Construct the speed characteristics of the same shunt motor when its field is separately excited from a 225-volt circuit and its armature is supplied (*a*) from a 115-volt source, (*b*) from a 440-volt source.

12. The shunt motor of Probs. 1, 3, 4, 5 is to be required to start its load with an initial armature current of 110 amp. Determine the resistances of a five-step starting rheostat so that when the starting resistance is all out of circuit the armature current will have its full-load value.

13. A 20-ton car is equipped with four motors, each of which has the characteristics shown in Fig. 47, page 525. The car is to be accelerated at the rate of 1.5 miles per hr. per sec. on a level track, and the frictional resistance may be assumed to be constant at 8 lb. per ton. The trolley voltage is 600 volts, the drive wheels are 24 in. in diameter, and the gear ratio is 15:58, as shown in Fig. 47. Assume that the starting resistance is controlled so that the current in each motor remains constant during the

acceleration period, while the starting resistance is being cut out of circuit, and that after the resistance is all out the motors take current directly from the trolley. What is the current input per motor during the initial acceleration period? Construct the speed-time curve of the car to show the speed from the moment of starting until a total time of 25 sec. has elapsed. [HINT: From the relation $a = dv/dt$, $t = \int_0^v (1/a)dv$].

14. Make an estimate of the moment of inertia of the armature of the machine specified in Prob. 2, Chap. IX. The commutator length may be taken as 5 in. Neglect the teeth, and assume that the armature winding is equivalent to a copper shell 1 in. thick and 13 in. long. Neglect also the spider construction of armature core and commutator.

15. If at a speed of 750 r.p.m. the armature of the machine of Prob. 14 is found to be slowing down at the rate of 15 r.p.m. per sec., what is the resisting torque and what is the power consumption?

16. A motor-generator set is provided with a solid cast-steel flywheel 8 ft. in diameter and 15 in. thick. How much energy is stored in the flywheel at a speed of 600 r.p.m.? How much power will the flywheel supply to the load if its speed is reduced at the rate of 10 r.p.m. per sec.?

Chapter XII

1. The armature of a four-pole machine has 57 slots, wound with four conductors per slot arranged as a simplex lap winding, 1 turn per element. The commutator has a diameter of 10 in., and the brush width is 0.75 in. in the tangential direction. Prepare a diagram showing the sequence of commutation in adjacent slots, on the assumption that the coils are insulated in pairs for insertion in the slots.

2. The armature winding of Prob. 1 is rearranged to form a simplex wave winding, all other conditions remaining the same. Prepare a diagram showing the sequence of commutation (*a*) in slots in which all four conductors are active, (*b*) in the slots in which the order is affected by the dummy coil.

3. What is the theoretical duration of short circuit in the armatures of Probs. 1 and 2 if the speed of rotation is 1750 r.p.m.?

4. A four-pole machine has a simplex wave winding arranged in 63 slots, four conductors per slot, two turns per element. The diameter of the commutator is 10 in., and the tangential thickness of the brushes is 1¼ in. The outside diameter of the armature core is 12 in., the axial length of the core is 5 in., and the slots are 0.3 in. wide by 1.0 in. deep. The conductors consist of No. 4 double cotton-covered wire (round) placed one over the other. The thickness of insulation at the bottom of the slot, and also between the top and bottom elements in a slot, is 15 mils. The total length of the end connections joining consecutive conductors is three times the pole pitch, and the pole arc is 70 per cent of the pole pitch. What is the average reactance voltage per element if the machine is rated at 15 kw., 125 volts, at 1150 r.p.m.?

5. An armature has semi-closed circular slots of radius r cm., as illustrated in Fig. 23, page 292. The opening at the top of the slot is r_1 cm. wide and r_2 cm. deep, the center of the circular slot being $(r + r_2)$ cm. below the

periphery. If the slot contains z conductors, distributed uniformly over the cross-section of the circular slot, what is the inductance due to slot leakage on the assumption that the leakage flux passes straight across the slot?

Chapter XIII

1. A 220-volt shunt motor has an armature resistance, exclusive of brush contact resistance, of 0.44 ohm and a shunt field resistance of 169 ohms. When running without load the armature current is 1.6 amp. and the speed is 1180 r.p.m. Find the conventional efficiency, the efficiency of conversion, and the mechanical efficiency when the armature current is 25 amp. What is the horsepower output corresponding to this armature current, and what is the speed if it is assumed that the flux is constant?

2. Find the efficiency of the machine of Prob. 1 corresponding to armature currents of 5, 10, 15, 20, and 35 amp., and plot a curve showing the relation between efficiency and horsepower output.

3. At what value of horsepower will the motor of Probs. 1 and 2 have maximum efficiency, and what is the maximum efficiency?

4. Compute the all-day efficiency of the motor of the preceding problems if it operates for 3 hr. at one-fourth load, 2 hr. at one-half load, 1½ hr. at three-fourths load, 3 hr. at full load, and ½ hr. at 1¼ times rated load, the length of the working day being 10 hr.

5. If it is required to design a shunt motor that shall have maximum efficiency at five-eighths of full load and an efficiency of 82 per cent at full load, what must be the variable and fixed losses expressed in percentage of full-load rating? What is the value of the maximum efficiency? What is the efficiency at one-fourth load?

6. A 230-volt 30-hp. shunt motor takes a line current of 7.3 amp. when running without load. The field current is 3.1 amp., and the armature resistance, not including the brush contacts, is 0.089 ohm. This motor is belted to a 120-volt 25-kw. shunt generator which has a full-load efficiency of 87 per cent. The belt loss is 0.41 hp. The field current of the generator is 5.6 amp., and the armature resistance, not including the brush contacts, is 0.025 ohm. What is the motor input when the generator is delivering full load?

7. A 230-volt 20-kw. compound machine has an armature resistance, not including the brush contacts, of 0.093 ohm, a shunt field resistance of 99.5 ohms, and a series field resistance of 0.038 ohm. When operated as a shunt motor without load, from 230-volt mains, the line current is 7.5 amp. If the machine is operated as a flat-compounded generator, what is its conventional efficiency at full load?

8. The machine of Prob. 7 has four poles, and its rated speed is 1150 r.p.m. Determine the approximate values of the diameter and length of the armature core on the assumption that the flux density in the airgap is 45,000 lines per sq. in. and that there are to be 500 ampere-conductors per in. of periphery, the ratio of pole arc to pole pitch being 0.7.

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